

Two equivalent ways of directing the spaces

Emmanuel Haucourt

CEA LIST,
Modeling and Analysing Interaction between Systems Laboratory

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The *Pakken-Vrijlaten* language

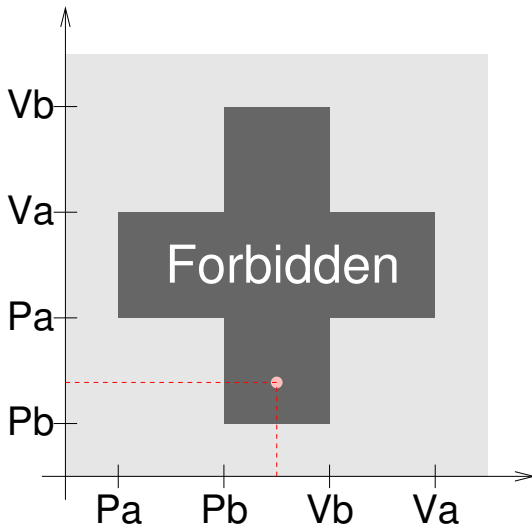
Edsger Wybe Dijkstra (1968)

#mutex a b

$P(a).P(b).V(b).V(a) \mid P(b).P(a).V(a).V(b)$

The geometric interpretation of the *PV* language

Scott D. Carson and Paul F. Reynolds (1987)



Partially Ordered Spaces $\mathcal{Po}$

Leopoldo Nachbin (1948,1965)

$$\text{pospace } \overrightarrow{X} : \begin{cases} X & \text{topological space} \\ \sqsubseteq & \text{partial order closed in } X \times X \end{cases}$$

morphism f from $\overrightarrow{X}$ to $\overrightarrow{X'}$: **continuous** and **order preserving** maps.

Directed real line $\overrightarrow{\mathbb{R}}$ and the sub-objects of its products.

The directed loops are **not allowed** in $\mathcal{Po}$.

Locally Ordered Spaces $\mathcal{Lpo}$

Lisbeth Fajstrup, Eric Goubault and Martin Raußen (1998)

$$\vec{X} : \begin{cases} X & \text{topological space} \\ \mathcal{U}_X & \text{open covering}^1 \text{ of } X \\ (U, \sqsubseteq_U) & \text{pospace for all } U \in \mathcal{U}_X \end{cases}$$

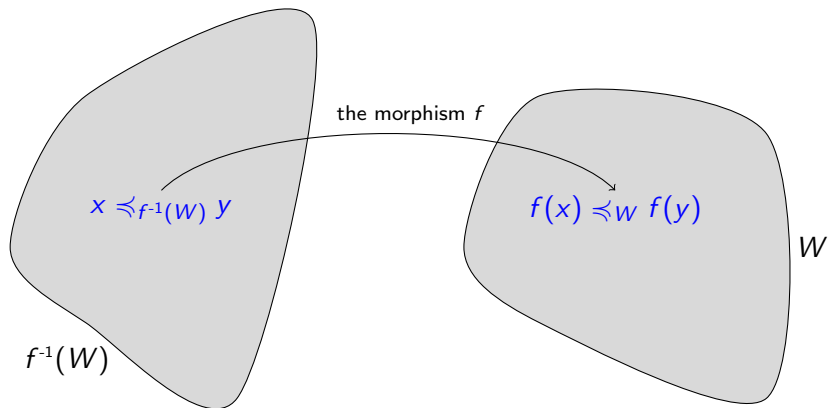
$$(\sqsubseteq_U)|_{U \cap V} = (\sqsubseteq_V)|_{U \cap V} \text{ for all } U, V \in \mathcal{U}_X$$

$f : \vec{X} \rightarrow \vec{X}'$ **continuous** and **locally order preserving** maps

i.e. $x \sqsubseteq_U y \Rightarrow f(x) \sqsubseteq_{U'} f(y)$ for all $U \in \mathcal{U}_X$ and $U' \in \mathcal{U}_{X'}$
such that $U \subseteq f^{-1}(U')$

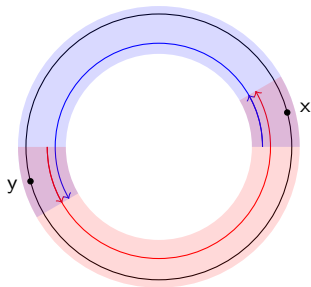
¹Actually one can even suppose that $\mathcal{U}_X$ is a $\sqsubseteq$ -ideal.

