Kurver og flader i geometri, arkitektur og design 8. lektion

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Kubiske parameterfremstillinger

Mål: parameterfremstilling

$$\mathbf{p}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3 \text{ således at}$$

$$\mathbf{p}(0) = \overrightarrow{OP_0},\ \mathbf{p}(1) = \overrightarrow{OP_1},\ \mathbf{p}'(0) = \mathbf{v}_0,\ \mathbf{p}'(1) = \mathbf{v}_1.$$

Løsning: $\mathbf{p}(t) = F_1(t)\overrightarrow{OP_0} + F_2(t)\overrightarrow{OP_1} + F_3(t)\mathbf{v}_0 + F_4(t)\mathbf{v}_1$

$$\begin{aligned} F_1(t) &:= 2t^3 - 3t^2 + 1 \\ F_2(t) &:= -2t^3 + 3t^2 \\ F_3(t) &:= t^3 - 2t^2 + t \\ F_4(t) &:= t^3 - t^2 \end{aligned}$$

de fire Hermitepolynomier.

Kubiske splines

Mål: C^2[1]-parameterfremstilling for en kurve gennem punkter

$$P_0, P_1, \dots, P_n.$$

Den består af n kubiske parameterfremstillinger (Hermite)

$$\begin{array}{c} \mathbf{p}_1(t) \text{ gennem } P_0 \text{ og } P_1 \\ \mathbf{p}_2(t) \text{ gennem } P_1 \text{ og } P_2 \\ \dots \\ \mathbf{p}_n(t) \text{ gennem } P_{n-1} \text{ og } P_n. \end{array}$$

C^2 svarer til : $\mathbf{p}'_i(1) = \mathbf{p}'_{i+1}(0) = \mathbf{v}_i$

$\mathbf{p}''_i(1) = \mathbf{p}''_{i+1}(0)$

Derudover ønskes: $\mathbf{p}''_1(0) = \mathbf{p}''_n(1) = \mathbf{0}$

(Krumning $= 0$ i P_0 og P_1.)

[1] 1. og 2. afledede kontinuert!

Oversættelse til lineær algebra

Ide: Find hastighedsvektorer $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n$ sål. at

$$\mathbf{p}_1{}'(0) = \mathbf{v}_0, \quad \mathbf{p}_i'(1) = \mathbf{p}_{i+1}'(0) = \mathbf{v}_i, \quad 0 \le i \le n-1, \quad \mathbf{p}_n'(1) = \mathbf{v}_n$$

$$\mathbf{p}_1{}''(0) = \mathbf{0}, \quad \mathbf{p}_i''(1) = \mathbf{p}_{i+1}''(0), \quad 0 \le i \le n-1, \quad \mathbf{p}_n''(1) = \mathbf{0}$$

$$\mathbf{p}_1{}''(0) = \mathbf{0} \Leftrightarrow 2\mathbf{v}_0 + \mathbf{v}_1 = 3\overrightarrow{P_0P_1}$$

$$\mathbf{p}_i''(1) = \mathbf{p}_{i+1}''(0) \Leftrightarrow \mathbf{v}_{i-1} + 4\mathbf{v}_i + \mathbf{v}_{i+1} = 3\overrightarrow{P_{i-1}P_{i+1}}$$

$$\mathbf{p}_n''(1) = \mathbf{0} \Leftrightarrow \mathbf{v}_{n-1} + 2\mathbf{v}_n = 3\overrightarrow{P_{n-1}P_n}$$

Lineært ligningssystem: $(n+1)$ ligninger i $(n+1)$ ubekendte $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n$

Koefficientmatriks $\mathbf{A}_{n+1} = \begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ 0 & . & . & . & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 4 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 2 \end{bmatrix}$

Beregning af hastighedsvektorerne

Hastighedsvektorerne $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ er løsningerne til:

$$\mathbf{A}_{n+1} \begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \cdot \\ \cdot \\ \mathbf{v}_{n-1} \\ \mathbf{v}_n \end{bmatrix} = 3 \begin{bmatrix} \overrightarrow{P_0P_1} \\ \overrightarrow{P_0P_2} \\ \cdot \\ \cdot \\ \overrightarrow{P_{n-2}P_n} \\ \overrightarrow{P_{n-1}P_n} \end{bmatrix} \leftarrow \text{højresiden}$$

$$\Leftrightarrow \begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \cdot \\ \cdot \\ \mathbf{v}_{n-1} \\ \mathbf{v}_n \end{bmatrix} = 3\mathbf{A}_{n+1}{}^{-1} \begin{bmatrix} \overrightarrow{P_0P_1} \\ \overrightarrow{P_0P_2} \\ \cdot \\ \cdot \\ \overrightarrow{P_{n-2}P_n} \\ \overrightarrow{P_{n-1}P_n} \end{bmatrix}$$

