Constructible sheaves and their cohomology for asynchronous logic and computation

14 January 2010

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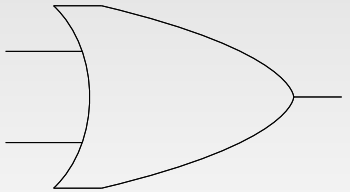
Acknowledgements

- This is a preliminary report on progress in a larger project on applied sheaf theory
  - More substantial results are to come!
- It's joint work with
  - Robert Ghrist (Penn)
  - Yasu Hiraoka (Hiroshima)
- The focus is on logic here, but is part of
  - AFOSR MURI on Information Dynamics in Networks
  - PI: Rob Calderbank (Princeton)

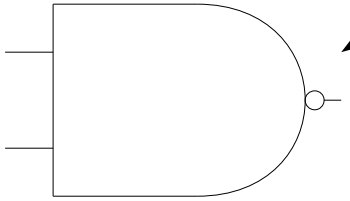
Logic gates

AND



OR

“bubble” indicates negation



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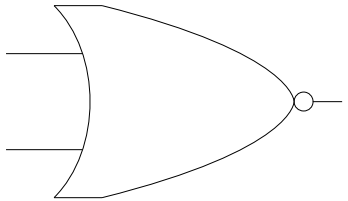
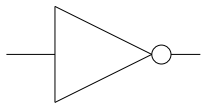
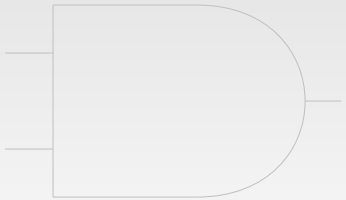


NOR

1  0

NOT

Logic gates



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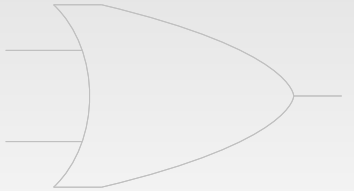

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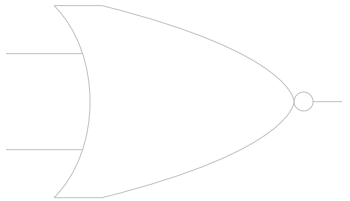
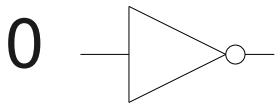


NOR

A change occurs...



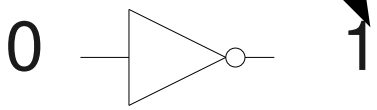
0 0

NOT

Logic gates

Propagation delay varies from device to device

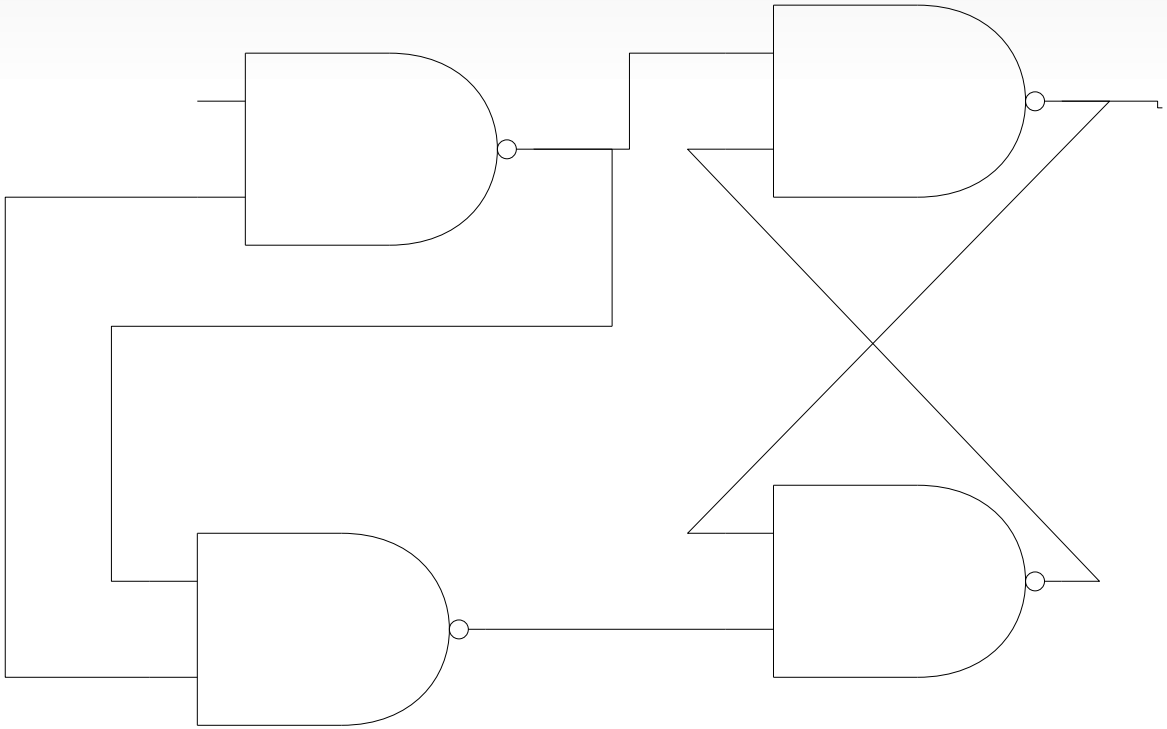
... eventually changes the output



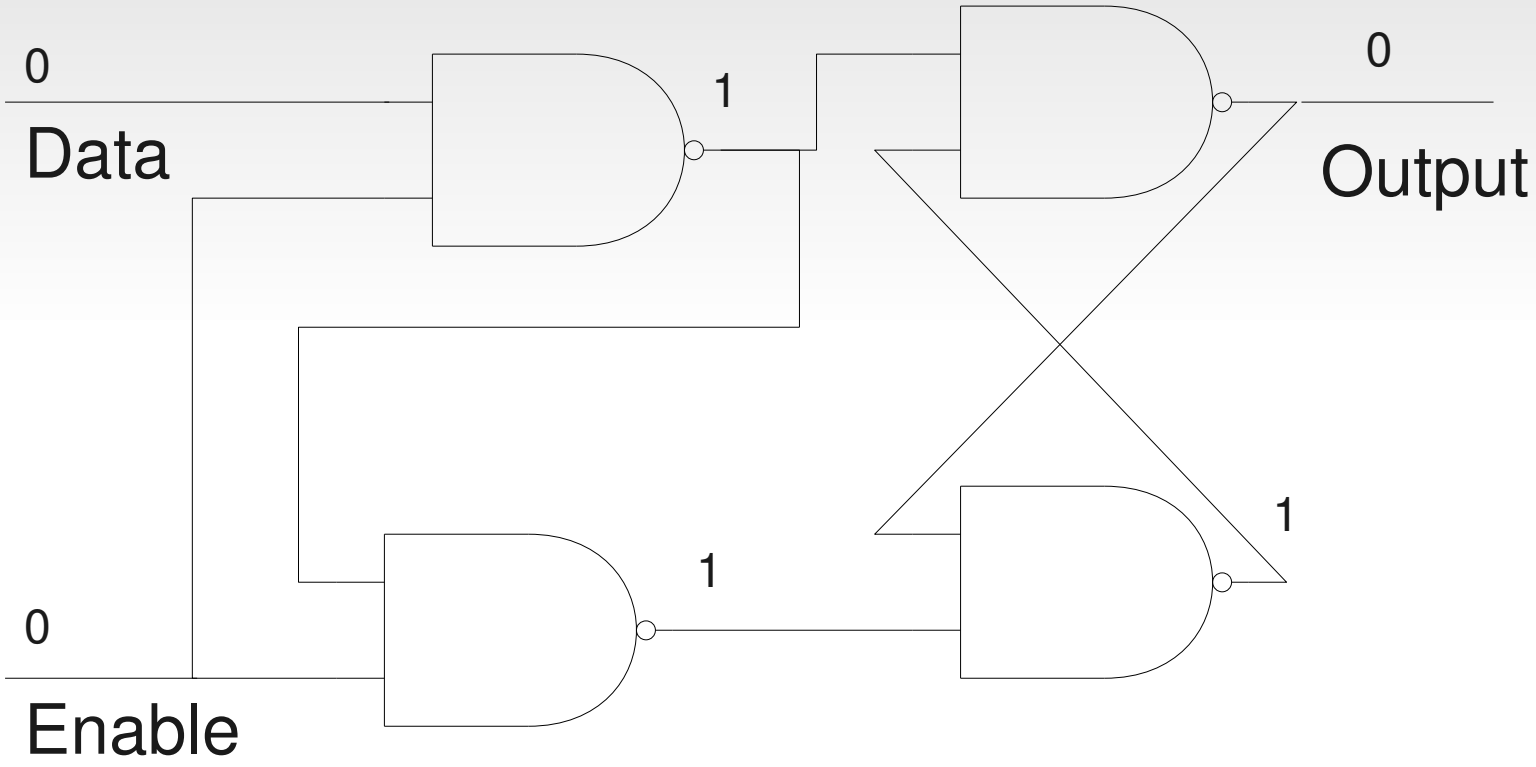
NOT

Problem: time-bound logic

- Propagation delays along connections and within gates!
- Feedback – can hold state
- Race conditions:
  - Hazards
  - Glitches
  - Oscillations
  - Lock-ups

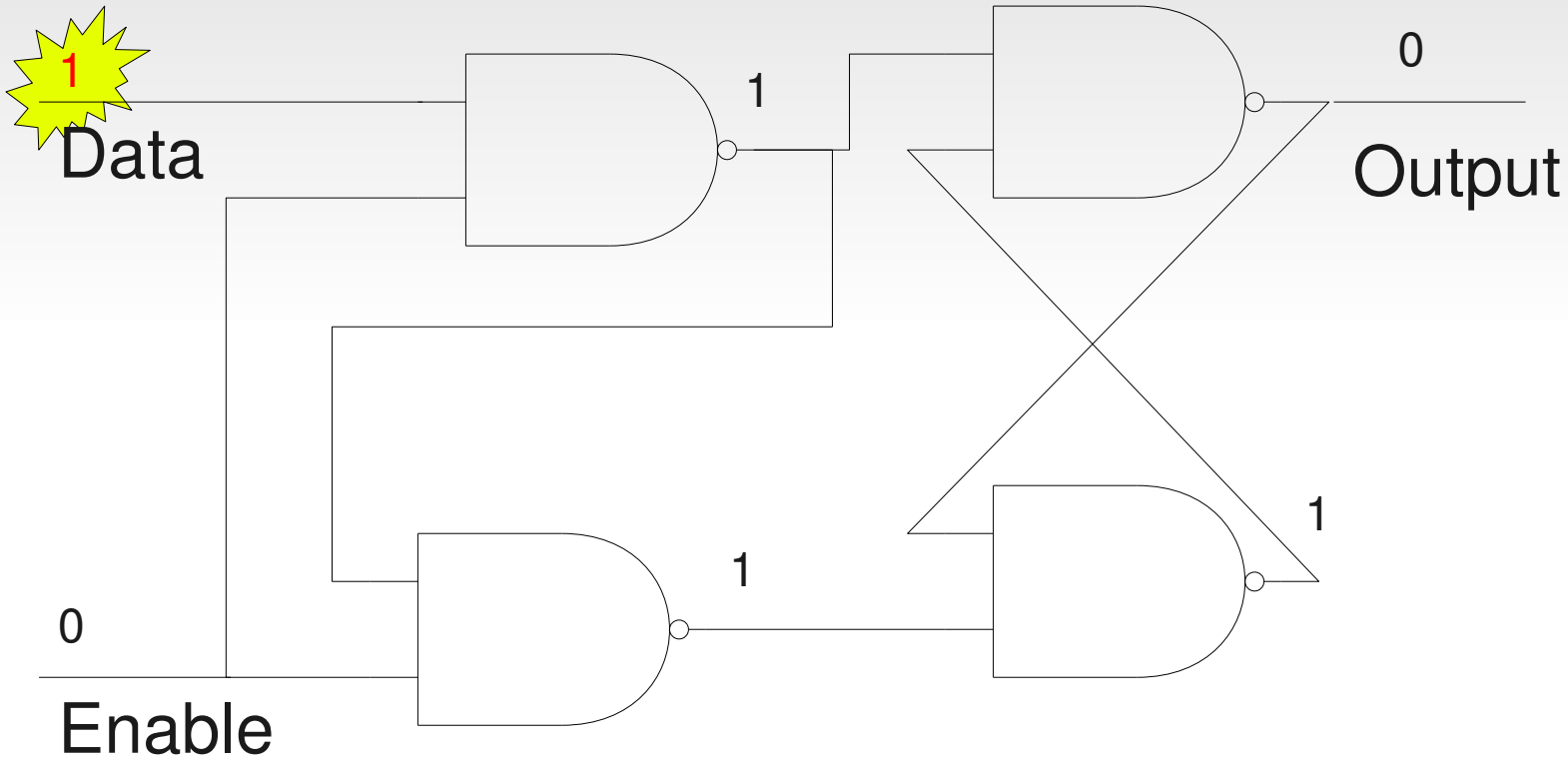


Example of timebound logic



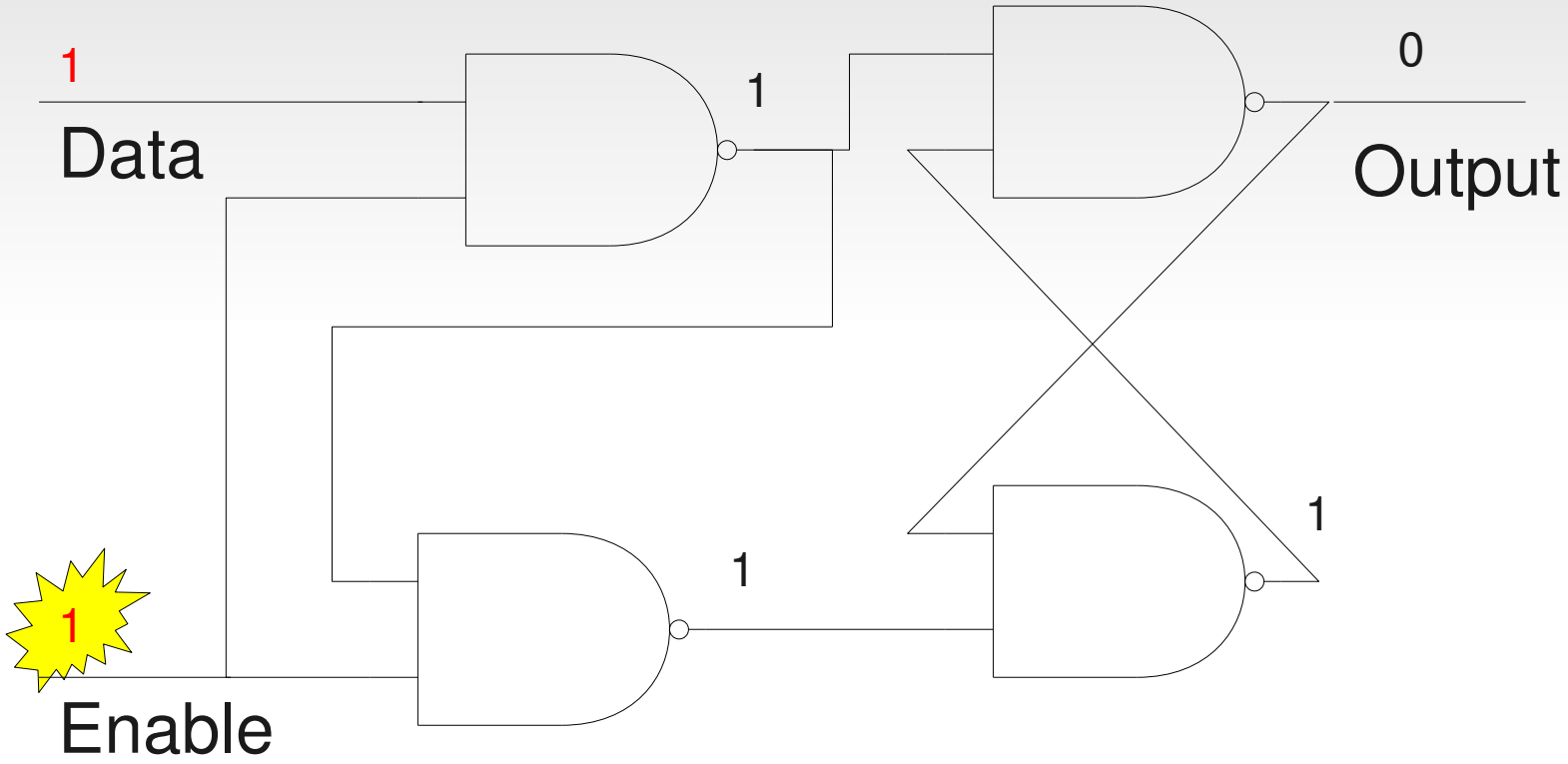
This is an E flip-flop circuit, a basic memory element. It's initially storing the value 0