Morphisms of $\mathcal{L}po$



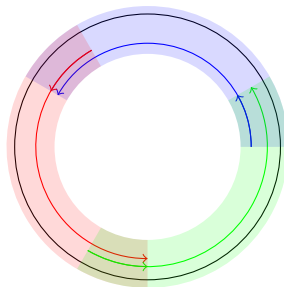
Locally Ordered Spaces

Directed circle $\overrightarrow{\mathbb{S}^1}$ and the sub-objects of its products



$$x \sqsubseteq y \text{ and } y \sqsubseteq x$$

Problem



Colimits in $\mathcal{L}po$ are ill-behaved

since $\mathcal{L}po$ does not allow vortex

- $\mathbb{C} \setminus \{|z| < 1\}$ has a local pospace structure such that
 $(r, \theta) \in \overline{[1, +\infty[} \times \overrightarrow{\mathbb{R}} \longmapsto re^{i\theta} \in \mathbb{C} \setminus \{|z| < 1\}$ is a morphism of $\mathcal{L}po$.
- $\mathbb{C}$ has **no** local pospace structure such that
 $(r, \theta) \in \overrightarrow{\mathbb{R}}_+ \times \overrightarrow{\mathbb{R}} \longmapsto re^{i\theta} \in \mathbb{C}$ is a morphism of $\mathcal{L}po$.
- The following is a pushout in $\mathcal{L}po$

$$\begin{array}{ccc}
 \overrightarrow{\mathbb{C} \setminus \{|z| < 1\}} & \xrightarrow{z \mapsto |z|} & \overrightarrow{\mathbb{R}}_+ \\
 \uparrow & & \uparrow \\
 \overrightarrow{\{|z| = 1\}} & \xrightarrow{!} & \{0\}
 \end{array}$$

Streams *Str*

Sanjeevi Krishnan (2006)

A **stream** is a topological space X equipped with a **circulation** i.e. a mapping defined over the collection Ω_X of open subsets of X

$$W \in \Omega_X \mapsto \preceq_W \text{ preorder on } W$$

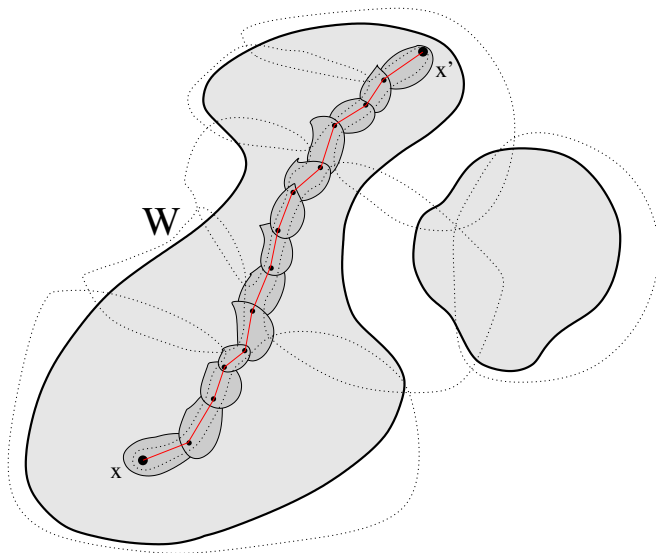
such that for all $W \in \Omega_X$ and all open coverings $(O_i)_{i \in I}$ of W

$$(W, \preceq_W) = \bigvee_{i \in I} (O_i, \preceq_{O_i})$$

$f : \vec{X} \rightarrow \vec{X}'$ **continuous** and **locally order preserving** maps

i.e. $x \preceq_{f^{-1}(W')} y \Rightarrow f(x) \preceq_{W'} f(y)$ for all $W' \in \Omega_{X'}$

The stream condition



Moore paths and Concatenation

on a topological space X

A **Moore path** is a continuous mapping $\delta : [0, r] \rightarrow X$ ($r \in \mathbb{R}_+$)

Its source $s(\delta)$ and its target $t(\delta)$ are $\delta(0)$ and $\delta(r)$

