Solutions exercises splines: 24 March 2014

1) As we have 4 points ($P_0 = (0, 0), P_1 = (3, 0), P_2 = (6, 0), P_3 = (6, 3)$), A is a 4×4 matrix, then by aid of the slides:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

2)

$$A \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = 3 \begin{bmatrix} P_1 - P_0 \\ P_2 - P_0 \\ P_3 - P_1 \\ P_3 - P_2 \end{bmatrix} = \begin{bmatrix} (9, 0) \\ (18, 0) \\ (9, 9) \\ (0, 9) \end{bmatrix}$$

We solve to get the velocity vectors (slopes) at the the points (P_0, P_1, P_2, P_3) separately for the x and y coordinates. Solve by using the program on the net (http://webspace.ship.edu/jwcraw/Gaussian2.html) or e.g. by finding the inverse of A by Matlab:

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}_x = A^{-1} \begin{bmatrix} 9 \\ 18 \\ 9 \\ 0) \end{bmatrix} = \begin{bmatrix} 14/5 \\ 17/5 \\ 8/5 \\ -4/5 \end{bmatrix}$$

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}_y = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 9 \\ 9 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -2/5 \\ 7/5 \\ 19/5 \end{bmatrix}$$

Corresponding polynomials and splines

A 3rd order polynomial ($p(t)$) can be written in terms of the Hermite polynomials ($F_1(t) = 1 - 3t^2 + 2t^3$; $F_2(t) = 3t^2 - 2t^3$; $F_3(t) = t - 2t^2 + t^3$; $F_4(t) = -t^2 + t^3$) as

$$p(t) = F_1(t)P_0 + F_2(t)P_1 + F_3(t)v_0 + F_4(t)v_1 \ (v_0 = P_0^{'} v_1 = P_1^{'})$$

So we find the three polynomials (parametrization with respect to t) connecting the $\underline{\mathbf{x}\text{ coordinates}}$ of $P_0 - P_1$ $P_1 - P_2$ and $P_2 - P_3$ as:

$$
\begin{aligned}
p_x^1(t) &= F_1(t)P_0^x + F_2(t)P_1^x + F_3(t)v_0^x + F_4(t)v_1^x \\
&= 0F_1(t) + 3F_2(t)3 + \frac{14}{5}F_3(t) + \frac{17}{5}F_4(t) \quad = 3F_2(t) + \frac{14}{5}F_3(t) + \frac{17}{5}F_4(t) \\
p_x^2(t) &= F_1(t)P_1^x + F_2(t)P_2^x + F_3(t)v_1^x + F_4(t)v_2^x \;= 3F_1(t) + 6F_2(t) + \frac{17}{5}F_3(t) + \frac{8}{5}F_4(t) \\
p_x^3(t) &= F_1(t)P_2^x + F_2(t)P_3^x + F_3(t)v_2^x + F_4(t)v_3^x \\
&= 6F_1(t) + 6F_2(t) + \frac{8}{5}F_3(t) + \frac{-4}{5}F_4(t) \\
&= 6(1 - 3t^2 + 2t^3) \;+\; 6(3t^2 - 2t^3) \;+\; \frac{8}{5}(t - 2t^2 + t^3) \;+\; \frac{-4}{5}(-t^2 + t^3) \\
&= 6 - 18t^2 + 12t^3 \;\; + 18t^2 - 12t^3 \;+\; \frac{8}{5}t - \frac{16}{5}t^2 + \frac{8}{5}t^3 \;+\; \frac{4}{5}t^2 - \frac{4}{5}t^3 \\
&= 6 + \frac{8}{5}t - \frac{12}{5}t^2 + \frac{4}{5}t^3
\end{aligned}
$$

Where for the 3rd polynomial we have inserted the expressions of the Hermite-polynomials to show its explicit form.

Below the polynomials p_x^1,p_x^2 and p_x^3 are shown.

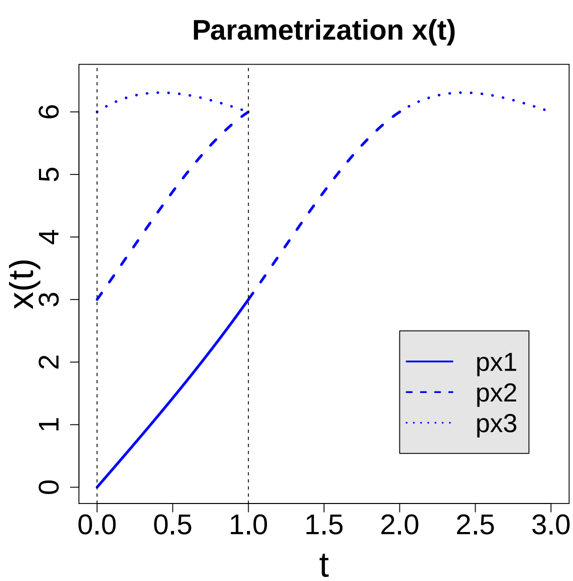

Figure 1: The three x-polynomials over the [0,1] interval and their shifted versions.