Example of timebound logic



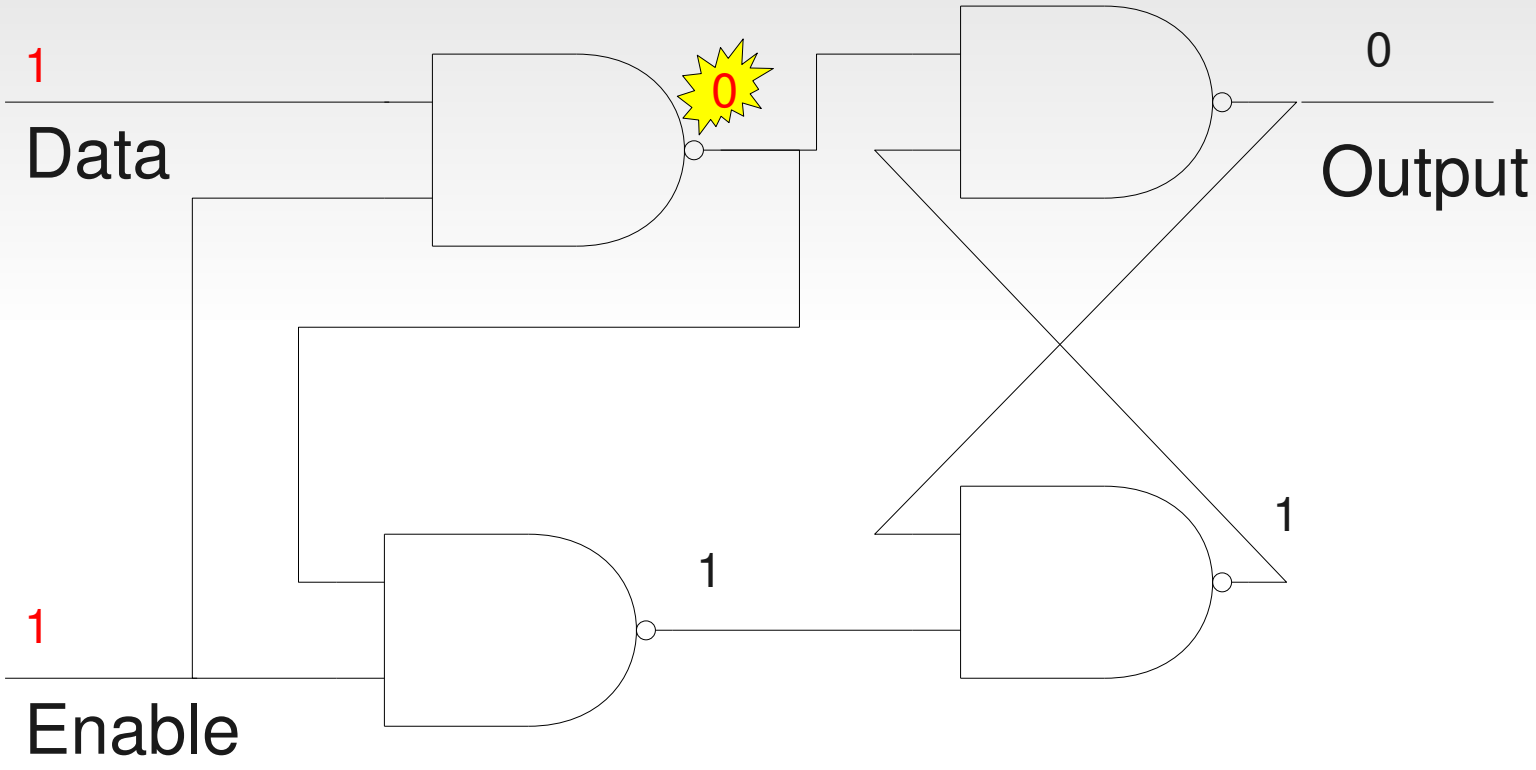
If we change the Data input to 1, nothing exciting happens...

Example of timebound logic



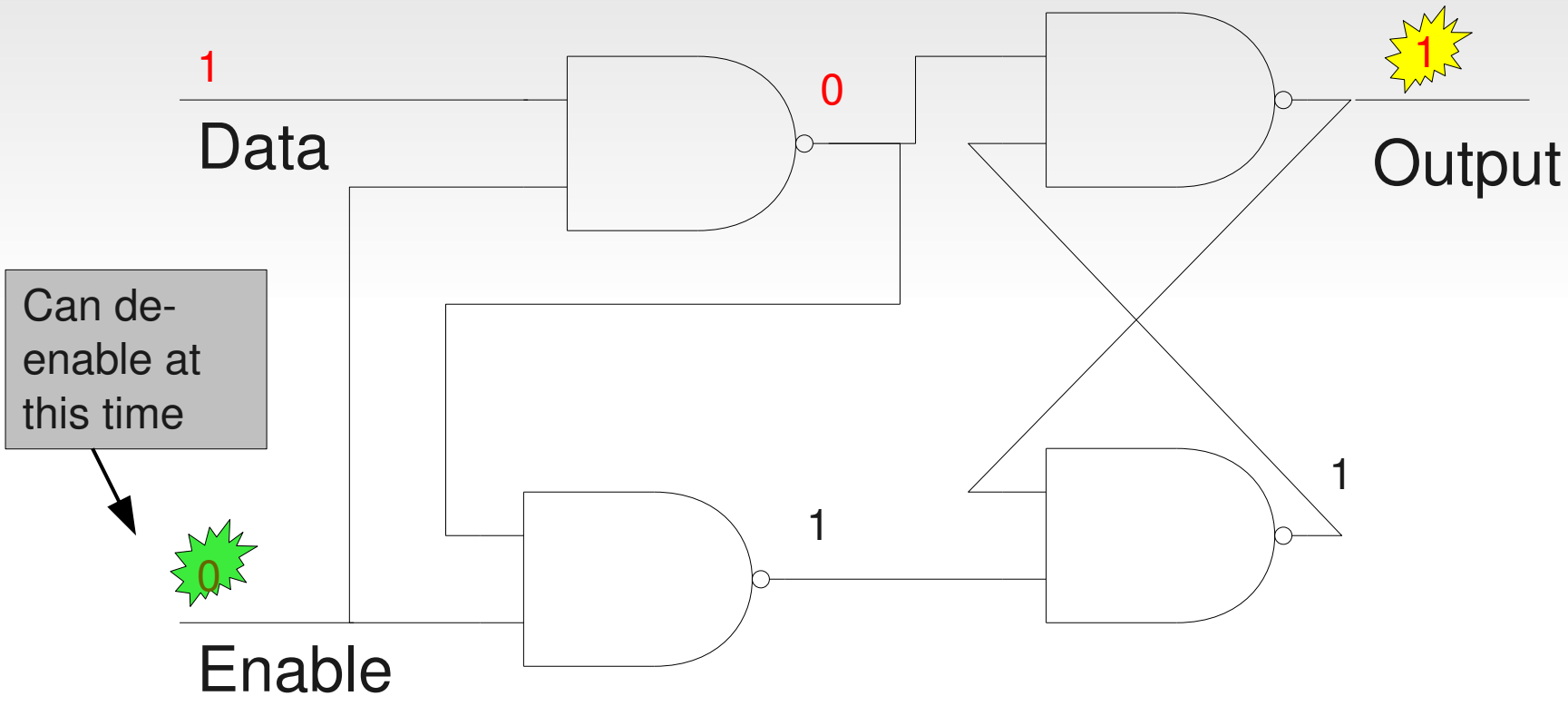
Pulsing the Enable input to 1 causes the Data input to be “read” and “stored”...

Example of timebound logic



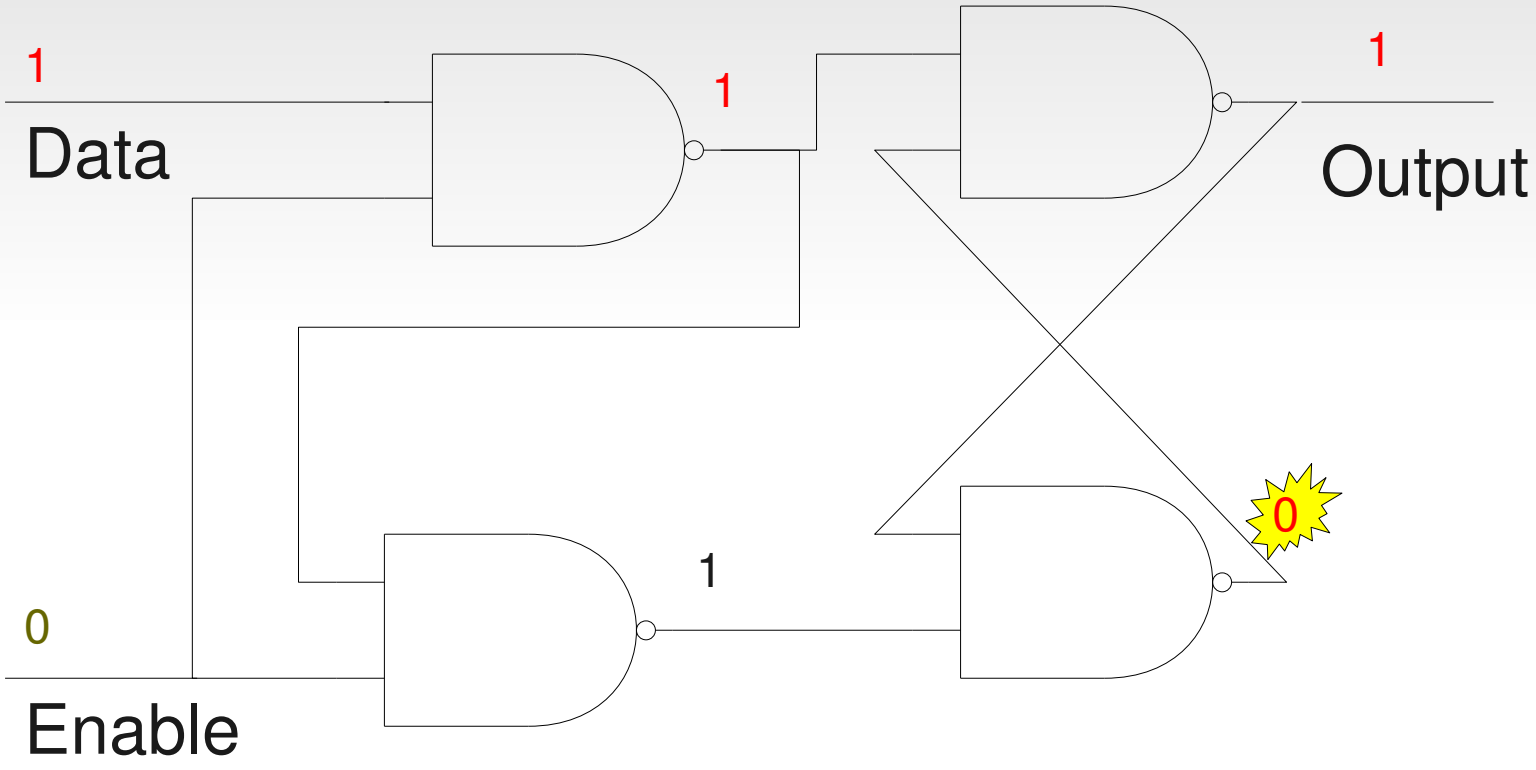
... but it takes time... t=1

Example of timebound logic



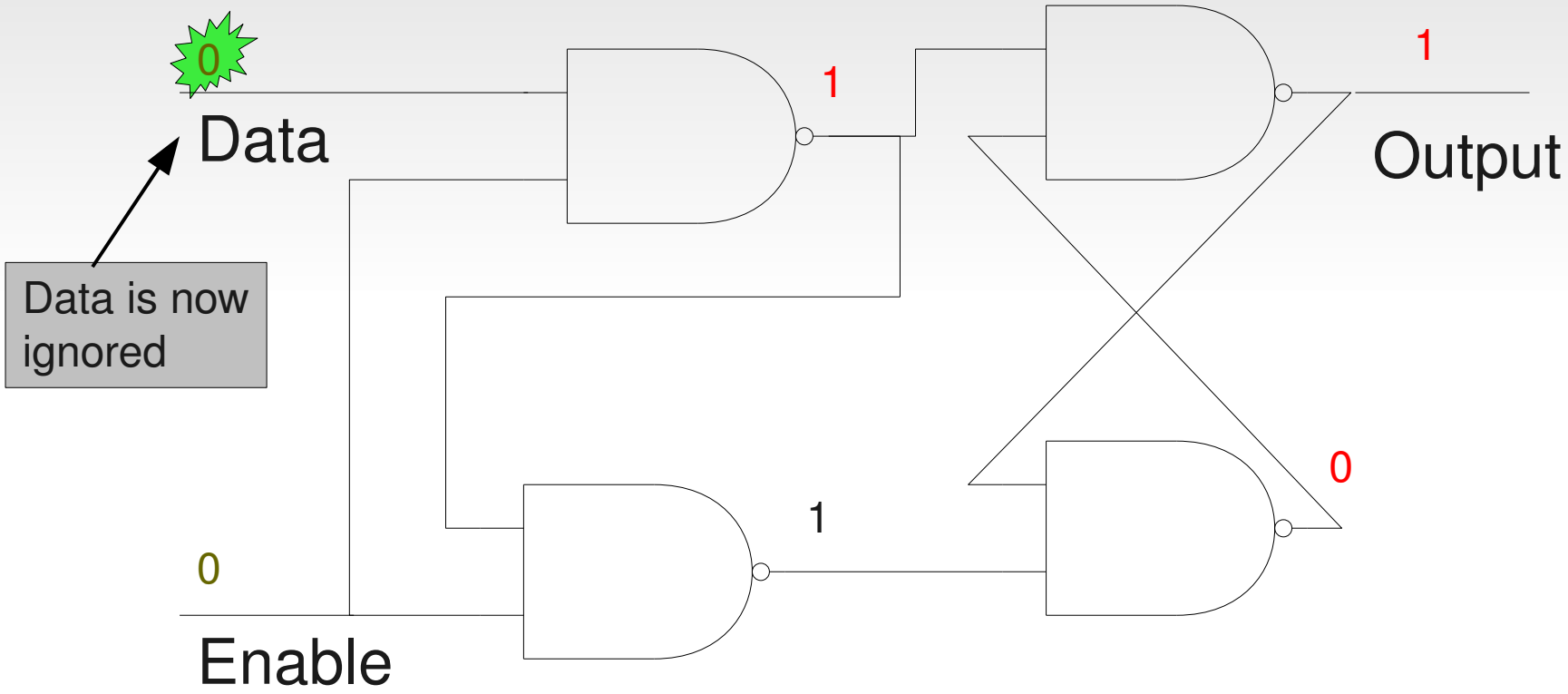
... but it takes time... t=2

Example of timebound logic



... but it takes time... t=3

Example of timebound logic

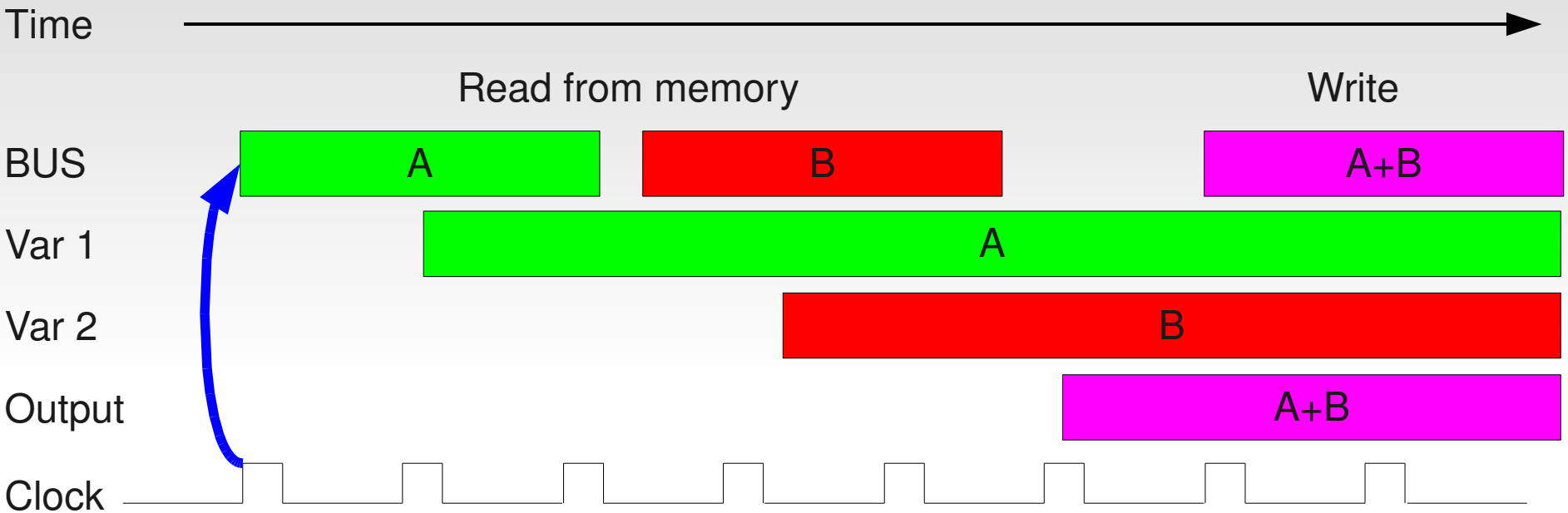


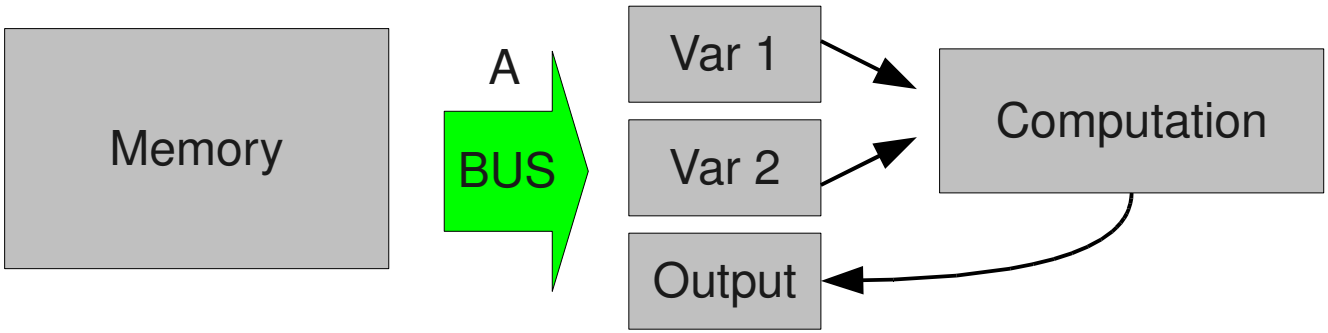
... and will hold the new value!