A **subpath** of δ is a path $\delta \circ \theta$ where $\theta : [0, r] \rightarrow [0, r']$ is **increasing**

Given a path $\gamma : [0, s] \rightarrow X$ such that $s(\gamma) = t(\delta)$ we have the concatenation of δ followed by γ

$$\gamma * \delta : [0, r + s] \longrightarrow X$$

$$t \longmapsto \begin{cases} \delta(t) & \text{if } t \in [0, r] \\ \gamma(t - r) & \text{if } t \in [r, r + s] \end{cases}$$

The path category functor

from $\mathcal{Top}$ to $\mathcal{Cat}$

- The **points** of X together with the **Moore paths** of X and their **concatenation** form a category $P(X)$ whose **identities** are the paths defined on $\{0\}$
- This construction is **functorial** $P : \mathcal{Top} \rightarrow \mathcal{Cat}$

d -Spaces $dTop$

Marco Grandis (2001)

A topological space X and a collection dX of paths on X s.t.

- dX contains all constant paths
- dX is stable under concatenation
- dX is stable under subpath

$f : \vec{X} \rightarrow \vec{X}'$ continuous and $f \circ \delta \in dX'$ for all $\delta \in dX$

Examples of d-spaces

- the compact interval $[0, r]$ with all the continuous increasing maps on it : denoted by $\uparrow\mathbb{I}_r$
- the Euclidean circle with paths $t \in [0, r] \mapsto e^{i\theta(t)}$ where θ is any increasing continuous map to $\mathbb{R}$: denoted by $\uparrow\mathbb{S}^1$
- the directed complex plane $\uparrow\mathbb{C}$ with paths $t \in [0, r] \mapsto \rho(t)e^{i\theta(t)}$ where ρ and θ are any increasing continuous map to $\mathbb{R}_+$ and $\mathbb{R}$

Examples of streams

- the compact interval $[0, r]$ with $x \preceq_U x'$ when $x \leq x'$ and $[x, x'] \subseteq U$: denoted by $\overrightarrow{\mathbb{I}}_r$
- the Euclidean circle with $x \preceq_U x'$ when $x \curvearrowright x' \subseteq U$ denoted by $\overrightarrow{\mathbb{S}^1}^2$

² $x \curvearrowright x'$ denotes the anticlockwise arc from x to x' .

Alternative approaches

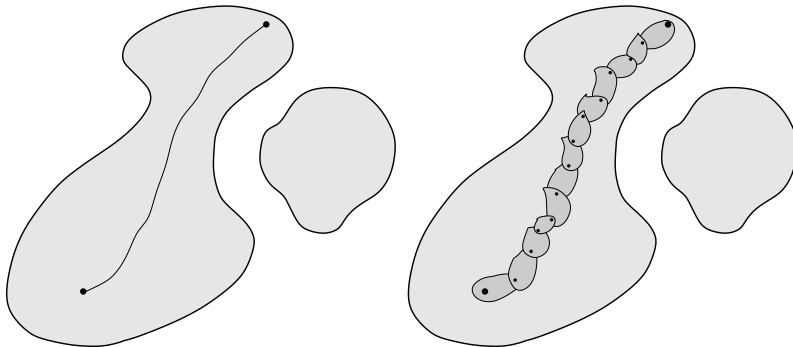
- Enriching small categories in $\mathcal{T}op$ ([Philippe Gaucher](#))
- Completing $\mathcal{L}po$ by means of Sheaves and Localization ([Krzysztof Worytkiewicz](#))

From $dTop$ to Str

The functor S

Let (X, dX) be a d-space and put $x \preccurlyeq_U x'$ when there exists $\delta \in dX$ such that

- $\exists t, t' \in \text{dom}(\delta)$ s.t. $t \leq t'$, $\delta(t) = x$ and $\delta(t') = x'$
- $\text{img}(\delta) \subseteq U$



