DYNAMICAL CLASSIFICATION

March 4, 9:00-11:45
Fredrik Bajers Vej 7E, room 3-109.

Lectures.

Aims and Content. We are often interested in a comparison of the global (long term) behavior of solutions of different differential equations. We say that two systems of differential equations are equivalent if their solutions altogether (its flow) share the same fate. For example a linear system with a (complex) spiral sink and a linear system with a (real) sink share the same fate: They tend to the origin as time goes to infinity. We say that these systems are conjugate.
A precise definition of this notion in greater generality will be introduced during the lecture. We shall give examples and a criterion for dynamical systems to be conjugate.

Lecturer: Rafael Wisniewski

References: [HSD] 4.2, 8.1.

Exercises: HSD, ch. 4:

- Ex.4 on page 72;
- Ex. 5a on page 72;
- Ex. 6 on page 73.

LINEARIZATION OF NONLINEAR SYSTEMS AROUND EQUILIBRIUM POINTS

March 4, 12:30 – 15:15

Lectures.

Aims and Content. Consider a planar nonlinear system, $X' = F(X)$ with equilibrium point X_0, i.e., $F(X_0) = 0$, and Jacobian DF_{X_0} at this equilibrium point . The flow of the nonlinear system given by F is locally conjugate to the flow of the linearized system $X' = DF_{X_0}(X)$ if the linearized system is hyperbolic, i.e., if that system does not have a purely imaginary eigenvector. In that case, close to X_0 the phase planes of the non-linear and of the linearized system look alike.

In particular, if X_0 is a (spiral) sink/source for the linearized system, then the same is true for the nonlinear system from the outset. Likewise, if X_0 is a saddle point for the linearized system, then the original system has a stable curve and an unstable curve, and flow lines nearby follow these curves.

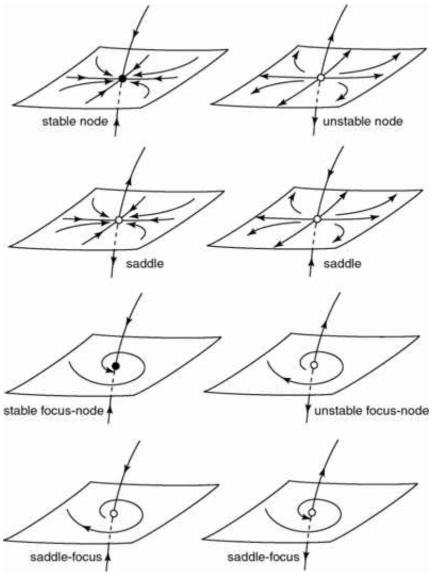
We will outline a proof for the case of a sink/source in 2D, and we will sketch a proof in the saddle point case; cf. an outline on the attached pages.

In higher dimensions, the so-called Hartman-Grobman Theorem tells us that linearization gives important information at hyperbolic equilibria in general. The original system and the linearized system are locally conjugate close to the equilbrium, and the analogues of stable/unstable curves are the stable/unstable manifolds.

FREDRIK BAJERSVEJ 7G
9220 AALBORG ØST

9940 8848
9940 8855
9940 8762

FAJSTRUP@MATH.AAU.DK
RAUSSEN@MATH.AAU.DK
RAF@ES.AAU.DK



Lecturers: Rafael Wisniewski and Martin Raussen

References:

HSD: ch. 8.1 – 8.3.
Wikipedia: Linearization
Wikipedia: Hartman-Grobman theorem

Exercises: [HSD], ch. 8, ex. 8.5. Please use one of the plot tools to illustrate some particular cases - to find the null clines, equilibrium point(s), their linearization etc.

Evaluation. Homework. As part of the evaluation of the course, you are required to work out Exploration 4.3 (pp. 71 – 72) in groups of two or three people. Please hand in a short report on your work.
Deadline for delivery: March 12 (the last day of the course)

LINEARIZATION AT EQUILIBRIUM POINTS - 2D

"Sinks are sinks".