Synchronous design

- Can avoid race conditions by polling after transients are finished
  - Unavoidable limitation: limited by the slowest circuit
- Synchronous solution: circuits poll their inputs only at specific points in time – **a global clock**
- But...
  - Biggest single drain of power in modern CPUs is the clock
  - Clock distribution and skew a major problem
  - Correcting clock skew requires additional circuitry and power usage

Example logic timeline (synchronous)





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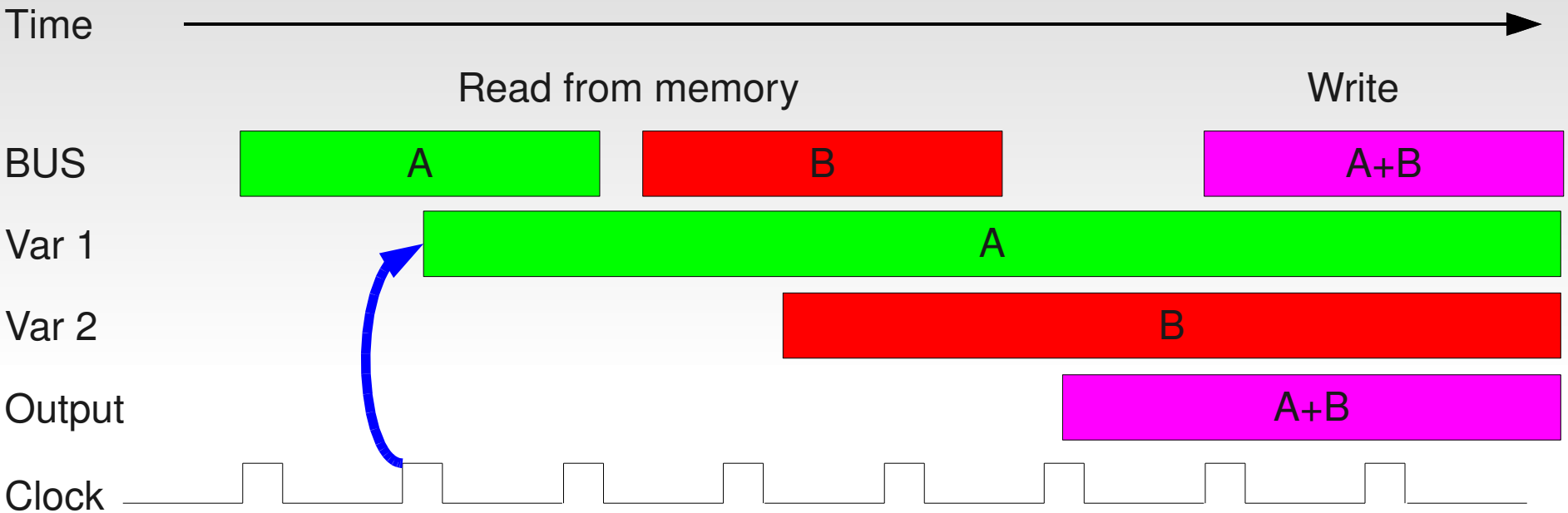

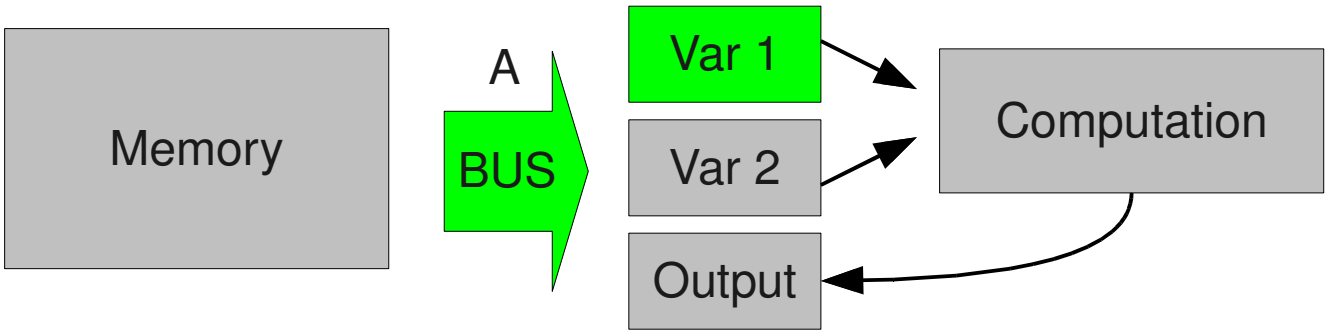

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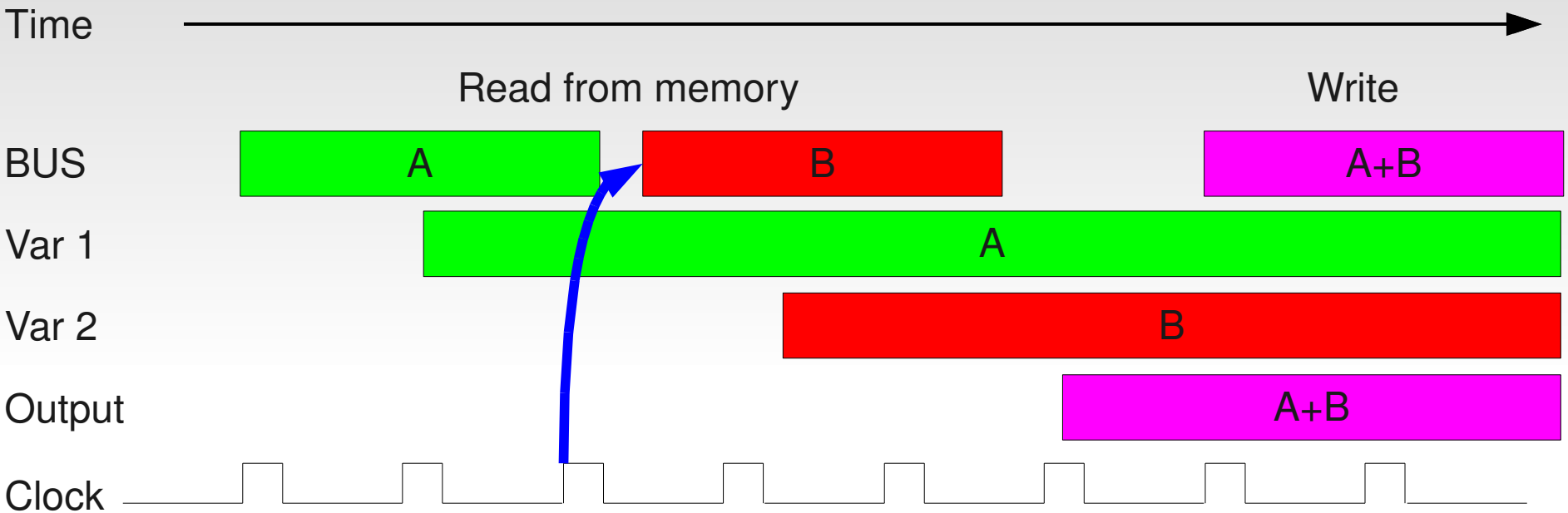

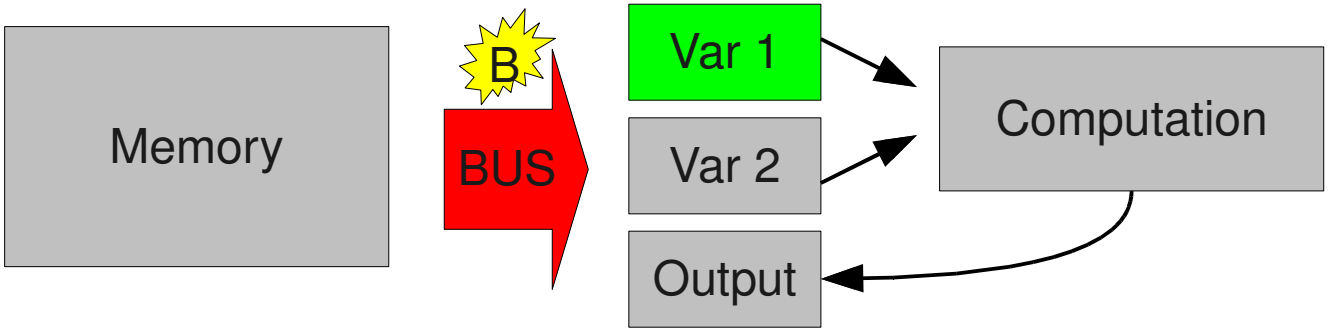

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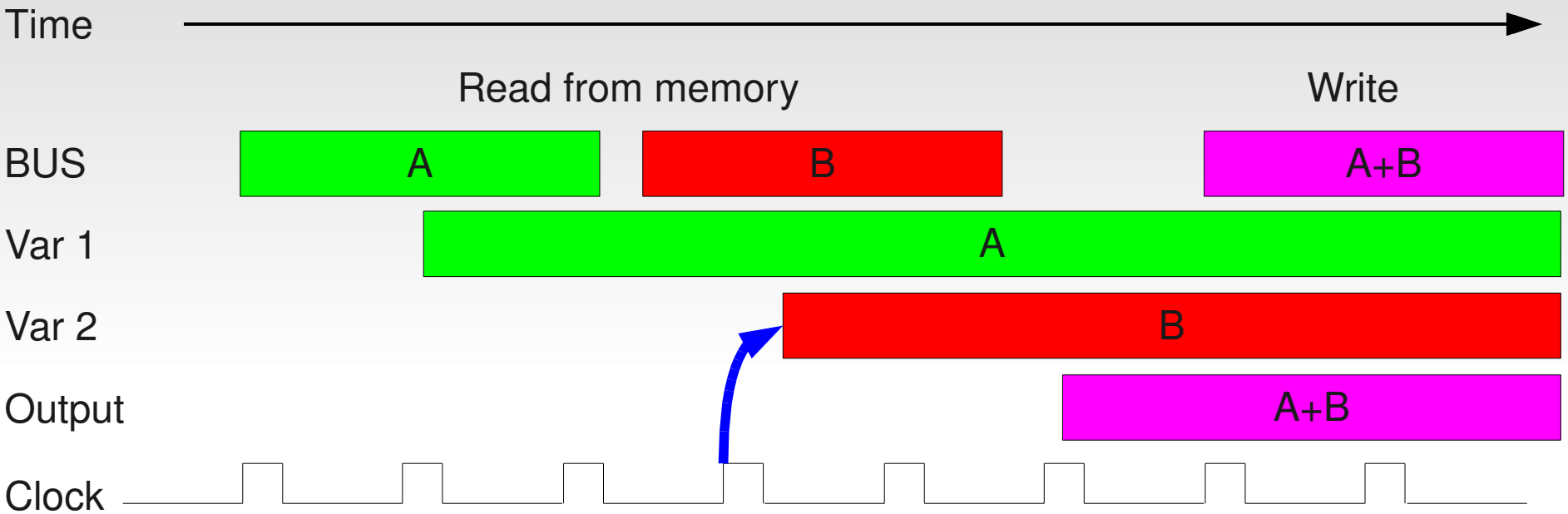

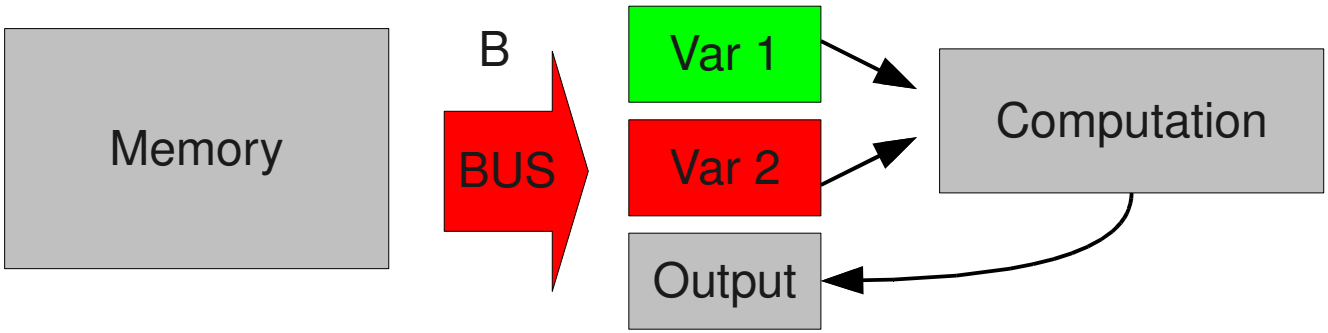

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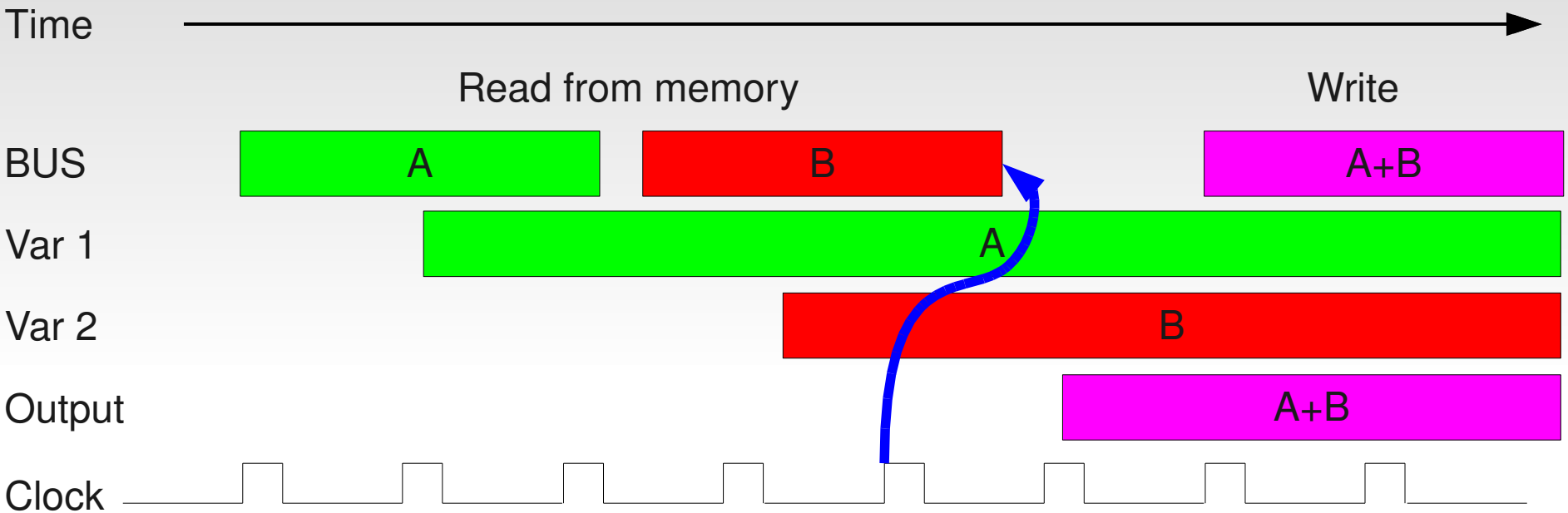

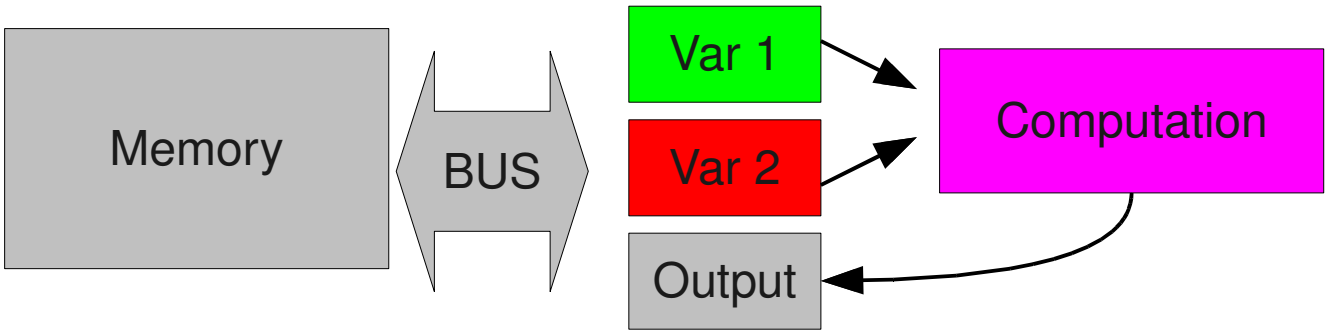

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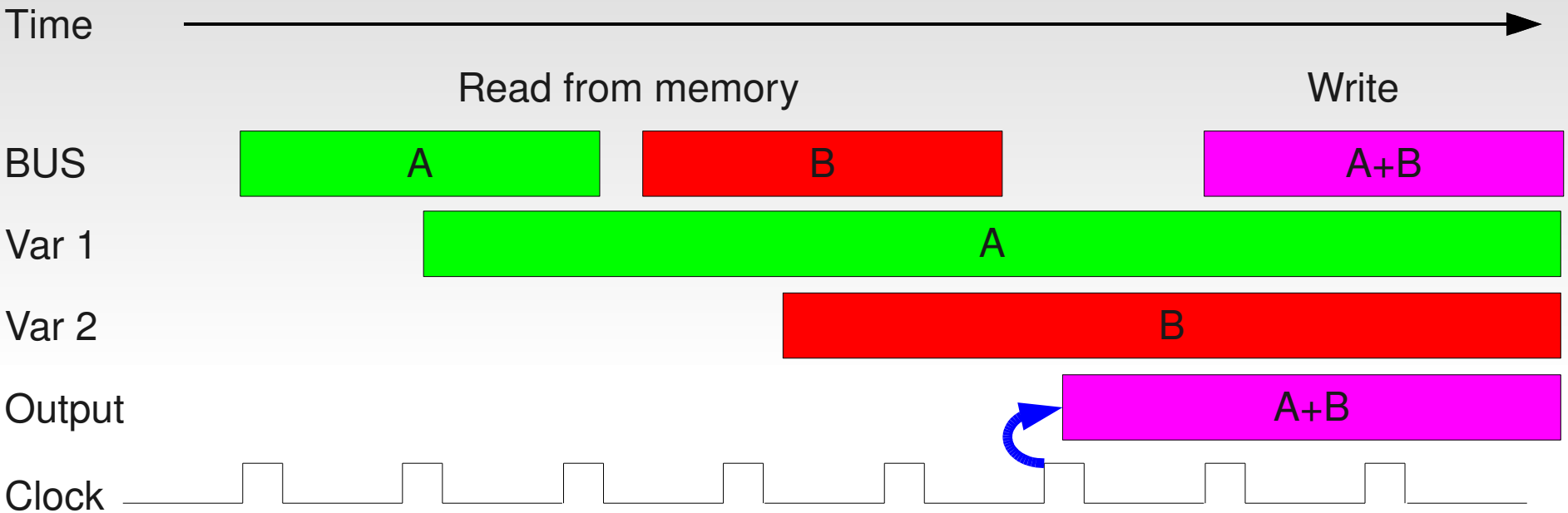

