**Homework Set 2**

SEPTEMBER 27, 2010

It is the aim of this homework set to achieve a calculation of the differential $\psi_*$ of the map $\psi : S^3 \to SO(3)$ from the first homework set. We will write $\psi_{*,\alpha} : T_\alpha S^3 \to T_{\psi(\alpha)}SO(3)$ when we indicate base points of the tangent spaces. Left invariant vector fields on both Lie groups play a decisive role, and that is why one needs to study left multiplication:

Left multiplication $l_\alpha : H \to H$ with a quaternion $\alpha := a + bi + cj + dk \in H$ can be seen as a linear map on the 4-dimensional real vector space $H$ with basis elements $1, i, j, k$. It is easy to check that, with respect to this basis, $l_\alpha$ corresponds to the matrix

$$L_\alpha := \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

For two quaternions, $\alpha, \beta \in H$, we have obviously $l_{\alpha\beta} = l_\alpha \circ l_\beta$ and hence $L_{\alpha\beta} = L_\alpha \circ L_\beta$. For a unit quaternion $\alpha \in S^3$, this has the consequence that $1 = l_1 = l_{\alpha\bar{\alpha}} = l_\alpha \circ l_{\bar{\alpha}}$ and hence $I = L_1 = L_\alpha L_{\bar{\alpha}} = L_\alpha L_\alpha^T$; in particular, $L_\alpha \in SO(4)$ for $\alpha \in S^3$.

Since $l_\alpha : \mathbf{R}^4 \cong H \to H \cong \mathbf{R}^4$ is a linear map, its differential $(l_\alpha)_{*,v} : T_v\mathbf{R}^4 \to T_{\alpha v}\mathbf{R}^4$ is equal to $l_\alpha$ for every $v \in \mathbf{R}^4$. Hence, for $\alpha \in S^3$, also the restriction of $l_\alpha$ to $S^3$ has $l_\alpha$ itself as its differential: $(l_\alpha)_{*,\beta} : T_\beta S^3 \to T_{\alpha\beta}S^3$ is given by $l_\alpha$ and can be expressed by multiplication with the matrix $L_\alpha$ on vectors in $T_\beta S^3$.

1. Given a matrix $A \in SO(n)$.

   - Show that the tangent space $T_A(SO(n))$ corresponds to the set of matrices $\{Y_{n,n} | Y^T A + A^T Y = 0\}$.
   - Show that the differential $(l_A)_{*,I} : T_I SO(n) = K_n = \{X_{n,n} | X^T + X = 0\} \to T_A(SO(n))$ of left multiplication $l_A$ by $A \in SO(n)$ is given by left multiplication as well: $(l_A)_{*,I}(X) = AX$.

   (Hint: Use the "curve method" from Ch. 16.3, p. 162).

2. Let $\varphi : S^3 \to SO(4)$ denote (the injective!) Lie group homomorphism from Exercise set 1.1. Let $\alpha = a + bi + cj + dk \in S^3 \subset H$ denote a unit quaternion. Check that

   - $T_\alpha S^3 = \{[k, l, m, n] | ak + bl + cm + dn = 0\} \subset \mathbf{R}^4$;
   - $\varphi \circ l_\alpha = l_{\varphi(\alpha)} \circ \varphi$;
   - the differential at $1 \in S^3$ of left multiplication $l_\alpha : S^3 \to S^3$ satisfies $\varphi_{*,\alpha}((l_\alpha)_{*,1}[0, e, f, g]^T) = \varphi(\alpha)\varphi_{*,1}[0, e, f, g]^T \in T_{\varphi(\alpha)}SO(3)$.
   - its inverse satisfies: $\varphi_{*,1}(l_{\alpha^{-1}})_{*,\alpha}[k, l, m, n]^T = \varphi(\alpha)^T \varphi_{*,\alpha}[k, l, m, n]^T \in T_I SO(3)$.

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(Comment: Variation over all matrices $\varphi(\alpha) \in SO(4)$ corresponding to quaternions $\alpha \in S^3$ yields the left invariant vector fields determined by the tangent vectors $[0, e, f, g] \in T_{\mathbf{1}}S^3$ pushed forward to $\varphi(S^3) \subset SO(4)$.)

Exercise 1.3 and 1.4 in the first homework set describe a natural smooth map
$\psi : S^3 \to SO(3)$ given by $\psi(\alpha) = \begin{bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 2(a^2+d^2)-1 \end{bmatrix}$.
This matrix representation was derived from the formula $\psi(\alpha)(\mathbf{v}) = \varphi(\alpha)\varphi(\mathbf{v})\varphi(\alpha)^{-1}$ with $\varphi(\mathbf{v}) = \begin{bmatrix} 0 & e & -g & -f \\ -e & 0 & -f & g \\ g & f & 0 & e \\ f & -g & -e & 0 \end{bmatrix}$ for $\mathbf{v} = [e, f, g] \in \mathbf{R}^3$.

3. Show that the differential $\psi_{*,1} : T_1S^3 \to T_I(SO(3))$ of the map $\psi$ at $1 \in S^3$ is given by
$$\psi_{*,1}[0, e, f, g] = \begin{bmatrix} 0 & -2g & 2f \\ 2g & 0 & -2e \\ -2f & 2e & 0 \end{bmatrix} \in K_3 = T_I(SO(3)).$$
(Hint: Differentiate the map $t \mapsto \varphi(C(t)\alpha C(t)^{-1}),\ \alpha \in S^3$, for a curve $C(t)$ in $S^3$ with $C(0) = 1$ and $C'(0) = [0, e, f, g] \in T_1S^3$. You find the columns of the matrix $\psi_{*,1}[0, e, f, g]$ by performing these calculations for $\alpha = i, j, k$.)

4. Finally, identify the differential $\psi_{*,\alpha} : T_\alpha S^3 \to T_{\psi(\alpha)}(SO(3))$ of the map $\psi$ at $\alpha$:
   - Show that $\psi \circ l_\alpha = l_{\psi(\alpha)} \circ \psi$ and hence that $\psi_{*,\alpha} \circ (l_\alpha)_{*,1} = (l_{\psi(\alpha)})_{*,I} \circ \psi_{*,1}$.
   - Using most of the results so far, show that
   $$\psi_{*,\alpha}[k, l, m, n]^T = \psi(\alpha)\psi_{*,1}(L_\alpha^T[k, l, m, n]^T) \in T_{\psi(\alpha)}(SO(3)) \text{ for } [k, l, m, n] \in T_\alpha S^3.$$
   (We write $[k, l, m, n]^T$ to indicate that the vector is regarded as a column vector.)

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We ask you to work out solutions to the questions/exercises above, preferably in groups of two or three participants.
Please hand your solutions in, either on paper or electronically, no later than Monday, October 11.