From Str to $dTop$

The functor D

Let $(X, (\preceq_U)_{U \in \Omega_X})$ be a stream and consider the following collection of paths on the underlying space of X

$$\bigcup_{r \in \mathbb{R}_+} Str[\overrightarrow{\mathbb{I}_r}, X]$$

Theorem (Sanjeevi Krishnan)

$$(S : dTop \rightarrow Str) \dashv (D : Str \rightarrow dTop)$$

Denote the **unit** and the **co-unit** by η and ε

The cores of $\mathcal{Str}$ and $d\mathcal{Top}$

- Let $\mathcal{Str}$ be the full subcategory of $\mathcal{Str}$ whose collection of objects is

$$\{S(X) \mid X \text{ d-space}\}$$

- Let $d\mathcal{Top}$ be the full subcategory of $d\mathcal{Top}$ whose collection of objects is

$$\{D(X) \mid X \text{ stream}\}$$

By restricting the codomains of S and D we have the functors

$$S' : d\mathcal{Top} \rightarrow \mathcal{Str} \text{ and } D' : \mathcal{Str} \rightarrow d\mathcal{Top}$$

Some objects of $d\overline{Top}$ and $S\overline{tr}$

Directed versions of some usual spaces

- Compact Interval : $S(\uparrow\mathbb{I}_1) = \overrightarrow{\mathbb{I}}_1$ and $\uparrow\mathbb{I}_1 = D(\overrightarrow{\mathbb{I}}_1)$
- Hypercubes : $S((\uparrow\mathbb{I}_1)^n) = (\overrightarrow{\mathbb{I}}_1)^n$ and $D((\overrightarrow{\mathbb{I}}_1)^n) = (\uparrow\mathbb{I}_1)^n$ for all $n \in \mathbb{N}$
- Euclidean Circle : $S(\uparrow\mathbb{S}^1) = \overrightarrow{\mathbb{S}}^1$ and $\uparrow\mathbb{S}^1 = D(\overrightarrow{\mathbb{S}}^1)$
- Complex plane : $S(\uparrow\mathbb{C}) = \overrightarrow{\mathbb{S}}^1$ and $\uparrow\mathbb{S}^1 = D(\overrightarrow{\mathbb{C}})$
- Riemann Sphere : $S(\uparrow\Sigma) = \overrightarrow{\Sigma}$ and $\uparrow\Sigma = D(\overrightarrow{\Sigma})$

Properties

- The natural transformations $\eta * D$, $S * \eta$, $D * \epsilon$ and $\epsilon * S$ are identities, in particular $S \circ D \circ S = S$ and $D \circ S \circ D = D$
- the adjoint pair $S \dashv D$ induces a pair of isomorphisms $(\overline{S}, \overline{D})$

$$\overline{S} \circ \overline{D} = id_{S\overline{T}} \quad \overline{D} \circ \overline{S} = id_{\overline{d}Top}$$

More properties

- $d'Top$ is a mono and epi reflective subcategory of $dTop$: the reflector being $\overline{D} \circ S'$
- Str is a mono and epi coreflective subcategory of Str : the coreflector being $\overline{S} \circ D'$
- $d'Top$ and Str are complete and cocomplete
- the following diagrams commute

$$\begin{array}{ccc}
 Str & \xrightarrow{D} & dTop \\
 \uparrow & & \downarrow \overline{D} \circ S' \\
 Str & \xrightarrow{\overline{D}} & d'Top
 \end{array}$$

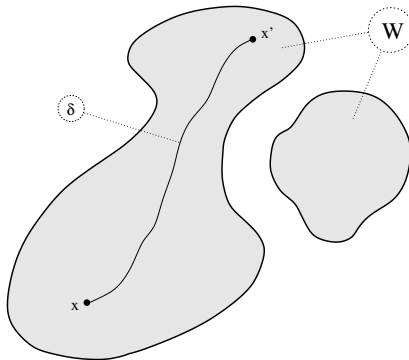
$$\begin{array}{ccc}
 Str & \xleftarrow{S} & dTop \\
 \downarrow \overline{S} \circ D' & & \uparrow \\
 Str & \xleftarrow{\overline{S}} & d'Top
 \end{array}$$

Describing the coreflector $\overline{S} \circ D'$

Let X be a stream and UX its underlying space

For all $W \in \Omega_{UX}$ we have $x \preceq_W^{(\overline{S} \circ D'(X))} x'$ iff