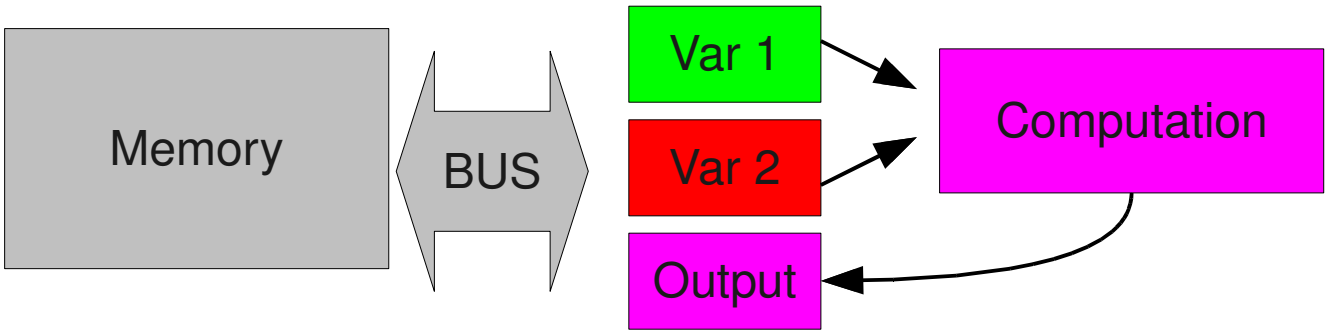

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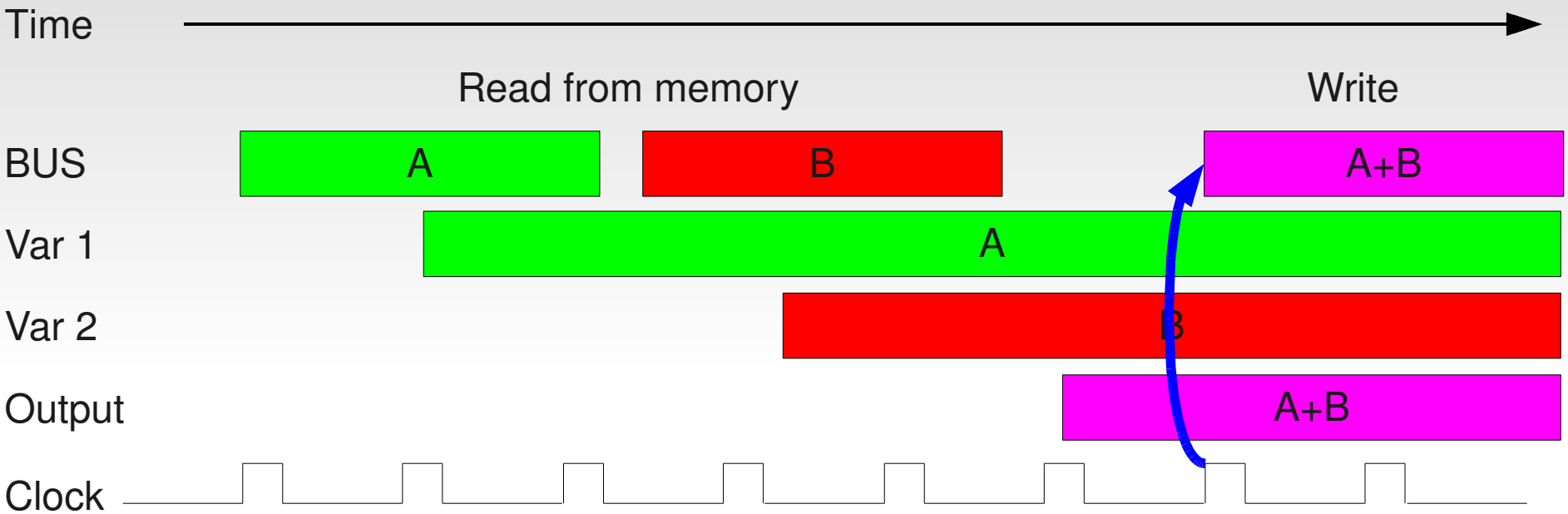
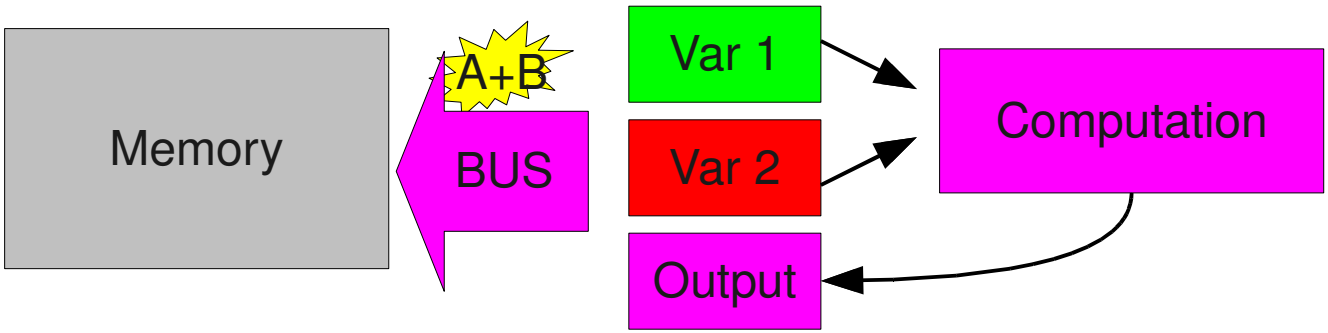
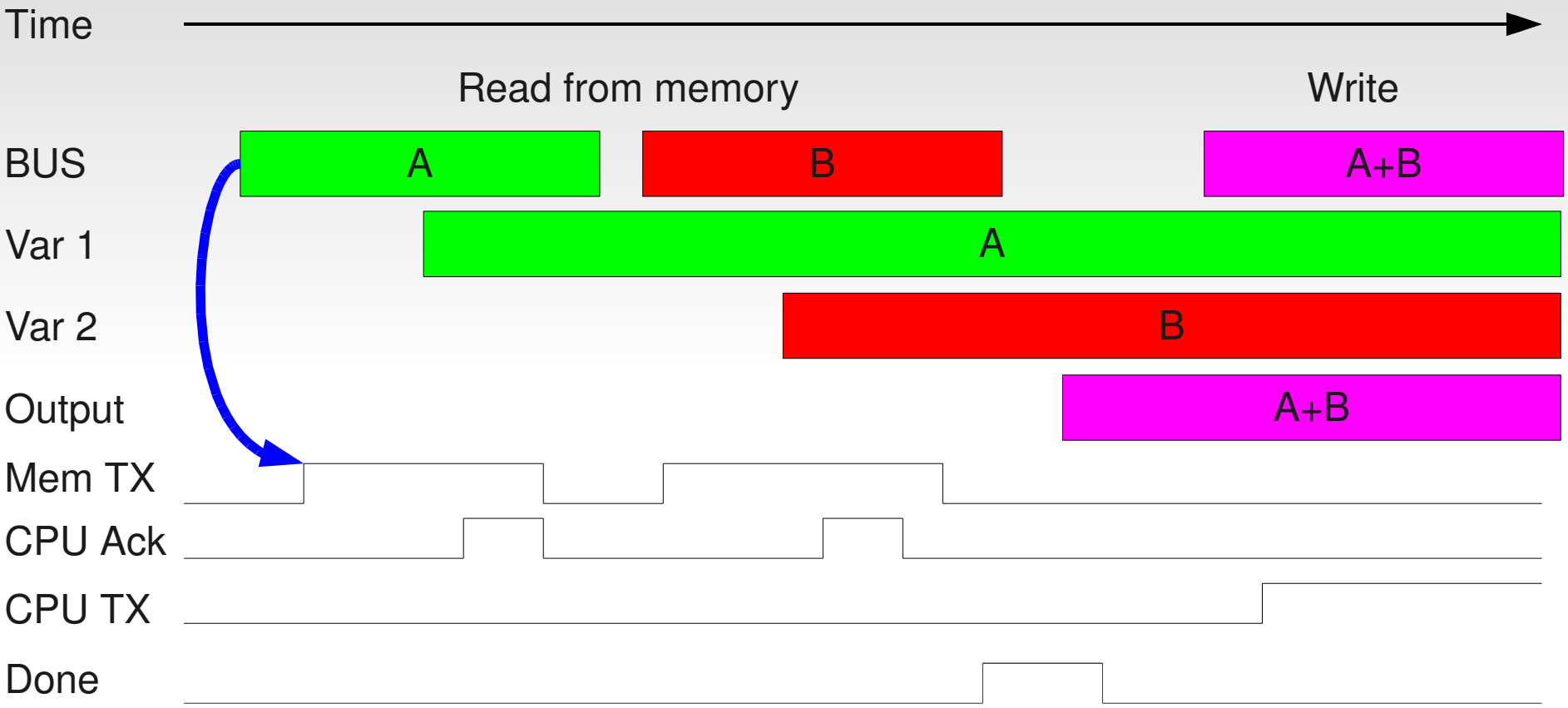
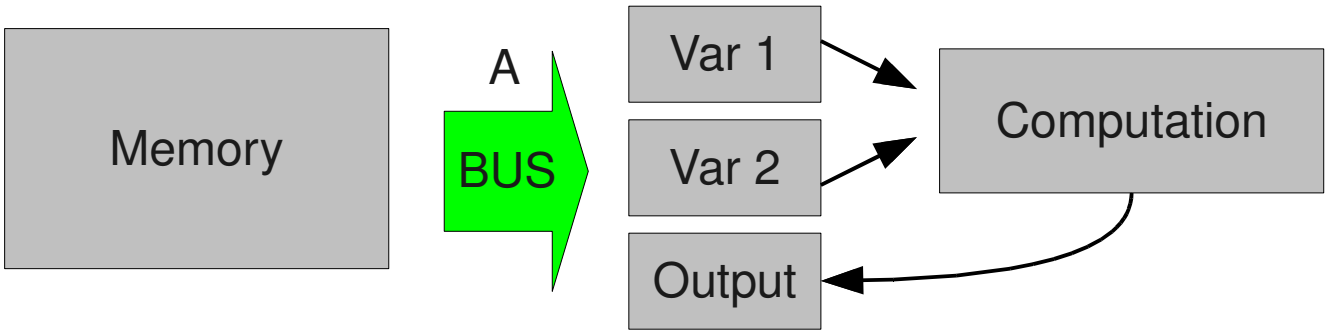




Example logic timeline (asynchronous)





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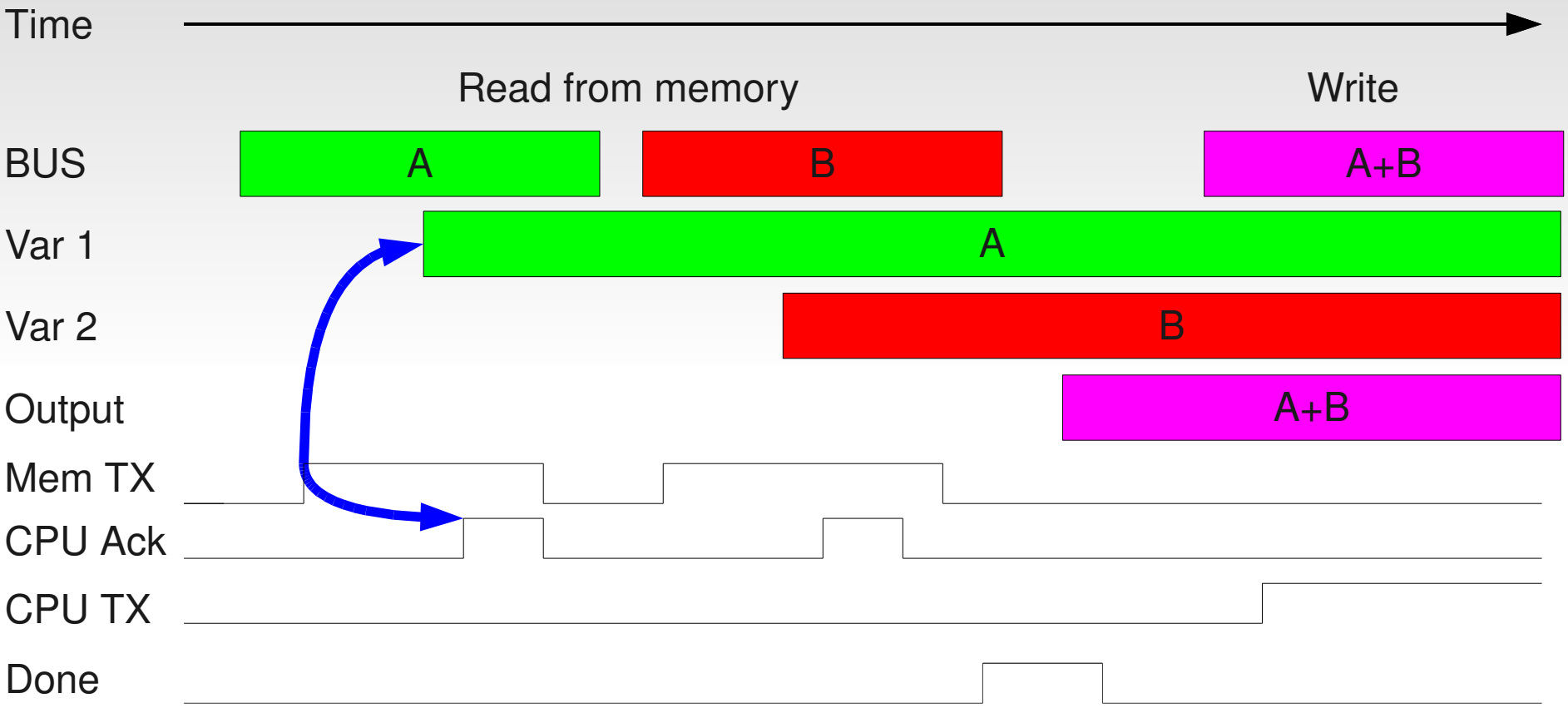

