Algebraic topology and Concurrency: Traces spaces and applications

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Outline

1. Directed algebraic topology
   - Motivations – mainly from Computer Science
   - Directed topology: Algebraic topology with a twist
2. Trace spaces and their organization
   - Trace spaces: definition, properties, applications
   - A categorical framework (with examples and applications)

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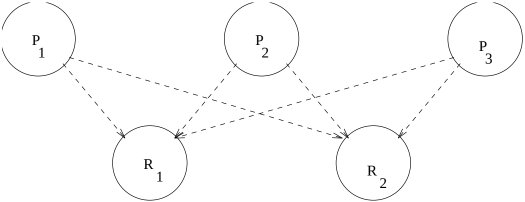
Conference: Algebraic Topological Methods in Computer Science III, July 2008, Paris

Motivation: Concurrency
Mutual exclusion

Mutual exclusion occurs, when *n* processes P_i compete for *m* resources R_j.



Only *k* processes can be served at any given time.

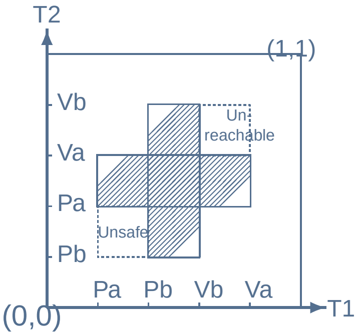
Semaphores!

Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)

Schedules in "progress graphs"

The Swiss flag example



PV-diagram from

$P_1 : P_aP_bV_bV_a$

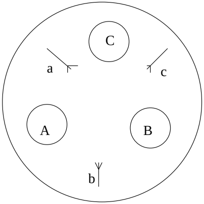
$P_2 : P_bP_aV_aV_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

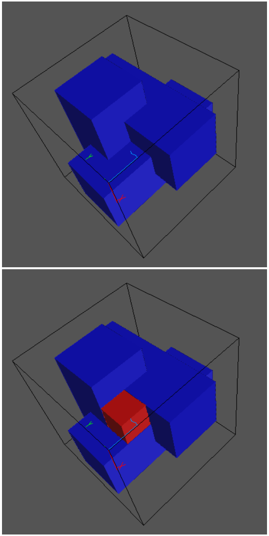
Deadlocks, unsafe and unreachable regions may occur.

Higher dimensional automata (HDA) 1

Example: Dining philosophers; dimension 3 and beyond



A=Pa.Pb.Va.Vb
B=Pb.Pc.Vb.Vc
C=Pc.Pa.Vc.Va

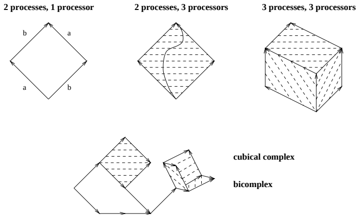


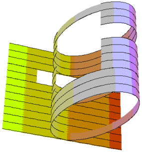
Higher dimensional complex with a forbidden region consisting of isothetic hypercubes and an unsafe region.

Higher dimensional automata (HDA) 2

seen as (geometric realizations of) pre-cubical sets

Vaughan Pratt, Rob van Glabbeek, Eric Goubault...





Squares/cubes/hypercubes are filled in iff actions on boundary are independent.

Higher dimensional automata are pre-cubical sets:

- like simplicial sets, but modelled on (hyper)cubes instead of simplices; glueing by face maps
- additionally: preferred directions – not all paths allowable.

Discrete versus continuous models

How to handle the state-space explosion problem?

Discrete models for concurrency (transition graph models) suffer a severe problem if the number of processors and/or the length of programs grows: The number of states (and the number of possible schedules) grows exponentially:
This is known as the state space explosion problem.
You need clever ways to find out which of the schedules yield equivalent results – e.g., to check for correctness – for general reasons. Then check only one per equivalence class.
Alternative: Infinite continuous models allowing for well-known equivalence relations on paths (homotopy = 1-parameter deformations) – but with an important twist!
Analogy: Continuous physics as an approximation to (discrete) quantum physics.

Concepts from algebraic topology 1

Homotopy

Top: the category of topological spaces and continuous maps.
$I = [0,1]$ the unit interval.

Definition

- A continuous map $H : X \times I \to Y$ (one-parameter deformation) is called a homotopy.
- Continuous maps $f, g : X \to Y$ are called homotopic to each other ($f \simeq g$) if there is a homotopy H with $H(x,0) = f(x), H(x,1) = g(x), x \in X$.
- $[X, Y]$ the set of homotopy classes of continuous maps from X to Y.
- A continuous map $f : X \to Y$ is called a homotopy equivalence if it has a "homotopy inverse" $g : Y \to X$ such that $g \circ f \simeq id_X, f \circ g \simeq id_Y$.

Concepts from algebraic topology 2

The fundamental group. Higher homotopy groups

Definition

- Variation: pointed continuous maps $f : (X, *) \to (Y, *)$ and pointed homotopies $H : (X \times I, * \times I) \to (Y, *)$.
- Loops in Y as the special case $X = S^1$ (unit circle).
- Fundamental group $\pi_1(Y, y) = [(S^1, *), (Y, y)]$ with product arising from concatenation and inverse from reversal.
- Higher homotopy groups $\pi_k(Y, y) = [(S^k, *), (Y, y)]$ – product from the pinch coproduct on S^k; abelian for $k > 1$.