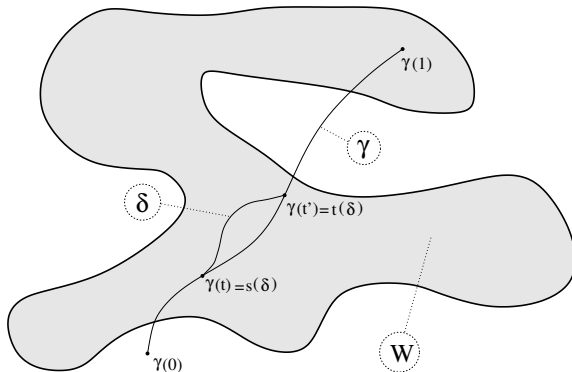
$\exists \delta \in \text{Str}[\vec{\mathbb{I}}_1, X]$ s.t. $s(\delta) = x$, $t(\delta) = x'$ and $\text{img}(\delta) \subseteq W$



Describing the reflector $\overline{D} \circ S'$

Let X be a d-space and UX its underlying space

Given a path $\gamma \in \mathcal{Top}[[0, r], UX]$, $\gamma \in d(\overline{D} \circ S'(X))$ iff
 $\forall W \in \Omega_{UX}, \forall t \leq t'$ s.t. $[t, t'] \subseteq \gamma^{-1}(W), \exists \delta \in dX$ s.t.
 $s(\delta) = \gamma(t), t(\delta) = \gamma(t')$ and $\text{img}(\delta) \subseteq W$



Realization of cubical sets in a cocomplete category $\mathcal{C}$

Let $K \in \mathcal{C}Set$ the category of cubical sets, we have

$$K \cong \underset{\substack{\square^n \rightarrow K \\ \text{in } \mathcal{C}Set \downarrow K}}{\text{colim}} \square^n$$

Let $\mathcal{C}$ be a cocomplete category and $F : \square \rightarrow \mathcal{C}$, we define the **geometric realisation** in $\mathcal{C}$ as

$$\downarrow K \downarrow_{\mathcal{C}} = \underset{\substack{\square^n \rightarrow K \\ \text{in } \mathcal{C}Set \downarrow K}}{\text{colim}} F(\square^n)$$

Directed Geometric Realization

of cubical sets

- Taking $F(\square^n) = (\overrightarrow{\mathbb{I}}_1)^n$ we have $1 \dashv_{Str}$ and $1 \dashv_{Str}$
- Taking $F(\square^n) = (\uparrow \mathbb{I}_1)^n$ we have $1 \dashv_{dTop}$ and $1 \dashv_{dTop}$

Relations

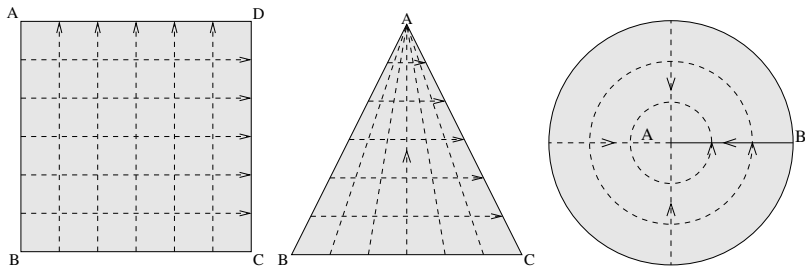
between the adjunction $S \dashv D$ and the directed geometric realizations

- for all $K \in \mathcal{cSet}$ $\overline{S}(1K \downarrow_{dTop}) = 1K \downarrow_{S\bar{r}}$ and $\overline{D}(1K \downarrow_{S\bar{r}}) = 1K \downarrow_{dTop}$
- for all $K \in \mathcal{cSet}$ $S(1K \downarrow_{dTop}) = 1K \downarrow_{Str}$ and $1K \downarrow_{S\bar{r}} = 1K \downarrow_{Str}$

$$S \left(\operatorname{colim}_{\substack{\square^n \rightarrow K \\ \text{in } \mathcal{cSet} \downarrow K}} (\uparrow \mathbb{I}_1)^n \right) = \operatorname{colim}_{\substack{\square^n \rightarrow K \\ \text{in } \mathcal{cSet} \downarrow K}} S((\uparrow \mathbb{I}_1)^n) = \operatorname{colim}_{\substack{\square^n \rightarrow K \\ \text{in } \mathcal{cSet} \downarrow K}} (\overrightarrow{\mathbb{I}}_1)^n$$