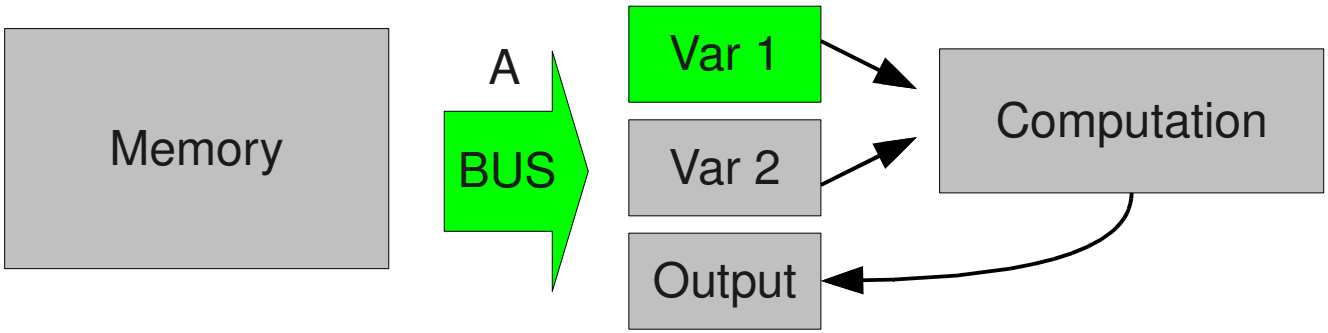

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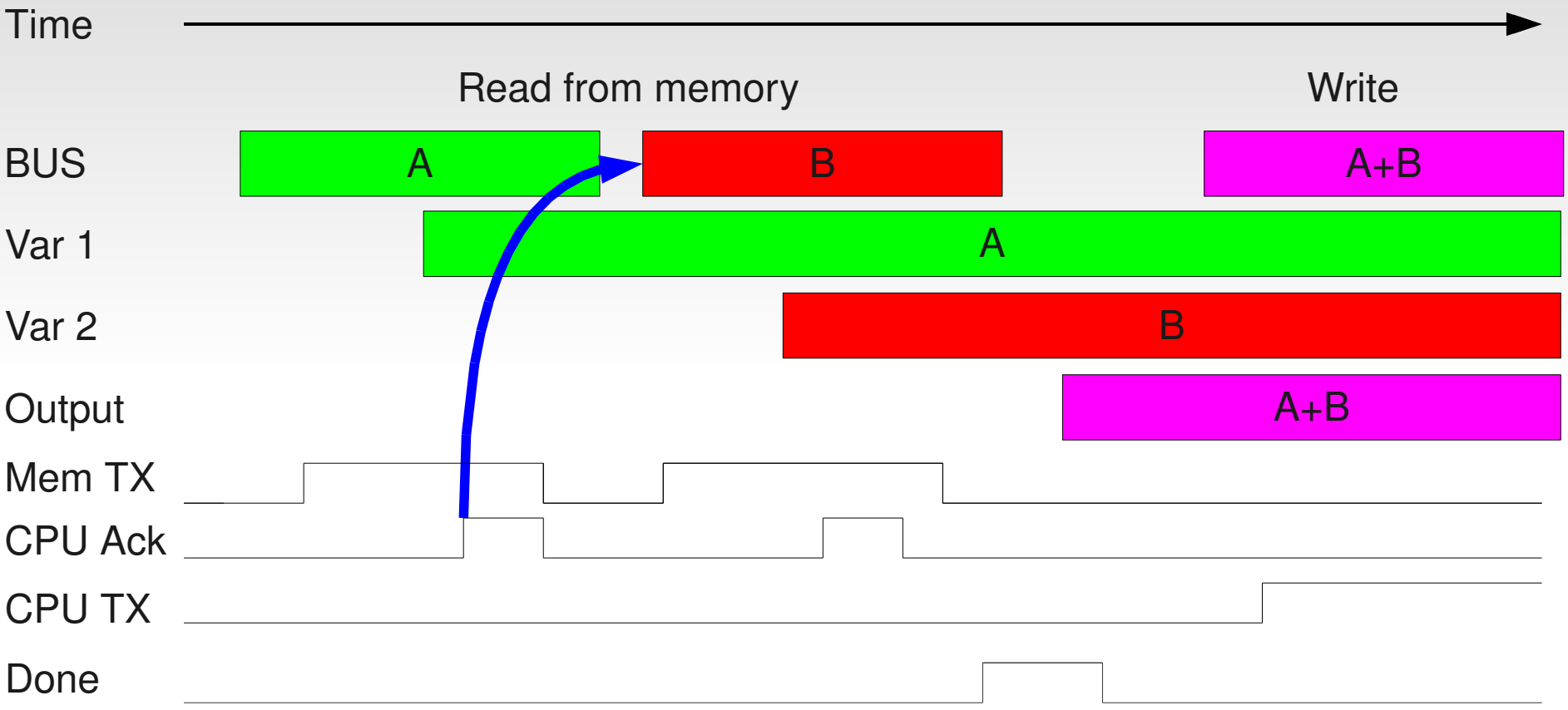

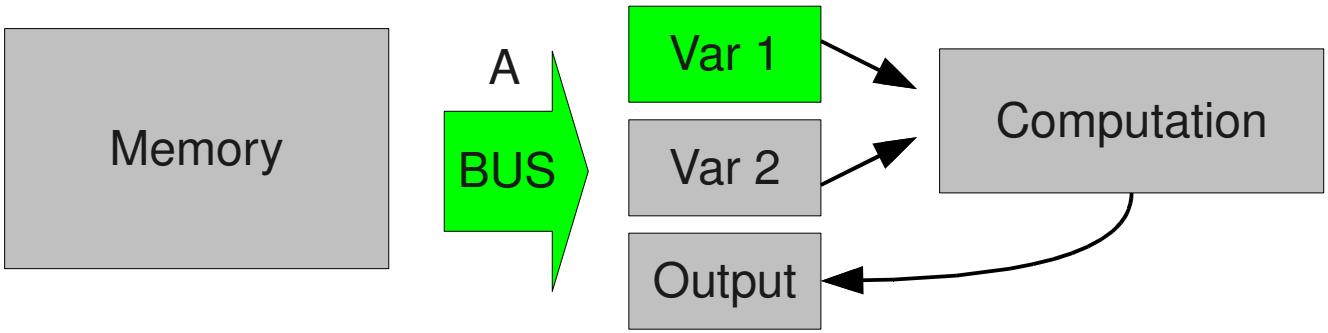

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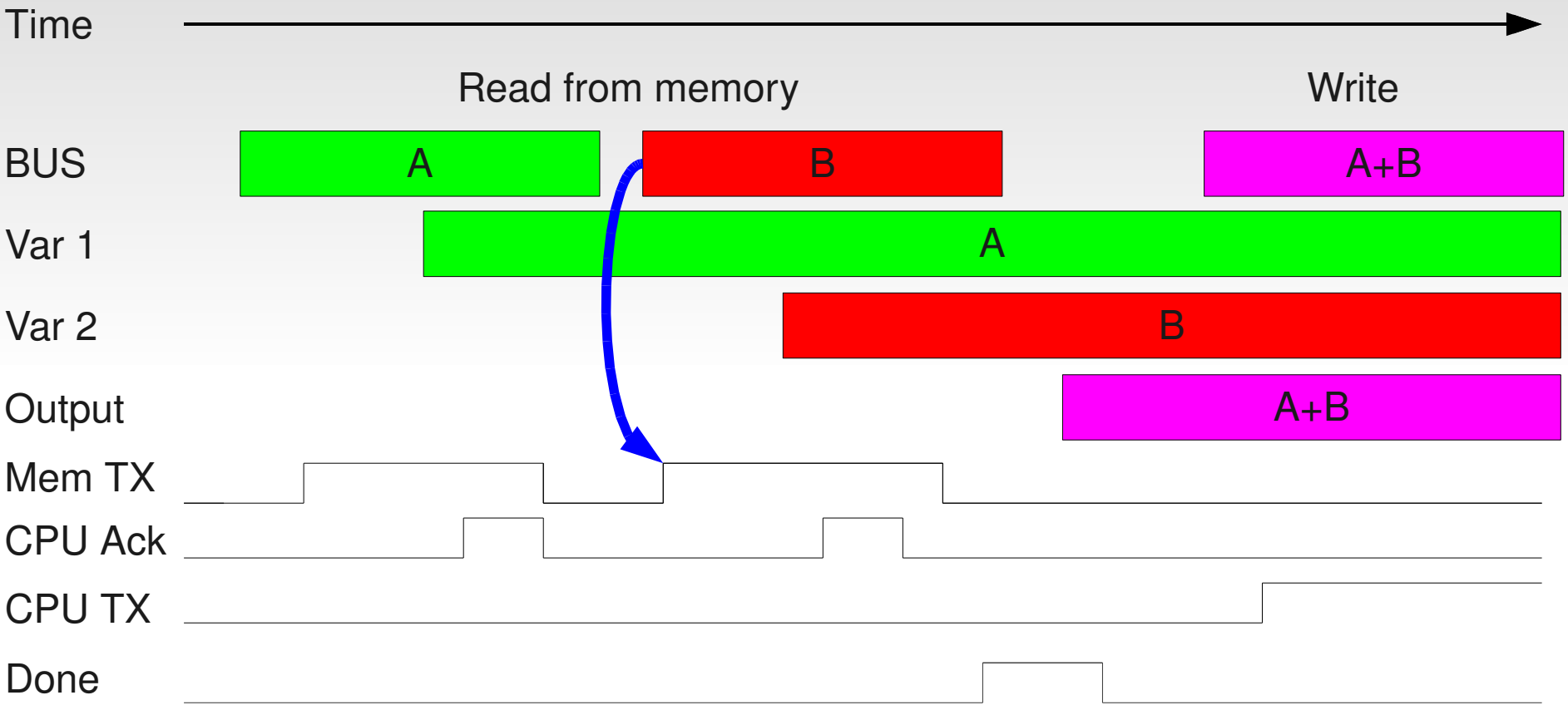

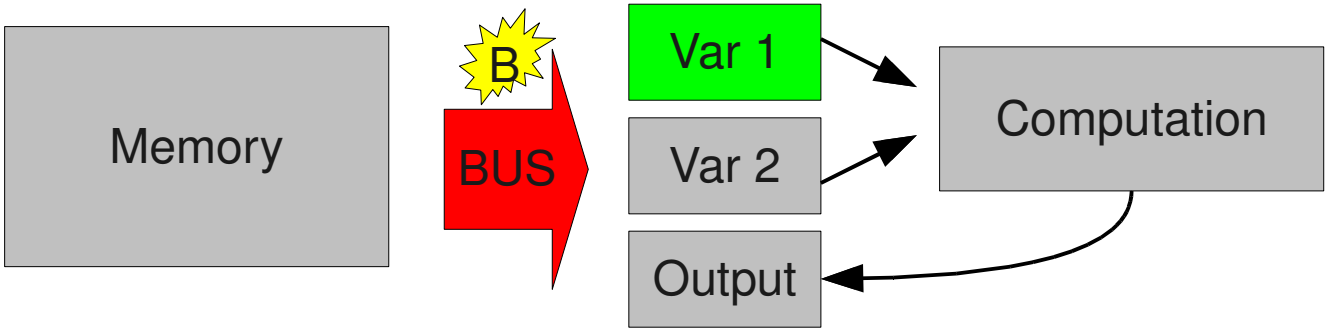

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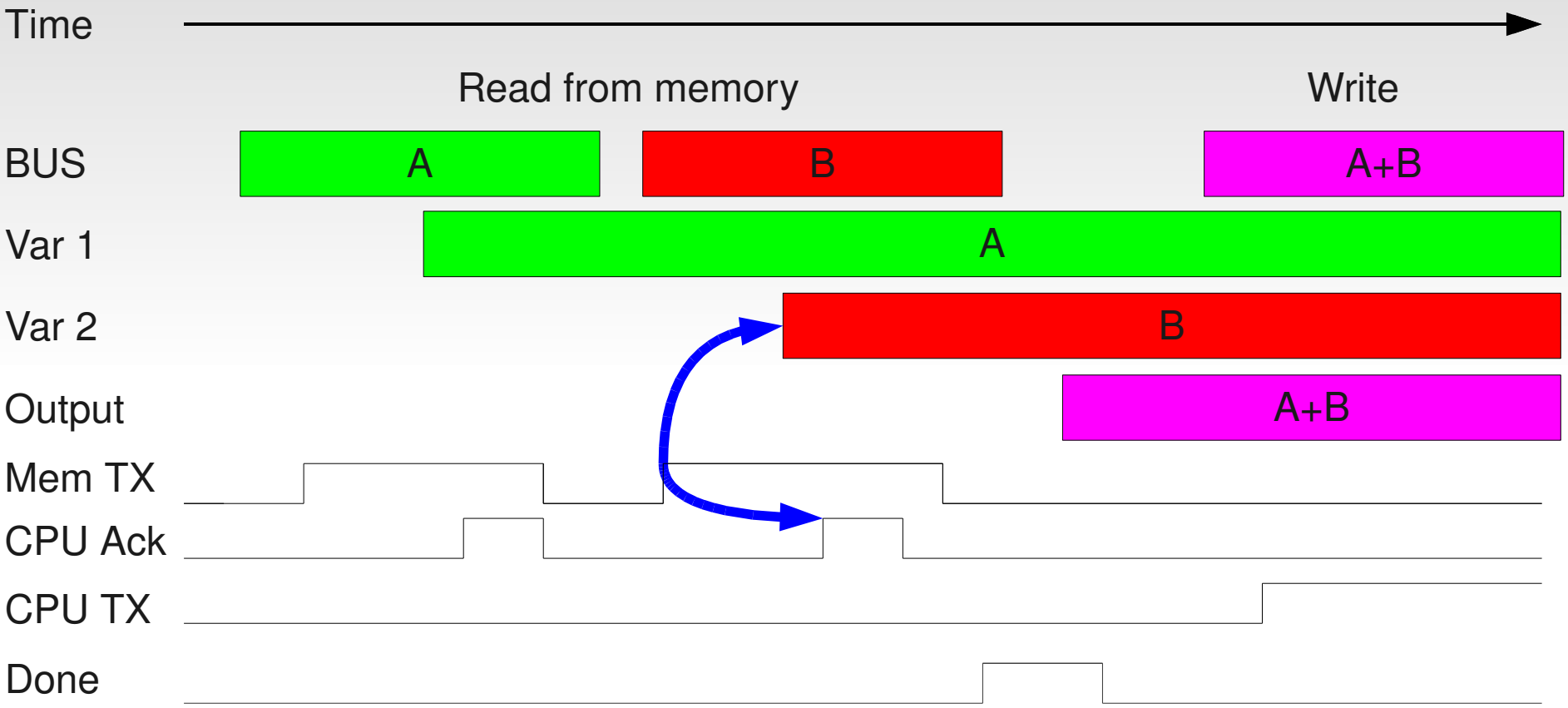

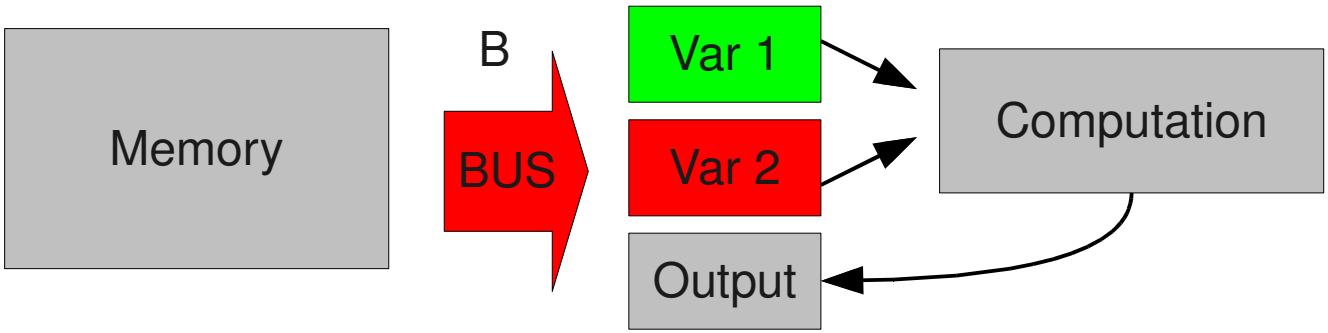

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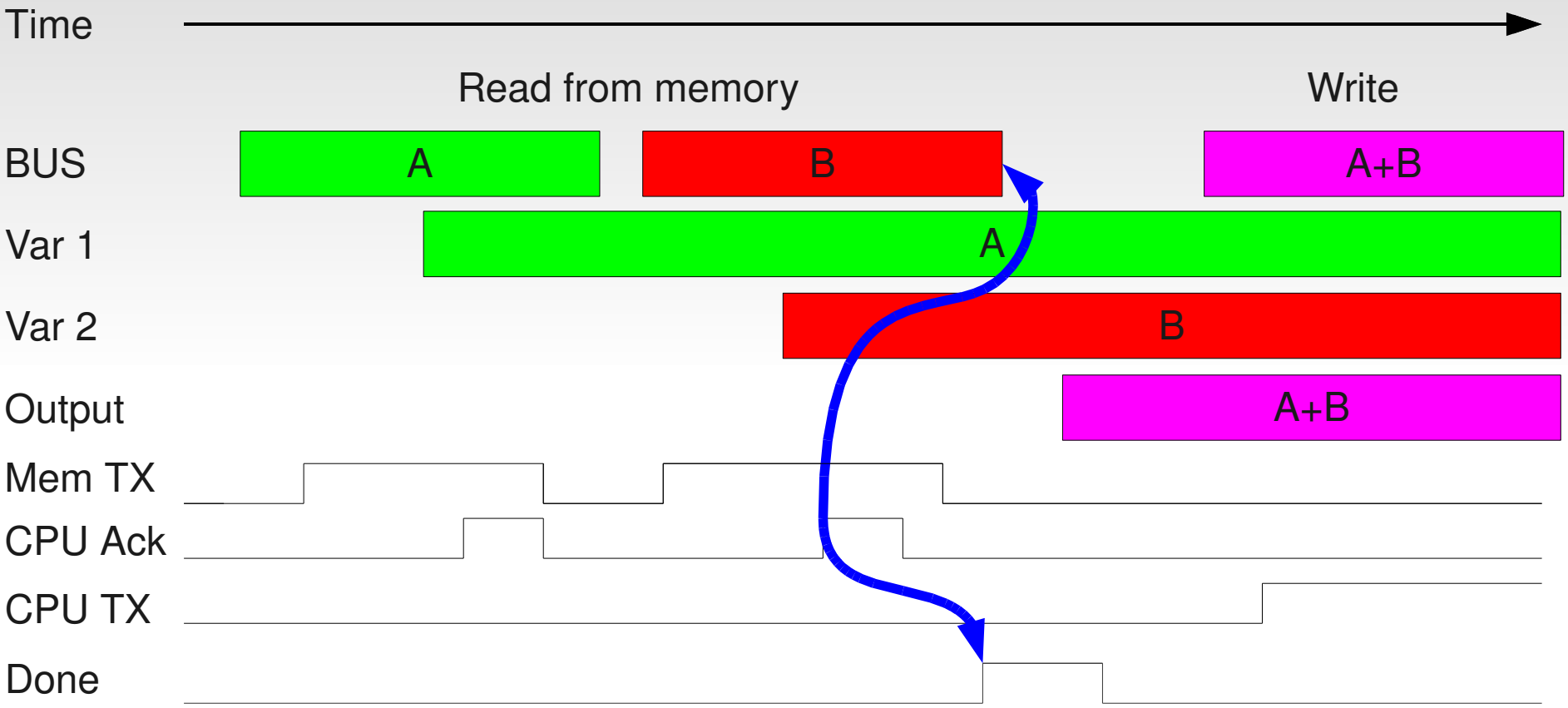