Et simplet tilfælde: $n = 3$

$$\mathbf{A}_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

$$\mathbf{A}_3^{-1} = \frac{1}{12}\begin{bmatrix} 7 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 7 \end{bmatrix}, 3\mathbf{A}_3^{-1} = \frac{1}{4}\begin{bmatrix} 7 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 7 \end{bmatrix}.$$

$$\begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(7\overrightarrow{P_0P_1} - 2\overrightarrow{P_0P_2} + \overrightarrow{P_1P_2}) \\ \frac{1}{2}(-\overrightarrow{P_0P_1} + 2\overrightarrow{P_0P_2} - \overrightarrow{P_1P_2}) \\ \frac{1}{4}(\overrightarrow{P_0P_1} - 2\overrightarrow{P_0P_2} + 7\overrightarrow{P_1P_2}) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{4}(5\overrightarrow{P_0P_1} - \overrightarrow{P_1P_2})^2 \\ \frac{1}{2}\overrightarrow{P_0P_2} \\ \frac{1}{4}(5\overrightarrow{P_1P_2} - \overrightarrow{P_0P_1}) \end{bmatrix}.$$

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[2] $7\overrightarrow{P_0P_1} - 2\overrightarrow{P_0P_2} + \overrightarrow{P_1P_2} = -7\overrightarrow{OP_0} + 2\overrightarrow{OP_0} + 7\overrightarrow{OP_1} - \overrightarrow{OP_1} - 2\overrightarrow{OP_2} + \overrightarrow{OP_2} = -5\overrightarrow{OP_0} + 6\overrightarrow{OP_1} - \overrightarrow{OP_2} = 5\overrightarrow{P_0P_1} - \overrightarrow{P_1P_2}$

Eksempel

$P_0 : (4,0), P_1 : (0,4), P_2 : (4,4)$;
$\overrightarrow{P_0P_1} = [-4,4], \overrightarrow{P_0P_2} = [0,4], \overrightarrow{P_1P_2} = [4,0]$;
$\mathbf{v}_0 = \frac{1}{4}([-20,20]-[4,0]) = [-6,5], \mathbf{v}_1 = \frac{1}{2}[0,4] = [0,2]$,
$\mathbf{v}_2 = \frac{1}{4}([20,0]-[-4,4]) = [6,-1]$.
Nu indsættes $P_0, P_1, \mathbf{v}_0, \mathbf{v}_1$ i parameterfremstilling $\mathbf{p}_1(t)$ for Hermitekurve nr. 1:
$\mathbf{p}_1(t) = (1-3t^2+2t^3)[4,0] + (3t^2-2t^3)[0,4] + (t-2t^2+t^3)[-6,5] + (-t^2+t^3)[0,2] = [4-6t+2t^3, 5t-t^3], t \in [0,1]$;
og tilsvarende $P_1, P_2, \mathbf{v}_1, \mathbf{v}_2$ i parameterfremstilling $\mathbf{p}_2(t)$ for Hermitekurve nr. 2:
$\mathbf{p}_2(t) = (1-3t^2+2t^3)[0,4] + (3t^2-2t^3)[4,4] + (t-2t^2+t^3)[-0,2] + (-t^2+t^3)[6,-1] = [6t^2-2t^3, -4+2t-3t^2+t^3]$,
$t \in [0,1]$.

Kurver med $n+1$ kontrolpunkter

n-te ordens Bézierkurver defineres ved hjælp af

Bernstein-polynomier $B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$ af grad n.

$$B_n(t) = B_{0,n}(t)Q_0 + \cdots + B_{i,n}(t)Q_i + \cdots + B_{n,n}(t)Q_n.$$

B-splines defineres ved hjælp af stykkevis polynomiale funktioner af grad k - som er 0 på en del af intervallet.

	støttepunkter	grad	glathed	lokal kontrol
kubisk spline	alle	3	C^2	⋮
Bézierkurve	endepunkter	n	C^∞	⋮
B-spline	endepunkter	k	C^{k-2}	delvis[3]

[3] Kontrolpunkt har indflydelse på område svarende til $k+1$ punkter