Realizing a vortex

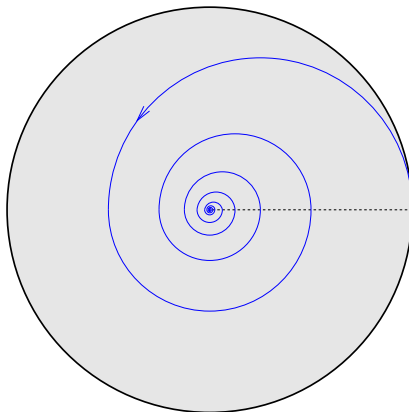
from the directed square



The downward spiral

There may be cubical sets K such that

$$D(1K|_{Str}) \neq 1K|_{dTop} \text{ and } 1K|_{dTop} \neq 1K|_{dTop}$$



Concrete category over $\mathcal{Top}$

Let $\mathcal{I}$ be the collection of all sub-intervals of $\mathbb{R}$ (including $\emptyset$ and the singletons)

- An adjunction $F \dashv U : \mathcal{C} \rightarrow \mathcal{Top}$ with U faithful.
- A family of objects $(\mathbb{I}_\iota)_{\iota \in \mathcal{I}}$ indexed by $\mathcal{I}$,
for $r \in \mathbb{R}_+$ the notation $\mathbb{I}_r$ stands for $\mathbb{I}_{[0,r]}$.

Axiom 1

Existence of Hypercubes

For all n -uple $(\iota_1, \dots, \iota_n)$ of elements of $\mathcal{I}$ the n -fold product $\mathbb{I}_{\iota_1} \times \dots \times \mathbb{I}_{\iota_n}$ exists and we suppose that $F(\{0\}) = \mathbb{I}_0$.
By convention the 0-fold product is the terminal object of $\mathcal{C}$.

Axiom 2

Coherence with respect to the product order of $\mathbb{R}^n$

For all continuous order³ preserving $\beta : \iota_1 \times \cdots \times \iota_n \rightarrow \iota'_1 \times \cdots \times \iota'_{n'}$
there exists a morphism $\alpha \in \mathcal{C}[\mathbb{I}_{\iota_1} \times \cdots \times \mathbb{I}_{\iota_n}, \mathbb{I}_{\iota'_1} \times \cdots \times \mathbb{I}_{\iota'_{n'}}]$ s.t.
$$U(\alpha) = \beta$$

As a consequence, for all $\iota \in \mathcal{I}$ we have $U(\mathbb{I}_\iota) = \iota$.

Given $x, r, s \in \mathbb{R}_+$ such that $x + r \leq s$, $i_{x,r}^s : \mathbb{I}_r \rightarrow \mathbb{I}_s$ is the unique morphism of $\mathcal{C}$ such that $U(i_{x,r}^s)$ is the following translation.

$$\begin{array}{ccc} [0, r] & \longrightarrow & [0, s] \\ t & \longmapsto & x + t \end{array}$$

³Here we mean product order.

Axiom 3

Concatenation via Pushout

The following diagram is a **pushout** square in $\mathcal{C}$

$$\begin{array}{ccc}
 & \mathbb{I}_{r+s} & \\
 i_{0,r}^{r+s} \nearrow & & \nwarrow i_{r,s}^{r+s} \\
 \mathbb{I}_r & & \mathbb{I}_s \\
 r \nwarrow & & \nearrow 0 \\
 & \mathbb{I}_0 &
 \end{array}$$

and for all $(\mathbb{I}_{r_1}, \dots, \mathbb{I}_{r_n})$ and all $i \in \{1, \dots, n\}$, it is **preserved** by the following endofunctor of $\mathcal{C}$

$$X \mapsto \mathbb{I}_{r_1} \times \dots \times \mathbb{I}_{r_{i-1}} \times X \times \mathbb{I}_{r_{i+1}} \times \dots \times \mathbb{I}_{r_n}$$

A structure satisfying the axioms 1, 2 and 3 is called a **framework for fundamental category** of **fffc**

Examples

of frameworks for fundamental categories

The categories $\mathcal{Top}$, $\mathcal{Po}$, $\mathcal{dTop}$, $\mathcal{Str}$, $\mathcal{dTop}$ and $\mathcal{Str}$ with their obvious forgetful functor and intervals are fffc's.