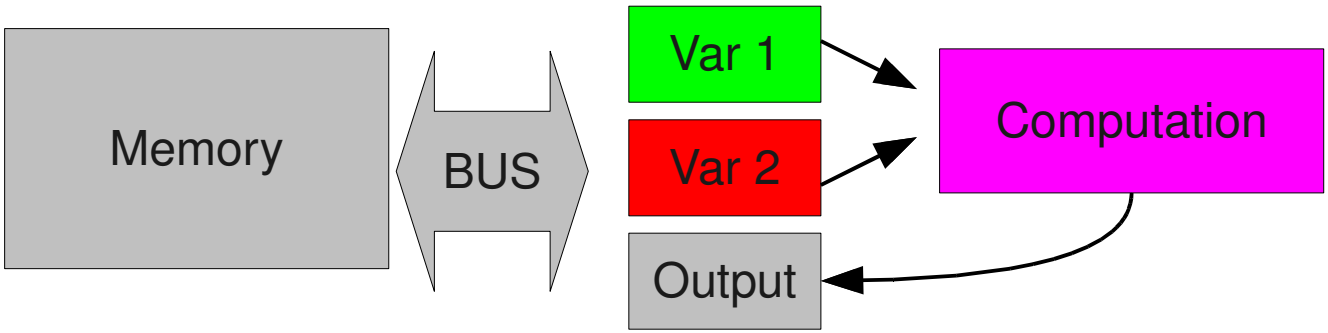

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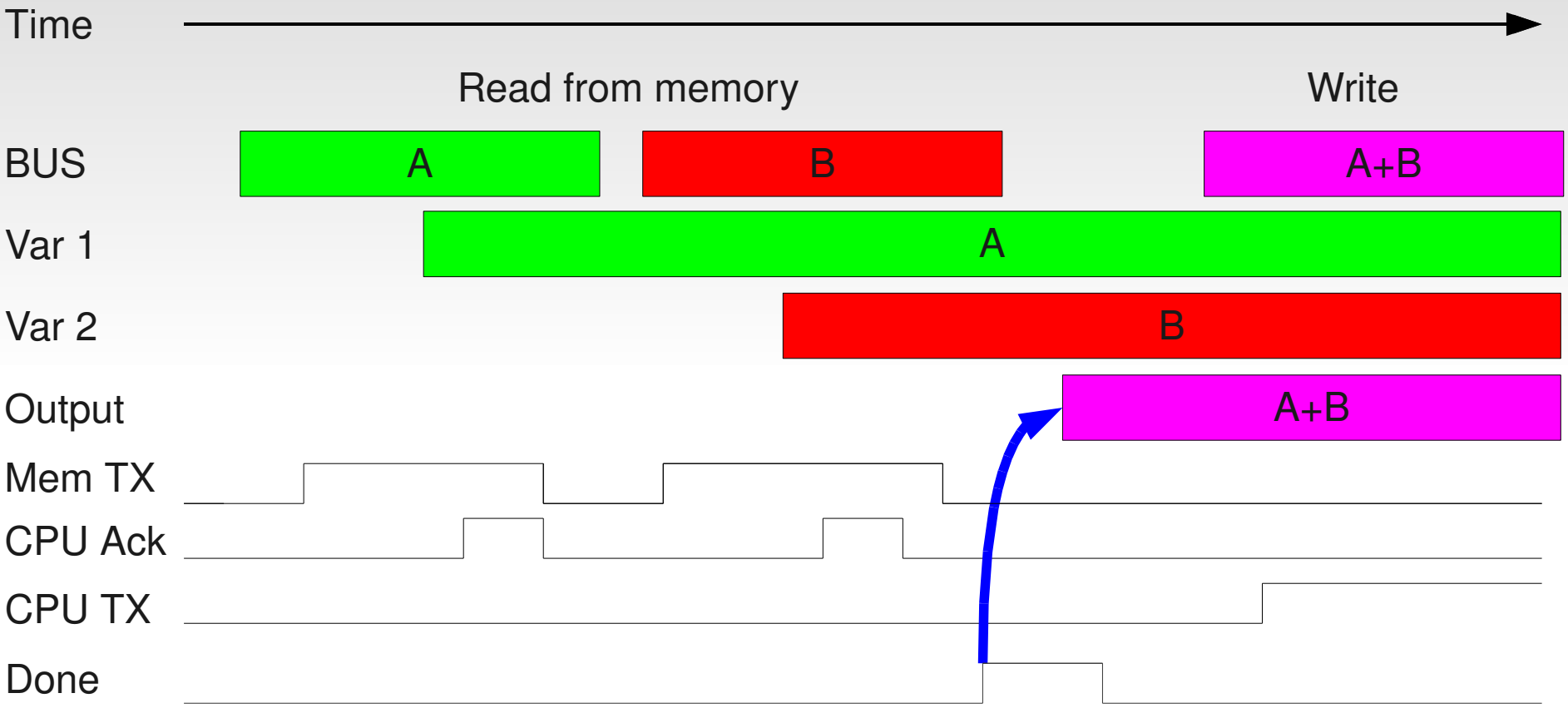

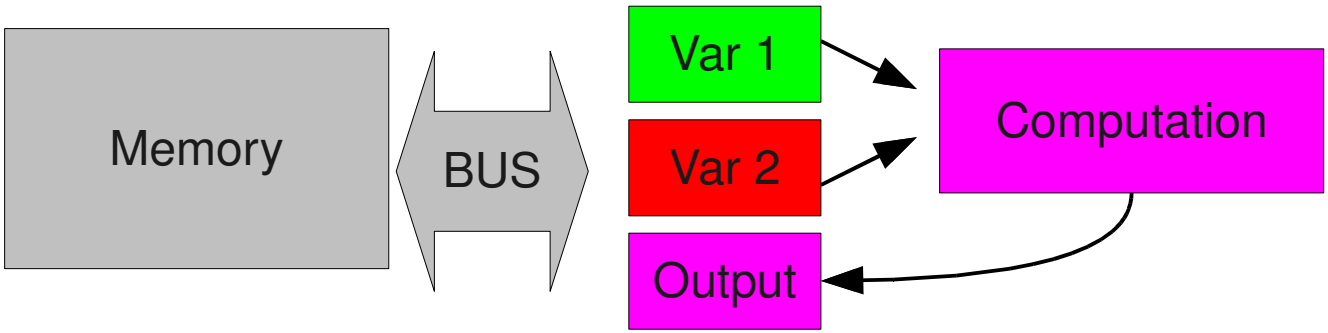

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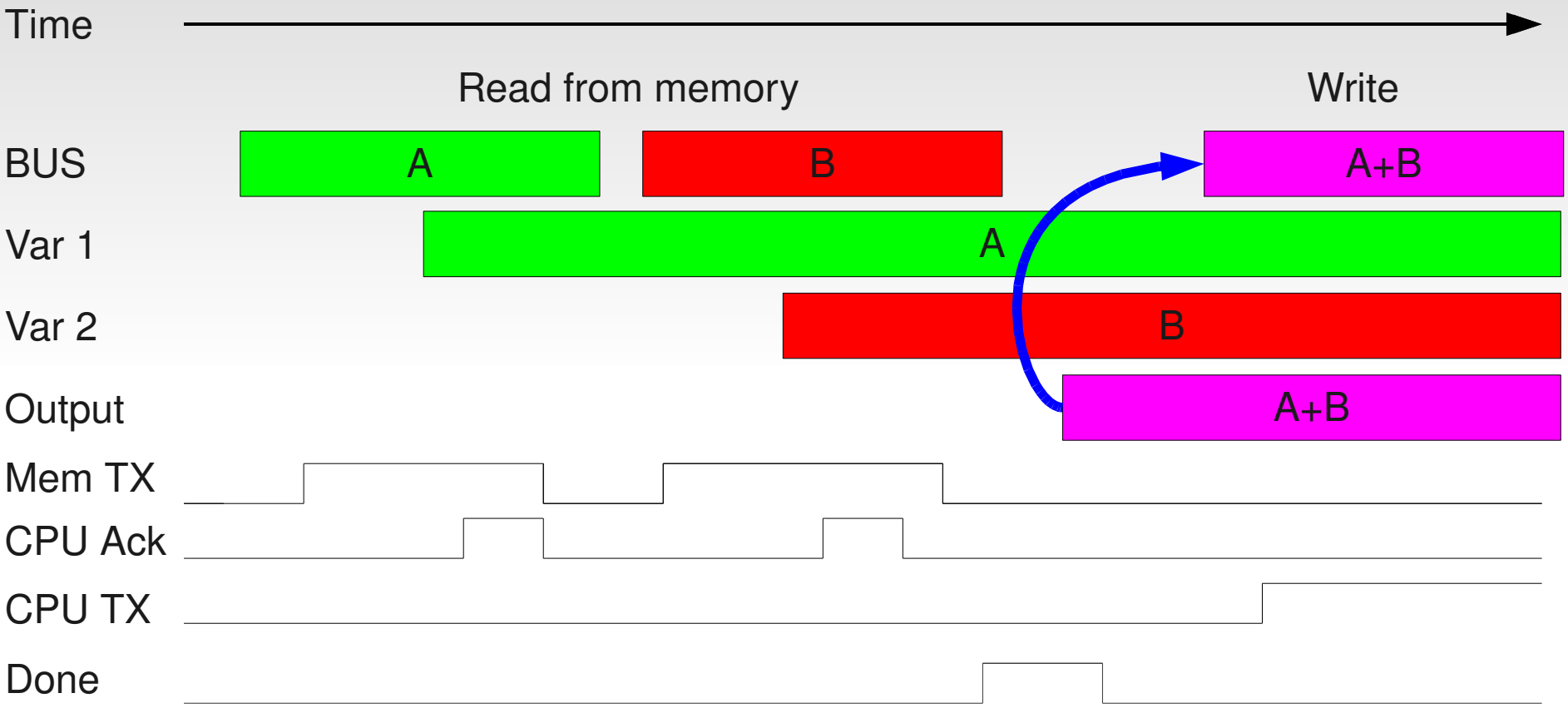

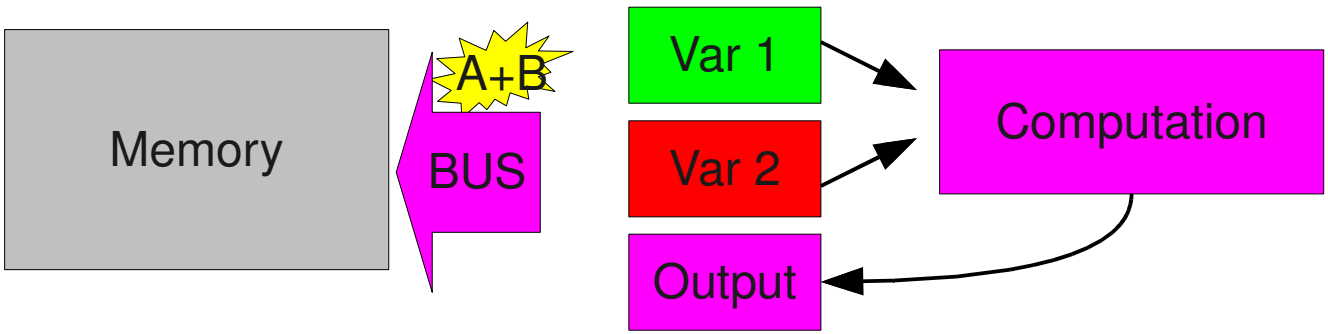

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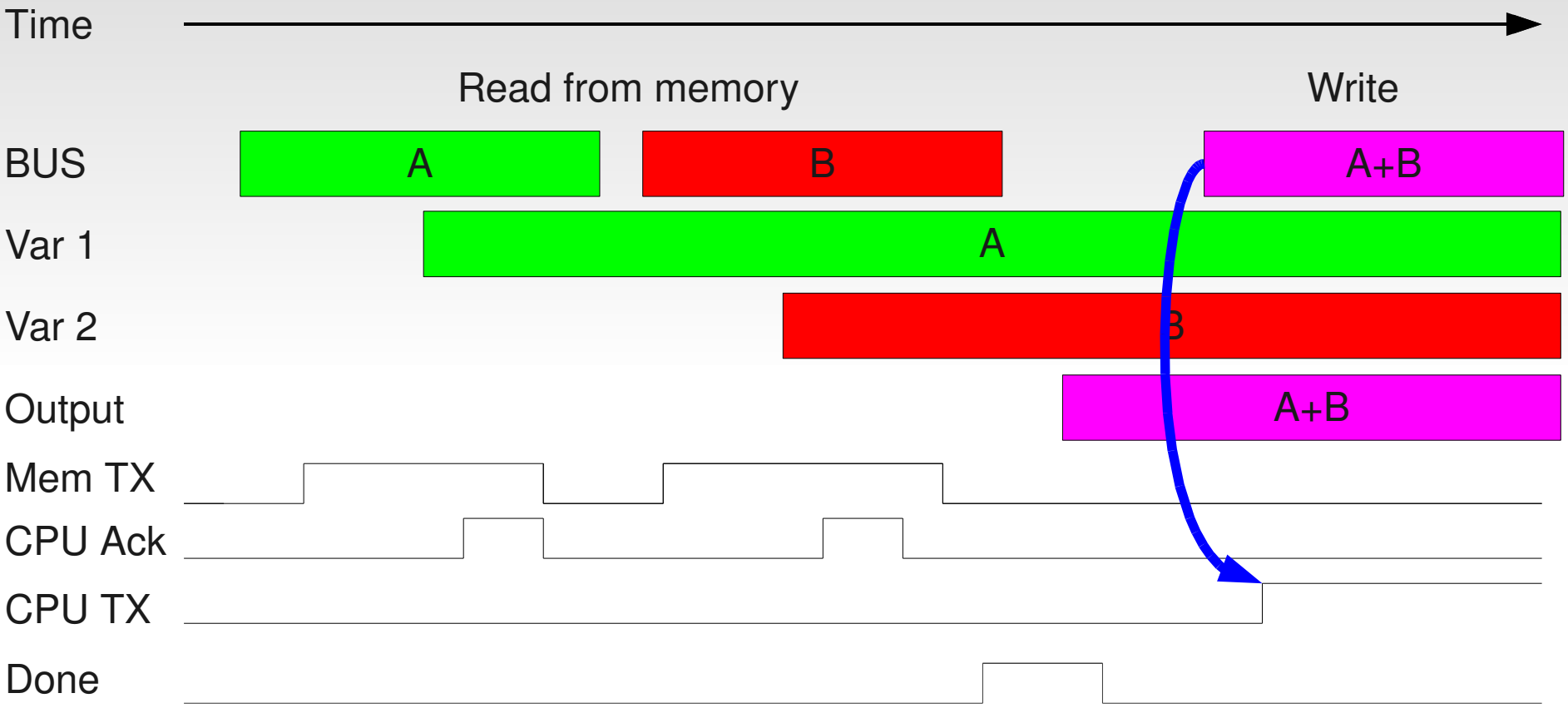
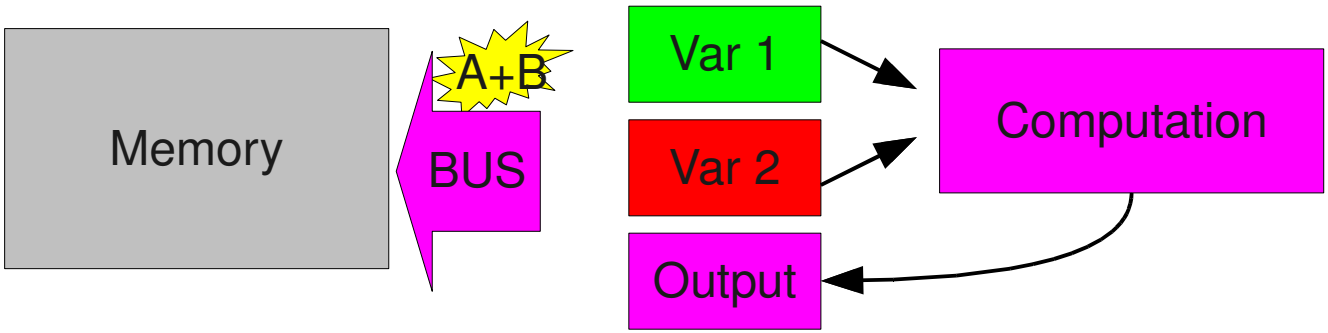




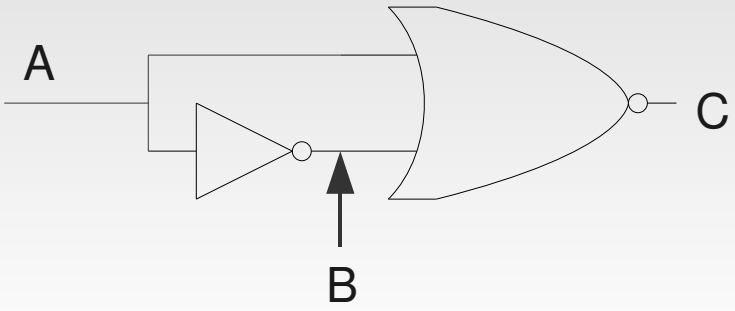
Asynchronous design

- Typical of older bus architectures and of networks
- Potential for significant power savings, space-on-die, and speed in certain areas
- Potential for better distribution of computation
- Design elegance: fewer transistors needed, less to break
- Network communication becomes more natural
  - Especially when latency is highly variable

Problems!

- Asynchronous circuits are hard to design!
- If you mistake a transient for the “final answer” of a circuit, you're faced with
  - Hazards (uncertainties in output value)
  - Glitches (very short pulses, which might confuse the underlying electronic technology)
  - Lock-ups (finite state machines getting stuck in a state where they cannot exit)
- Generally, all are the result of race conditions

Example of a glitch



Race condition between A and B causes glitch!

