The Rank of a Smooth Map

When? Tue, September 21; 8:45 – 11:45
Where? Fredrik Bajersvej 7G5-109

Lectures

Aims and Content

The important Regular Level Set Theorem (Thm. 9.11. in the textbook) can be generalized vastly. If just the smooth map $f : N \to M$ has constant rank (i.e. all Jacobians have the same rank) close to a level set $f^{-1}(c)$, then that level set is a regular submanifold of N.

Particularly important smooth maps are immersions (injective differential at all points in N) and submersions (surjective differential at all points in N). Those look locally, but not globally! like an injection, resp. a projection. To get one manifold properly placed as a submanifold of another one needs an embedding (Def. 11.14) $f : N \to M$. For such an embedding, the image $f(N) \subset M$ is a regular submanifold.

Spices and pickles:

- the orthogonal group $O(n) \subset Gl(n, \mathbf{R})$;
- immersions that are not embeddings (for various reaons);
- group multiplication in $Gl(n, \mathbf{R})$ and $Sl(n, \mathbf{R})$ (important in the discussion of Lie groups later on);
- the tangent plan of a 3D-level surface.

Lecturer:

Martin Raussen

References:

LWT Ch. 11 and ch. B.3;

Wikipedia Constant rank

Exercises:

[LWT]

Ch. 8, p. 88 8.8

Ch. 9, p. 99 9.1 – 9.7 (Start with those that you did not finish during or after the last lecture!)

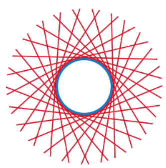
The Tangent Bundle

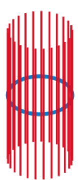
Tue, September 23; 12:30 – 15:30

Lectures

Aims and Content

The tangent spaces at all points of a given manifold M form together the tangent bundle TM of that manifold. The charts from the definition of a manifold can be used to give the set TM first a topology and then the structure of a (very special) differentiable manifold.





In particular, associating to every tangent vector its base point, describes a smooth map $\pi : TM \to M$ with the property that the inverse image $\pi^{-1}m$ of a point $m \in M$ is the vector space T_pM, i.e., the tangent space to M at p.
The tangent bundle is the right object to look at if one wants to study linear approximation, vector fields etc.

A tangent bundle is a special case of a vector bundle $\pi : E \to B$ characterized by the fact that every fiber $E_b := \pi^{-1}(b)$ is a vector space for each $b \in B$ and, moreover, by local triviality. A section of such a vector bundle associates to every $b \in B$ an element of the fibre E_b in a continuous (or smooth) manner.
In particular, a tangent field associates to every $p \in M$ a tangent vector $s(p) \in T_pM$ and thus encodes a first order differential equation on the manifold!
Several sections together that are linearly independent at every base point form a frame on the vector bundle (a smoothly varying basis in every fiber). Global frames do not always exist: For example, there is not even a 1-frame (nowhere vanishing tangent field) on the tangent bundle of a sphere of even dimension! On the other hand, they do exist on Lie groups and can be usefully exploited to yield simple representations for sections.

Lecturer:

Lisbeth Fajstrup

References:

LWT , ch. 12; ch. 2.4.

Wikipedia Tangent bundle

Exercises:

- **LWT, ch. 11, pp. 116 – 117:** 11.1, 11.2[1], 11.3, 11.5

[1] A condition on dimensions is missing. Which?

FREDRIK BAJERSVEJ 7 | 9940 8848 | FAJSTRUP@MATH.AAU.DK
9220 AALBORG ØST | 9940 8855 | RAUSSEN@MATH.AAU.DK
9940 8762 | RAF@ES.AAU.DK