We associate each object X of a given fffc $\mathcal{C}$ with the following d-space

$$\bigcup_{r \in \mathbb{R}_+} \mathcal{C}[\mathbb{I}_r, X]$$

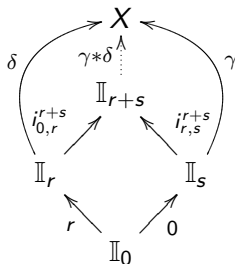
thus defining a faithful functor from $\mathcal{C}$ to $\mathcal{dTop}$

The category of directed paths of an object X of $\mathcal{C}$ denoted by $\vec{P}(X)$

Objects and Identities : $\mathcal{C}[\mathbb{I}_0, X]$ (points of X)

Morphisms : $\bigcup_{r \in \mathbb{R}_+} \mathcal{C}[\mathbb{I}_r, X]$ (directed paths on X)

Concatenation :

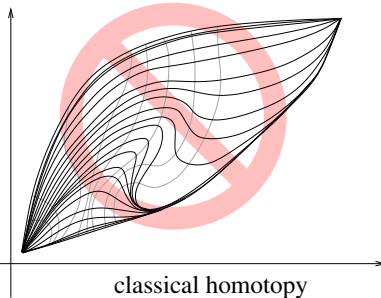
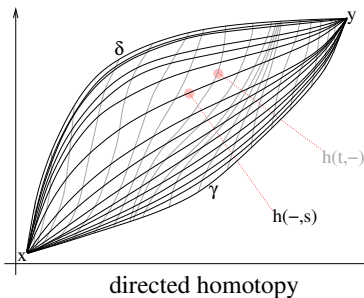


The construction is **functorial** and there is a **natural embedding**
of $\vec{P}(X)$ into $P \circ U(X)$

Directed Homotopy

between γ and δ two directed paths on X

Write $\gamma \preceq \delta$ when there exists two constant paths c_γ, c_δ and some $h \in C[\mathbb{I}_r \times \mathbb{I}_\rho, X]$ such that $U(h)$ is a usual homotopy from $U(c_\gamma * \gamma)$ to $U(c_\delta * \delta)$ ⁴



⁴The constant paths are needed so we can relate two directed paths whose domains of definition differ.

$\overrightarrow{\pi_1}(X)$

The Fundamental Category of X

Denote by $\sim$ for the equivalence relation generated by $\preceq$,
it yields to a **congruence** over $\overrightarrow{P}(X)$.

Then define the **fundamental category** of X as the quotient

$$\overrightarrow{\pi_1}(X) := \overrightarrow{P}(X) / \sim$$

The construction is functorial

$$\overrightarrow{\pi_1} : \mathcal{C} \rightarrow \mathbf{Cat}$$

and there is a **natural morphism** from $\overrightarrow{\pi_1}(X)$ to $\pi_1 \circ U(X)$

The Seifert-Van Kampen Theorem

generic version

We call **inclusion** any $\alpha \in \mathcal{C}[X, Y]$ s.t. $U(\alpha) = U(X) \hookrightarrow U(Y)$.

Then $U^\circ(X)$ is the topological interior of $U(X) \subseteq U(Y)$.

Theorem (Seifert - Van Kampen)

A square of inclusions such that $U^\circ(X_1)$ and $U^\circ(X_2)$ cover $U(X)$ and $U(X_0) = U(X_1) \cap U(X_2)$ is sent to pushout squares of $\mathcal{Cat}$ by the functors $\vec{P}$ and $\vec{\pi}_1$.

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

$$\begin{array}{ccc} \vec{P}(X_0) & \longrightarrow & \vec{P}(X_1) \\ \downarrow & & \downarrow \\ \vec{P}(X_2) & \longrightarrow & \vec{P}(X) \end{array}$$

$$\begin{array}{ccc} \vec{\pi}_1(X_0) & \longrightarrow & \vec{\pi}_1(X_1) \\ \downarrow & & \downarrow \\ \vec{\pi}_1(X_2) & \longrightarrow & \vec{\pi}_1(X) \end{array}$$