Limitations in current methods

- Traditional asynchronous design requires either
  - Very careful and exhaustive reasoning (time-dependent theorem-provers, concurrency theory), or
  - Detailed high-fidelity simulation (at sampling rate determined by the "GCD" of the propagation speeds)
- Bookkeeping is difficult, but essential
  - Difficult to test in stages, especially in testing *response* of circuitry to glitches
  - Exhaustive simulation is essentially impossible for large designs (*e.g.* CPUs)

Sheaf theory in logic circuits

- Provides some computational and conceptual tools
  - It's primarily a bookkeeping mechanism
- Building-up local models (gates and wires) into global ones (computational units)
  - The primary tool for this local-to-global transition is called *cohomology*
  - Sheaf cohomology organizes the computations effectively, and extracts lots of information!
  - Hierarchical design can be examined by *local* sheaf cohomology and sheaf direct image functors

Past work

A decidedly non-exhaustive list of some highlights:

- Sheaves over categories of interacting objects
  - Bacławski, Goguen (1970s)
- Concurrency & sheaf theory (not cohomological)
  - Lillius (1993), Van Glabbeek (2006)
- Constructible sheaves
  - Rota, Shapira, MacPherson (1960s)
- Quantum graphs (original motivating example)
  - Gutkin, Smilanski (2001), Kuchment (2003)
- Our focus is more strongly on *cohomology*

Sheaves: definition

A sheaf on a topological space X consists of

- A contravariant functor F from **Open**(X) to some subcategory of **Set**; this is a "sheaf of sets"
  - $F(U)$ for open U is called the *space of sections* over U
  - The inclusion map $U \subset V$ is sent to a *restriction map* $F(V) \to F(U)$. Usually it is the restriction of functions.
  - Given a point $p \in X$, the direct limit of $F(U_\alpha)$, for all U_α satisfying $p \in U_\alpha$ is called the *stalk* at p. It's a generalization of the germ of a smooth function
- And a gluing rule...

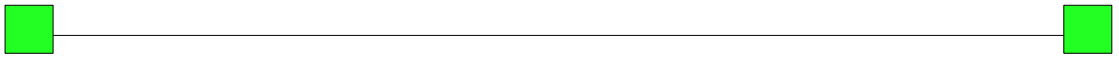
Sheaves: gluing

- The gluing rule: if U and V are open sets, then two sections defined on U and V that agree on $U \cap V$ come from a unique section defined on $U \cup V$

Sheaves: gluing

- The gluing rule: if U and V are open sets, then two sections defined on U and V that agree on $U \cap V$ come from a unique section defined on $U \cup V$

Base topological space X



Sheaves: gluing

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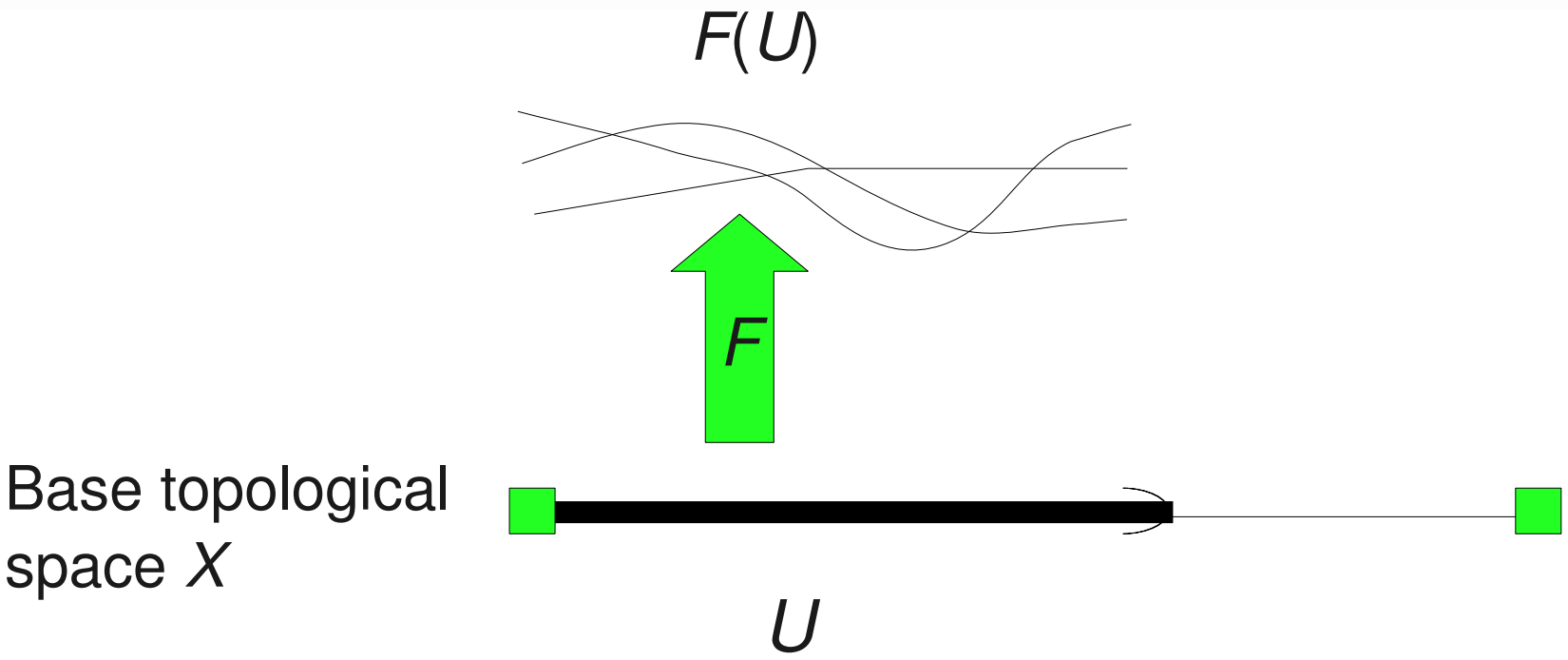

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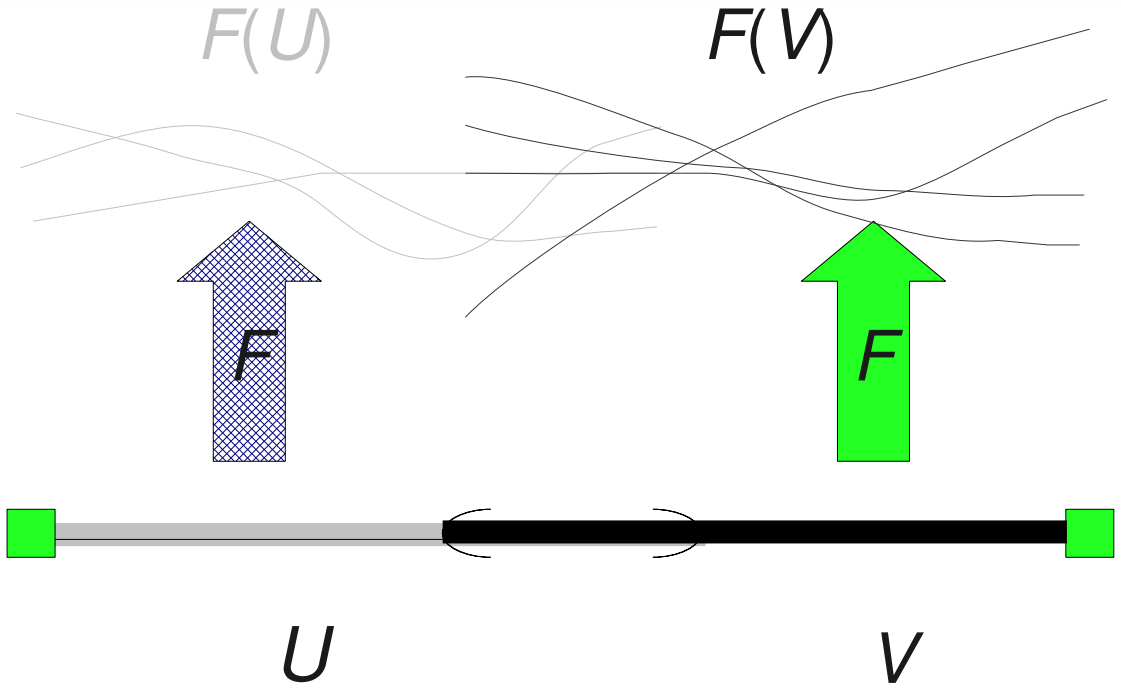

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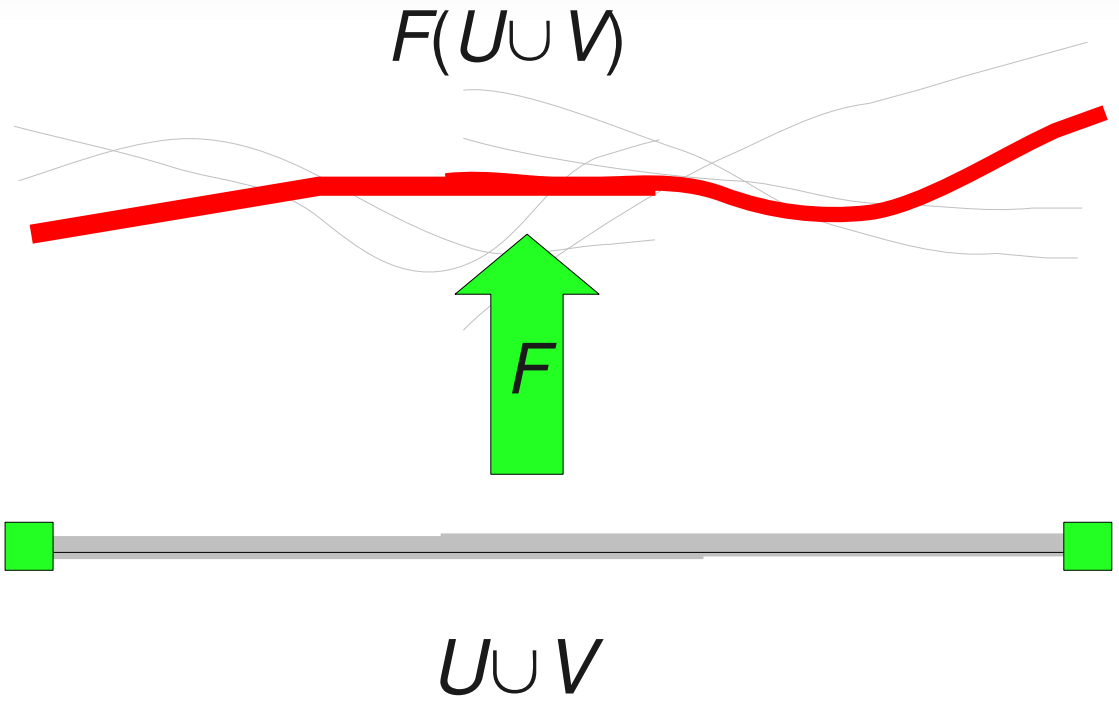


Examples and non-examples

Examples of sheaves:

- Locally constant functions on a topological space
- Continuous functions
- Analytic functions on a manifold

Non-examples (they violate the gluing rule):

- Constant functions
- L^2 functions on unbounded domains

Constructible sheaves

- Suppose X has a filtration, $X_0 \subset X_1 \subset \ldots \subset X_k$ in which each X_i is "tame"
- A sheaf F on X is constructible (with respect to the filtration) if it is locally constant on each stratum: $X_i \backslash X_{i-1}$
- Constructible sheaves have constrained structure, especially if the filtration is finite
- In the case of topological graphs, we'll use the natural filtration structure induced by the graph

Cohomology

- The cohomology functor is a tool for extracting global information from a sheaf
  - Provided it's a sheaf of abelian groups
  - It is homotopy invariant
- It tells you all of the global sections, and obstructions for extending local sections to global ones
  - For instance $H^0(X;F) \cong F(X)$ (all global sections)

Čech cohomology

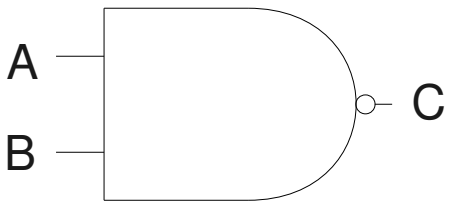
- Select a cover $\{U_\alpha\}$ of X and form the sequence of spaces and maps (the Čech cochain complex)

$$0 \to \oplus F(U_\alpha) \to \oplus F(U_\alpha \cap U_\beta) \to \oplus F(U_\alpha \cap U_\beta \cap U_\gamma) \to \ldots$$

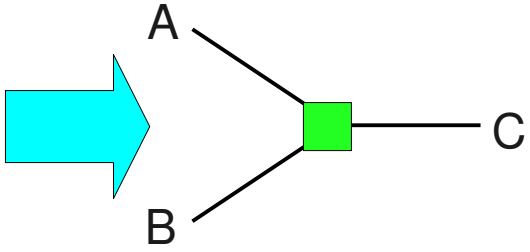
- The maps are called "coboundaries" and come from the differences between restrictions maps
- The homology of this sequence is the Čech cohomology of F with respect to $\{U_\alpha\}$.
  - Theorem: (Leray) If the cover is "good", then the Čech cohomology is a homotopy invariant, and therefore independent of the choice of cover

Problems with logic and sheaves

- If we use binary-valued ($\mathbb{Z}_2$-valued) sheaves in the obvious way, we run into a problem: most logical operations don't support the functoriality of any sheaf in a way that's compatible with cohomology
  - Put another way, logical operations aren't all $\mathbb{Z}_2$-linear!



Logic circuit



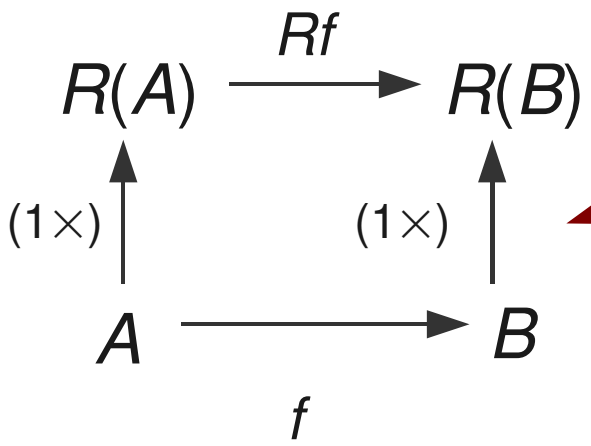
Connection graph

Not linear!

A	B	C
0	0	1
0	1	1
1	0	1
1	1	0

A (standard) algebraic trick!

- Instead, consider any function between sets $f{:}A \to B$
- Let R be a ring with unit, and $R(A)$ be the R-module generated by A
  - That is, generators of $R(A)$ are elements of A
- Then f lifts uniquely to an R-module homomorphism

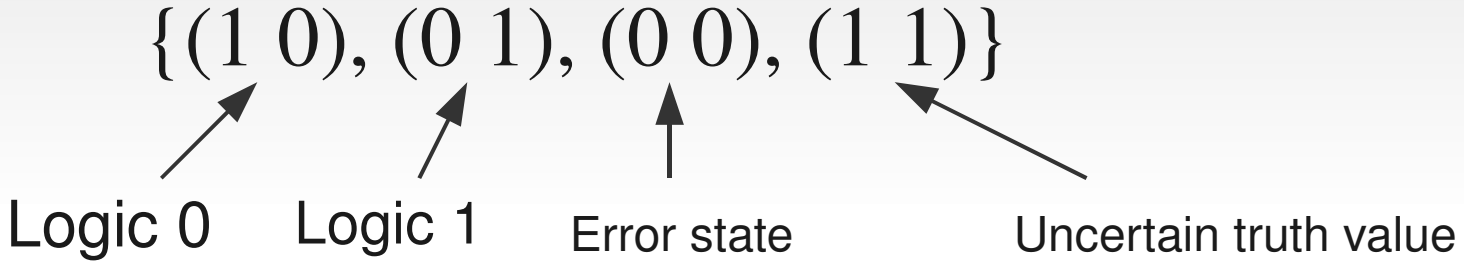


Notice that generally we cannot recover a unique element of B from R(B).

But we can if we've used $Rf \circ (1\times)$

Lifted logic values

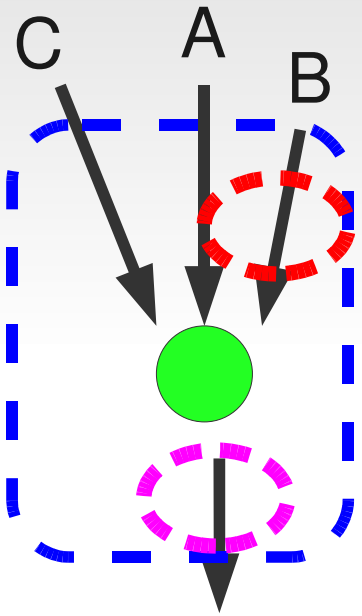
- Our logical value is represented by an element of $\mathbb{Z}_2^{\,2}$ (or R^2 where R is a ring with unit):



- Put another way, a logical value is $aq+bQ$, where
  - q=(1 0), represents a logic 0
  - Q=(0 1), represents a logic 1
  - $a,b\in\mathbb{Z}_2$ can be interpreted as a flag of whether Q or its inverted copy q is a *possible realization* of this value

Switching sheaves

- A *switching sheaf* over a directed graph is constructible with respect to stratification by the graph structure and
  - Stalks over points in an edge are $\mathbb{Z}_2^2$
  - A stalk over a vertex is the tensor product of n copies of $\mathbb{Z}_2^2$, where n is the incoming degree
  - Restriction maps from an open set containing a single vertex to a connected set in the interior of an edge are given by the diagram at right



Contraction of A, C

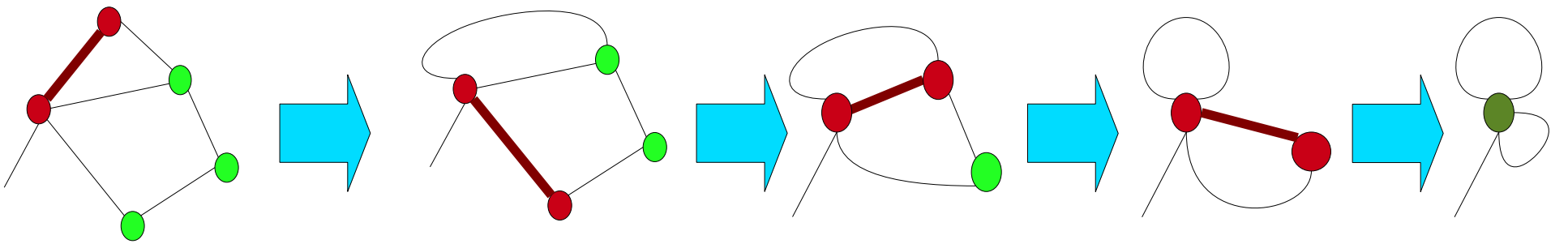
The lift into $\mathbb{Z}_2^8 \to \mathbb{Z}_2^2$ of a logic function

Edge collapse

- The benefit of the sheaf formalism is that useful sheaf functors are already well-known.
- The direct image functor (pushforward) relates to hierarchical design:
- Consider a continuous map $X \to Y$ that collapses an edge with distinct ends. This takes a constructible sheaf F on X to a constructible sheaf $f_* F$ on Y.
  - For switching sheaves, this also induces an isomorphism on cohomology (by the Vietoris mapping theorem)
- Big win conceptually and computationally!