Theorem 1. *Let $X' = F(X)$ denote a 2D-dynamical system with equilibrium point $\mathbf{x}_0$, i.e., $F(\mathbf{x}_0) = \mathbf{0}$. Assume that the linearization $DF(\mathbf{x}_0)$ – expressed by the Jacobian of F in the standard basis – has eigenvalues with negative real part. Then there is a neighbourhoood U of $\mathbf{x}_0$ such that all flow lines $\Phi(t, z)$ with initial conditions $z \in U$ converge to $\mathbf{x}_0$, i.e.,*

$$\lim_{t\to\infty} \Phi(t, z) = \mathbf{x}_0.$$

Proof. We give a proof in the case of two real negative eigenvalues: After coordinate change, we may assume that the equilibrium is in the origin O and that the differential equations are given by

$$\begin{aligned} x' &= -\lambda x + h_1(x, y) \\ y' &= -\mu y + h_2(x, y) \end{aligned}$$

with $\lambda \geq \mu > 0$ the numerical values of the eigenvalues of the linearization and remainder terms $h_i(x, y)$ satisfying the property

$$\lim_{(x,y)\to(0,0)} \frac{h_i(x, y)}{r} = 0. \tag{1}$$

As usual, $r = \sqrt{x^2 + y^2}$ denotes the first polar coordinate, i.e., the distance from the origin. Property (1) of the remainder term relies on the fact, that the functions describing the right hand side of the dynamical system are differentiable.

Our aim is to see that every flow line (solution curve) close to the origin converges to the origin for $t \to \infty$. To that aim let $(x(t), y(t))$ denote a parametrization of a flow line. We investigate the real function $r^2(t) = x^2(t) + y^2(t)$, i.e., the distance of points on the flow line from the origin (as function of the time parameter t). Let us calculate the derivative of that function with respect to t:

$$\begin{aligned} \frac{d}{dt} r^2(t) &= 2xx' + 2yy' \\ &= -2\lambda x^2 - 2\mu y^2 + 2xh_1(x, y) + 2yh_2(x, y) \\ &\leq -2\mu r^2 + 2r(h_1(x, y) + h_2(x, y)) \\ &= 2r^2(-\mu + \frac{h_1(x, y) + h_2(x, y)}{r}). \end{aligned}$$

The inequality relies on $\lambda \geq \mu$ and $x, y \leq r$. When r is small, the term in parenthesis is negative, and hence $r^2(t)$ decreases. But this is not quite enough to show that it tends to zero. To that end, consider the term

$$\frac{\frac{d}{dt} r^2(t)}{r^2(t)} \leq -2\mu + 2\frac{h_1(x, y) + h_2(x, y)}{r} \to -2\mu$$

for $t \to 0$. In particular, there is a distance $R_0 > 0$ (from O) with the property

$$|(x,y)| \leq R_0 \Rightarrow \frac{h_1(x,y) + h_2(x,y)}{r} < -\mu$$

and hence – for $|(x(t_0), y(t_0)| \leq R_0$ and $t \geq t_0$ –

$$\frac{\frac{d}{dt} r^2(t)}{r^2(t)} \leq \mu.$$

Integrating both sides we obtain for $T \geq t_0$:

$$\log r^2(T) - \log r^2(t_0) = \int_{t_0}^{T} \frac{\frac{d}{dt} r^2(t)}{r^2(t)} dt \leq \int_{t_0}^{T} -\mu dt = -\mu(T - t_0)$$

whence $r^2(T) \leq r^2(t_0)e^{-\mu(T-t_0)} \to 0$ for $T \to \infty$. □

Saddle points are saddle points.

Theorem 2. *Let $X' = F(X)$ denote a 2D-dynamical system with equilibrium point $\mathbf{x}_0$, i.e., $F(\mathbf{x}_0) = \mathbf{0}$. Assume that the linearization $DF(\mathbf{x}_0)$ – expressed by the Jacobian of F in the standard basis – has a positive and a negative real eigenvalue, i.e.,* $\det DF(\mathbf{x}_0) < 0$. *Then there exist*

- **a stable curve:** $\{\mathbf{z} | \lim_{t\to\infty} \Phi(t, \mathbf{z}) = \mathbf{x}_0\}$ *is a smooth curve through $\mathbf{x}_0$ whose tangent line at $\mathbf{x}_0$ is parallel with the eigenvector corresponding to the negative eigenvector.*
- **an unstable curve:** $\{\mathbf{z} | \lim_{t\to-\infty} \Phi(t, \mathbf{z}) = \mathbf{x}_0\}$ *is a smooth curve through $\mathbf{x}_0$ whose tangent line at $\mathbf{x}_0$ is parallel with the eigenvector corresponding to the positive eigenvector.*
- **Other flow lines:** *close to $\mathbf{x}_0$ "follow" first a stable curve and continue close to an unstable curve.*

The proof is more tricky and uses cones around the equilibrium point and the fact that two different solutions "cannot stay close to each other for all time".

Both theorems are special cases of a much more general theorem that holds in all dimensions in the neighbourhood of hyperbolic equilibrium points. This is the so-called Hartman-Grobman theorem.