Relations

between $S \dashv D$, $\lfloor - \rfloor$ and $\overrightarrow{\pi_1}$

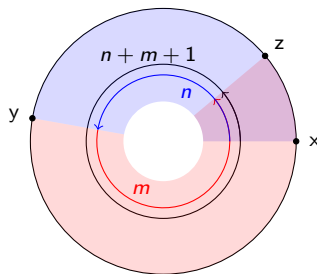
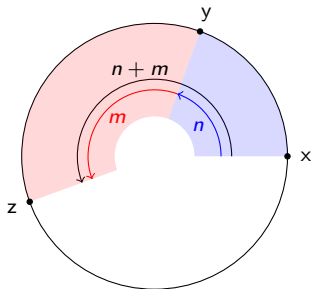
- For all topological spaces X , $\overrightarrow{\pi_1}(X)$ is the **fundamental groupoid** of X
- For all streams X , $\overrightarrow{\pi_1}(D(X)) = \overrightarrow{\pi_1}(X)$
- For all d-spaces X , if there exists a stream X' such that $X = D(X')$, then $\overrightarrow{\pi_1}(S(X)) = \overrightarrow{\pi_1}(X)$
- For all $X \in \mathcal{S}\bar{r}$ and all $Y \in d\overline{Top}$
 $\overrightarrow{\pi_1}(\overline{D}(X)) = \overrightarrow{\pi_1}(X)$ and $\overrightarrow{\pi_1}(\overline{S}(Y)) = \overrightarrow{\pi_1}(Y)$
- For all cubical sets K following have the same fundamental category : $D(\lfloor K \rfloor_{Str})$, $\lfloor K \rfloor_{Str}$, $\lfloor K \rfloor_{\mathcal{S}\bar{r}}$, $S(\lfloor K \rfloor_{d\overline{Top}})$, $\lfloor K \rfloor_{d\overline{Top}}$
- Question : what about $\lfloor K \rfloor_{d\overline{Top}}$?

The Fundamental Category of the directed hypercubes

The fundamental category of the directed hypercube $\overrightarrow{\mathbb{I}}_r$ is the product poset $([0, r], \leq)^n$.

The Fundamental Category of the Circles

directed or classical



$$\vec{\pi}_1(\vec{S}^1)[x, y] = \{x\} \times \mathbb{N} \times \{y\}$$

$$\pi_1(S^1)[x, y] = \{x\} \times \mathbb{Z} \times \{y\}$$

Define $\omega(x, n, y) := n$

The fundamental category of the directed complex plane

Let $z, z', z'' \in \mathbb{C}$

Define $p : z \in \mathbb{C} \setminus \{0\} \mapsto \frac{z}{|z|} \in \mathbb{S}^1$

$$\vec{\pi}_1(\vec{\mathbb{C}})[z, z'] = \begin{cases} \emptyset & \text{if } |z| > |z'| \\ \{\perp_{z'}\} & \text{if } z = 0 \\ \{z\} \times \mathbb{N} \times \{z'\} & \text{if } z \neq 0 \text{ and } |z| \leq |z'| \end{cases}$$

$(z, n, z') \circ \perp_z = \perp_{z'}$ i.e. **0** is the **initial** object of $\vec{\pi}_1(\vec{\mathbb{C}})$

$$(z', m, z'') \circ (z, n, z') = (z, \omega((pz', m, pz'') \circ (pz, n, pz')), z'')$$

The fundamental category of the directed Riemann sphere

Let $z, z', z'' \in \Sigma$

Extend $p : z \in \Sigma \setminus \{0, \infty\} \mapsto \frac{z}{|z|} \in \mathbb{S}^1$

$$\vec{\pi}_1(\vec{\mathbb{C}})[z, z'] = \begin{cases} \emptyset & \text{if } |z| > |z'| \\ \{\perp_{z'}\} & \text{if } z = 0 \\ \{\top_z\} & \text{if } z' = \infty \\ \{z\} \times \mathbb{N} \times \{z'\} & \text{if } z \neq 0 \text{ and } |z| \leq |z'| \end{cases}$$

$$\perp_\infty = \top_0$$

$(z, n, z') \circ \perp_z = \perp_{z'}$ i.e. **0** is the **initial** object of $\vec{\pi}_1(\vec{\Sigma})$

$\top_{z'} \circ (z, n, z') = \top_z$ i.e. **∞** is the **terminal** object of $\vec{\pi}_1(\vec{\Sigma})$

$$(z', m, z'') \circ (z, n, z') = (z, \omega((pz', m, pz'') \circ (pz, n, pz')), z'')$$