Collapsed graphs

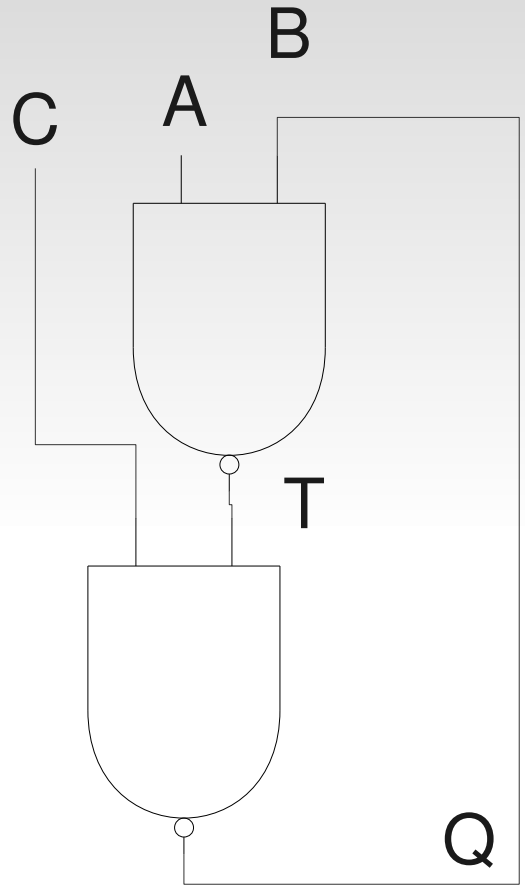
- We construct a spanning tree T for X, and a sequence of trees $T_1, T_2, \dots, T_N = T$ such that $T_{i+1} \setminus T_i$ consists of exactly one edge
- We can work with collapsed graphs X/T_i, on which the cohomology is easier to compute
  - Vietoris Mappring theorem: isomorphic cohomology



Cohomology of switching sheaves

- As noted earlier, $H^0(X;F) \cong F(X)$, so H^0 is generated by all of the allowable states of the logic circuit
  - Switching sheaves don't incorporate time explicitly, but one can still extract time-dependent information in H^0...
  - Appears to track hazard-related transitions between states
- $H^k(X;F)=0$ for $k>1$, since dim $X = 1$
- $H^1(X;F)$ appears to describe the states related to hazards

Example: flip-flop

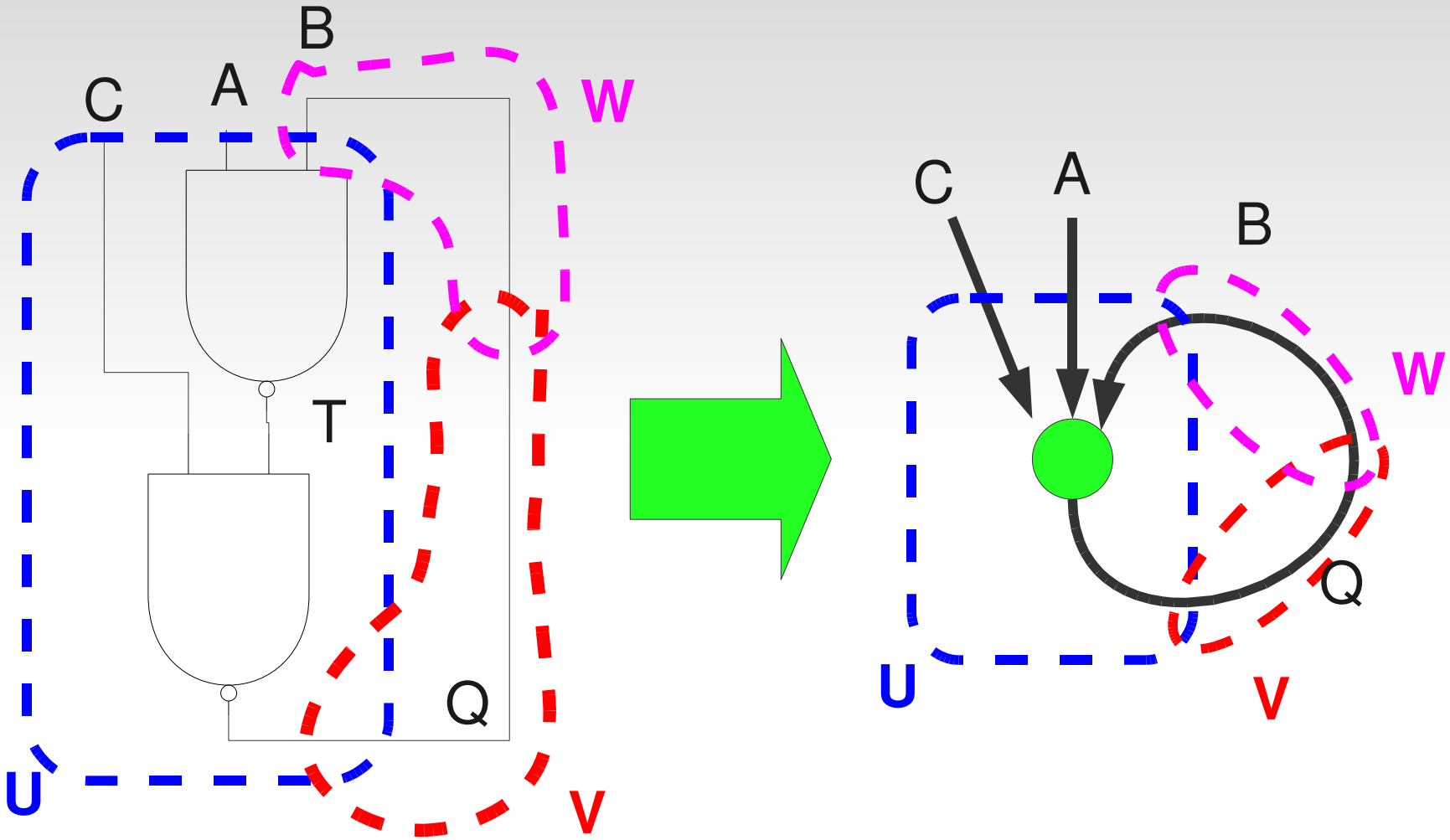


C	A	B	T	Q	
0	0	0	1	1	
0	0	1	1	1	Hazard!
0	1	0	1	1	
0	1	1	0	1	Set
1	0	0	1	0	Reset
1	0	1	1	0	
1	1	0	1	0	Hold
1	1	1	0	1	

Transition out of the hazard state to the a hold state causes a race condition

This is what traditional analysis gives...
5 possible states

Conversion to graph

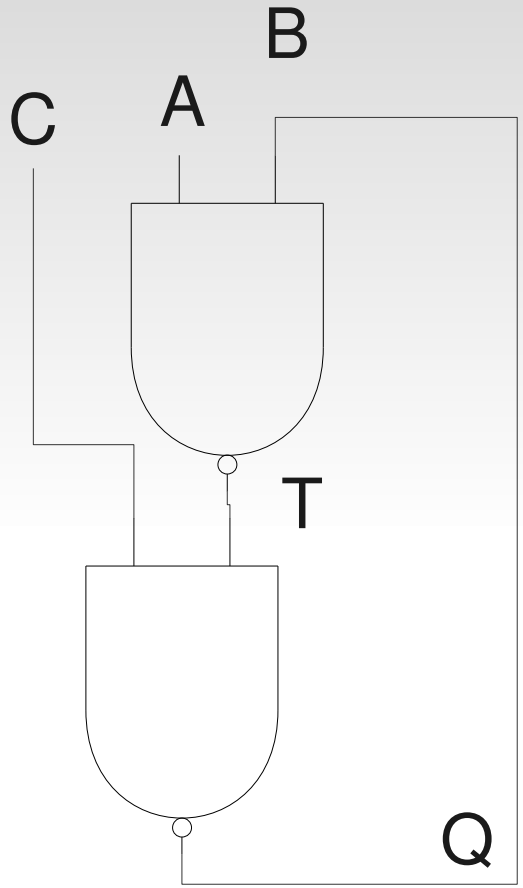


Čech cochain complex:

$$0 \to F(U) \oplus F(V) \oplus F(W) \to F(U \cap V) \oplus F(U \cap W) \oplus F(V \cap W) \to 0$$

$$0 \to \mathbb{Z}_2^{\,8} \oplus \mathbb{Z}_2^{\,2} \oplus \mathbb{Z}_2^{\,2} \to \mathbb{Z}_2^{\,2} \oplus \mathbb{Z}_2^{\,2} \oplus \mathbb{Z}_2^{\,2} \to 0$$

Flip-flop cohomology



$H^1(X;F) \cong \mathbb{Z}_2$

$H^0(X;F) \cong \mathbb{Z}_2^{\,7}$

Generated by:

a⊗B⊗c
A⊗B⊗c
a⊗b⊗C
A⊗b⊗C
A⊗B⊗C

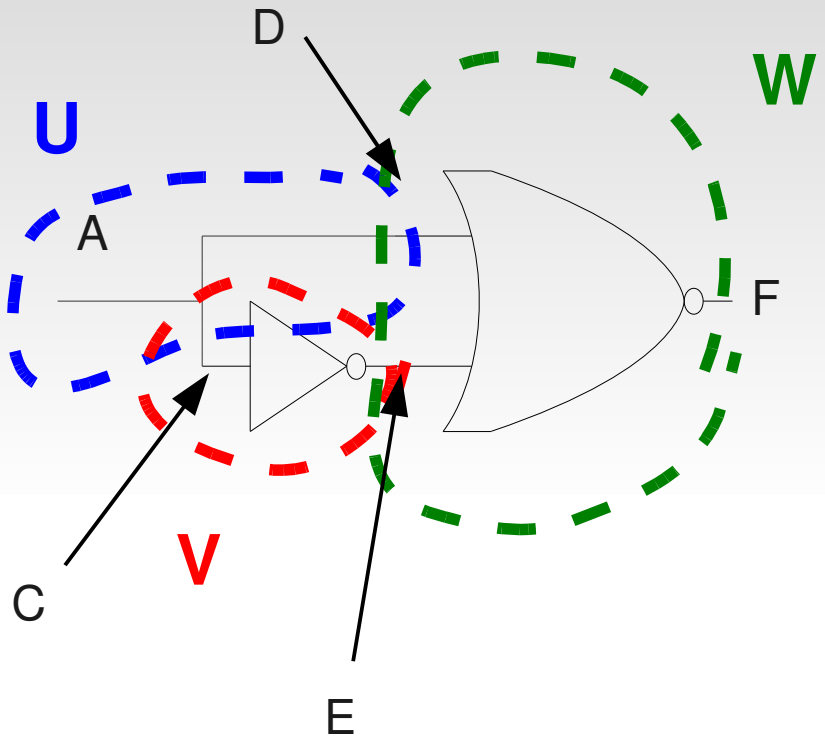
a⊗b⊗c+a⊗B⊗C
a⊗b⊗c+A⊗b⊗c

These states describe the possible transitions out of the hazard state – something that takes a bit more trouble to obtain traditionally

States from the traditional approach

(upper case means the generator corresponding to logical 1)

Example: glitch generator



$H^0(X;F)$ is generated by

A+C+D⊗e
a+c+d⊗E

A+a+C+c+d⊗e+D⊗E

$H^1(X;F)\cong\mathbb{Z}_2$

Hazard transition state

Čech cochain complex:

$$0\to F(U)\oplus F(V)\oplus F(W)\to F(U\cap V)\oplus F(U\cap W)\oplus F(V\cap W)\to 0$$

$$0\to \mathbb{Z}_2{}^2\oplus\mathbb{Z}_2{}^2\oplus\mathbb{Z}_2{}^4\to\mathbb{Z}_2{}^2\oplus\mathbb{Z}_2{}^2\oplus\mathbb{Z}_2{}^2\to 0$$

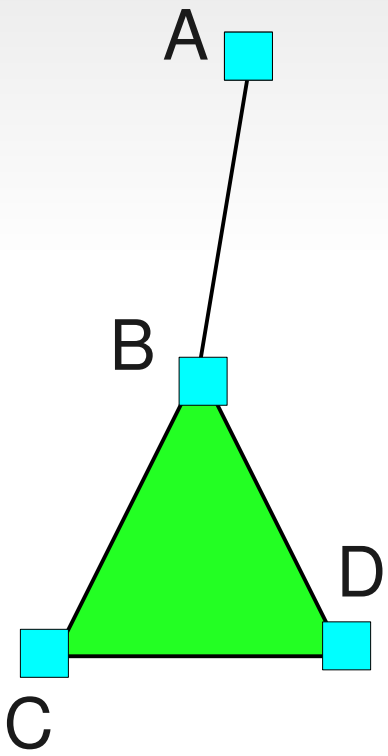
Computational aspects

- It's not immediately clear how one might store a (representation of a) constructible sheaf in a computer
- One needs to specify a vector space for each open set; there are various ways of doing this
  - The most obvious way to do this is to write the sheaf as a function, but then how does one store a vector space?
  - Possibly use type-level programming in Haskell? We could instantiate the sheaf as a type of Functor...
- Seriously, though, it seems to be an impediment to automating computation in constructible sheaves

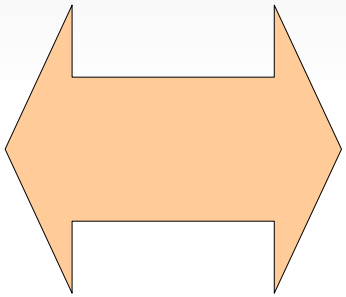
Category theory to the rescue!

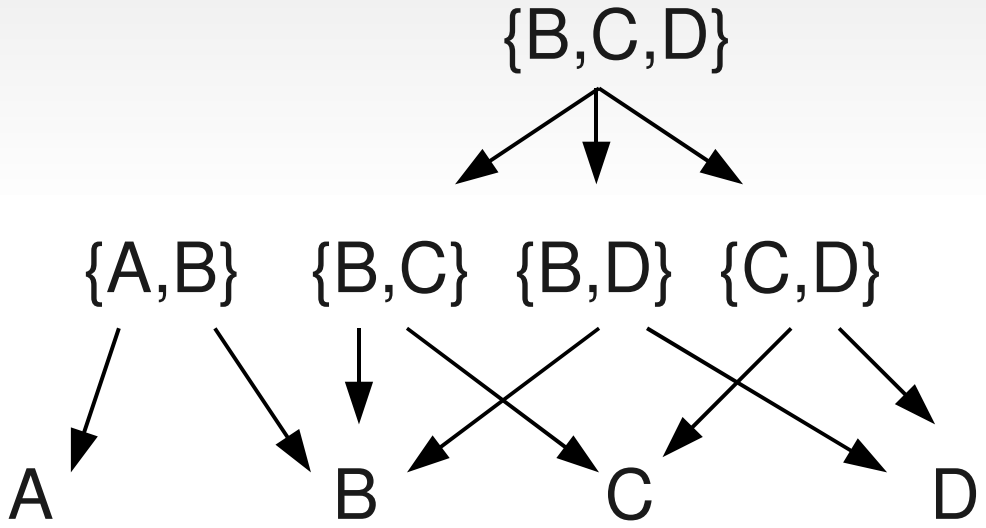
- It turns out that there's a different way:
- Theorem: (MacPherson) The category of constructible sheaves on an abstract simplicial complex K is isomorphic to the category of *pre*sheaves over a certain category associated to K
  - By presheaf, we mean a contravariant functor from a category to a subcategory of **Set**
  - The category in question here is the face category: objects are simplices, and morphisms describe boundaries (*i.e.* $A \to B$ if B is a face of A)

Simplicial complexes and the face category



Simplicial complex





Face category

Presheaves on a face category

- If our graph is a cell complex, we therefore only need to know the restriction maps and the stalks over each cell.
- This seems like a minimal amount of information
- Further, the construction is functorial, so we can transfer computation of sheaf cohomology to this context
- This relates to HDA in concurrency theory!

What's next?

- Theoretical directions
  - Figure out how exactly glitches and hazards are represented in the cohomology of a switching sheaf
  - Related: what is the physical meaning of $H^1(X;F)$?
  - Extend edge collapse methodology to other direct images; aiming towards a hierarchical approach to sheaf cohomology computation
- Computational directions
  - Run some more complicated examples of cohomology computations
  - Implement the cohomology computation on a computer