In the same way the three polynomials connecting the **y** coordinates of $P_0 - P_1$ $P_1 - P_2$ and $P_2 - P_3$ are:

$$
\begin{aligned}
p_y^1(t) &= F_1(t)P_0^y + F_2(t)P_1^y + F_3(t)v_0^y + F_4(t)v_1^y \\
&= 0F_1(t) + 0F_2(t) + \frac{1}{5}F_3(t) + \frac{-2}{5}F_4(t) \;\; = \;\; \frac{1}{5}F_3(t) + \frac{-2}{5}F_4(t) \\
p_y^2(t) &= F_1(t)P_1^y + F_2(t)P_2^y + F_3(t)v_1^y + F_4(t)v_2^y \\
&= 0F_1(t) + 0F_2(t) + \frac{-2}{5}F_3(t) + \frac{7}{5}F_4(t) \;\; = \;\; \frac{-2}{5}F_3(t) + \frac{7}{5}F_4(t) \\
p_y^3(t) &= F_1(t)P_2^y + F_2(t)P_3^y + F_3(t)v_2^y + F_4(t)v_3^y \\
&= 0F_1(t) + 3F_2(t) + \frac{7}{5}F_3(t) + \frac{19}{5}F_4(t) \;\; = \;\; 3F_2(t) + \frac{7}{5}F_3(t) + \frac{19}{5}F_4(t)
\end{aligned}
$$

Below the polynomials p_y^1,p_y^2 and p_y^3 are shown.

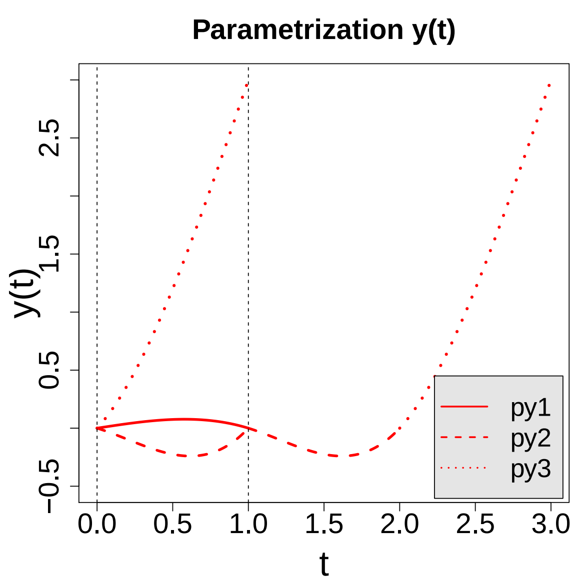

Figure 2: The three y-polynomials over the [0,1] interval and their shifted versions.

Plotting $y(t)$ versus $x(t)$ gives the desired spline solution.

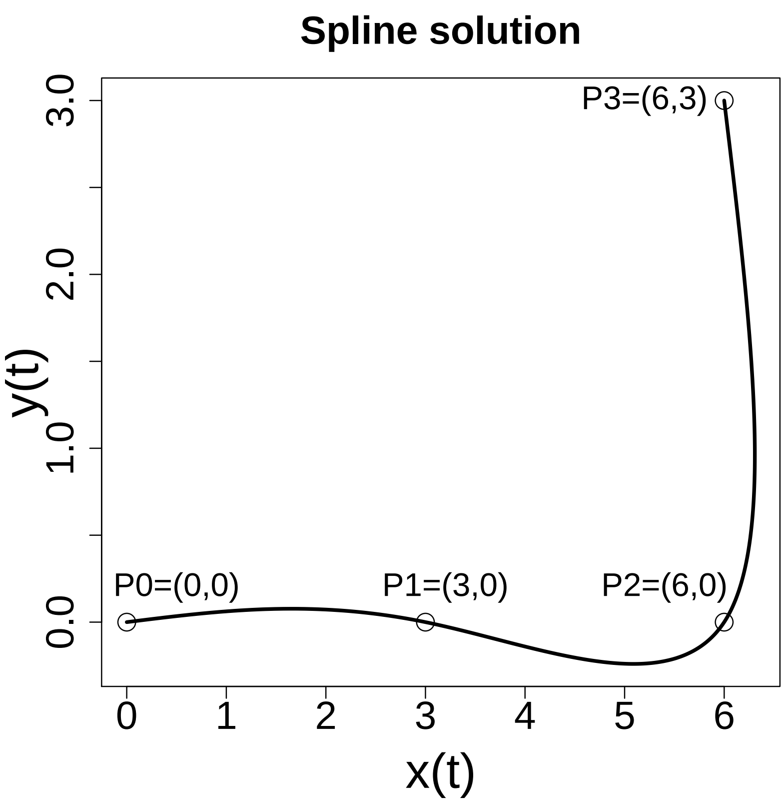

Figure 3: Spline solution, t=[0,3]