A framework for directed topology

d-spaces, M. Grandis (03)

X a topological space. $\vec{P}(X) \subseteq X^I = \{p : I = [0,1] \to X \text{ cont.}\}$ a set of d-paths ("directed" paths $\leftrightarrow$ executions) satisfying

- { constant paths } $\subseteq \vec{P}(X)$
- $\varphi \in \vec{P}(X)(x,y), \psi \in \vec{P}(X)(y,z) \Rightarrow \varphi * \psi \in \vec{P}(X)(x,z)$
- $\varphi \in \vec{P}(X), \alpha \in I^I$ a nondecreasing reparametrization $\Rightarrow \varphi \circ \alpha \in \vec{P}(X)$

The pair $(X, \vec{P}(X))$ is called a d-space.
Observe: $\vec{P}(X)$ is in general not closed under reversal:

$$\alpha(t) = 1 - t, \varphi \in \vec{P}(X) \not\Rightarrow \varphi \circ \alpha \in \vec{P}(X)!$$

Examples:

- An HDA with directed execution paths.
- A space-time(relativity) with time-like or causal curves.

d-maps, dihomotopy

A d-map $f : X \to Y$ is a continuous map satisfying

- $f(\vec{P}(X)) \subseteq \vec{P}(Y)$.

Let $\vec{P}(I) = \{\sigma \in I^I | \sigma \text{ nondecreasing reparametrization}\}$, and $\vec{I} = (I, \vec{P}(I))$. Then

- $\vec{P}(X) =$ set of d-maps from $\vec{I}$ to X.

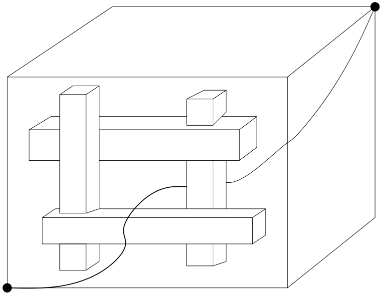
A dihomotopy $H : X \times I \to Y$ is a continuous map such that

- every H_t a d-map

i.e., a 1-parameter deformation of d-maps.

Dihomotopy is finer than homotopy with fixed endpoints

Example: Two L-shaped wedges as the forbidden region

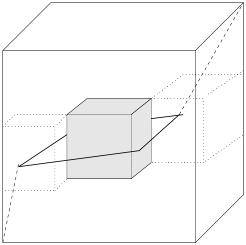


All dipaths from minimum to maximum are homotopic.
A dipath through the "hole" is not dihomotopic to a dipath on the boundary.

The twist has a price

Neither homogeneity nor cancellation nor group structure

Ordinary topology:
Path space = loop space (within each path component).
A loop space is an H-space with concatenation, inversion, cancellation.



"Birth and death" of d-homotopy classes

Directed topology:
Loops do not tell much; concatenation ok, cancellation not!
Replace group structure by category structures!

D-paths, traces and trace categories

Getting rid of reparametrizations

X a (saturated) d-space.
$\varphi, \psi \in \vec{P}(X)(x,y)$ are called reparametrization equivalent if there are $\alpha, \beta \in \vec{P}(I)$ such that $\varphi \circ \alpha = \psi \circ \beta$ ("same oriented trace").

Theorem

(Fahrenberg-R., 07): Reparametrization equivalence is an equivalence relation (transitivity!).

$\vec{T}(X)(x,y) = \vec{P}(X)(x,y)/_{\simeq}$ makes $\vec{T}(X)$ into the (topologically enriched) trace category – composition associative.
A d-map $f : X \to Y$ induces a functor $\vec{T}(f) : \vec{T}(X) \to \vec{T}(Y)$.

Two main objectives

- Investigation/calculation of the homotopy type of trace spaces $\vec{T}(X)(x,y)$ for relevant d-spaces X
- Investigation of topology change under variation of end points:

$$\vec{T}(X)(x',y) \stackrel{\sigma^*_{x'x}}{\longleftarrow} \vec{T}(X)(x,y) \stackrel{\sigma_{yy'*}}{\longrightarrow} \vec{T}(X)(x,y')$$

Categorical organization, leading to components of end points

Application: Enough to check one d-path among all paths through the same components!

Aim: Decomposition of trace spaces

Method: Investigation of concatenation maps

Let $L \subset X$ denote a (properly chosen) subspace.
Investigate the concatenation map
$c_L : \vec{T}(X)(x_0, L) \times_L \vec{T}(X)(L, x_1) \to \vec{T}(X)(x_0, x_1),\ (p_0, p_1) \mapsto p_0 * p_1$

onto? fibres? Topology of the pieces?

Generalization: $L_1, \cdots, L_k$ a sequence of (properly chosen) subspaces. Investigate the concatenation map on
$\vec{T}(X)(x_0, L_1) \times_{L_1} \cdots \times_{L_j} \vec{T}(X)(L_j, L_{j+1}) \times_{L_{j+1}} \cdots \times_{L_k} \vec{T}(X)(L_n, x_1)$.

onto? fibres? Topology of the pieces?

An important special case

All fibres contractible and locally contractible

Corollary

Let $f: X \to Y$ denote a proper surjective map between locally compact separable metric spaces. Let moreover X be locally contractible, and for each $y \in Y$, let $f^{-1}(y)$ be contractible and locally contractible. Then

1. *Y is locally contractible, and*
2. *f is a weak homotopy equivalence.*

Applications to trace spaces I

A simple case as illustration

Definition

A subset $L \subseteq X$ of a d-space X is called

achronal if all $p \in \vec{P}(L) \subset \vec{P}(X)$ are constant.

order convex if $[x_0, x_1] = \{p(t) \mid p \in \vec{P}(X)(x_0, x_1), t \in I\} \subseteq A$; in particular, $p^{-1}(A)$ is either an interval or empty for all $p \in \vec{P}(X)$;

unavoidable from $B \subset X$ to $C \subset X$ if $\vec{P}(X \setminus A)(B, C) = \emptyset$.

Theorem

Let X denote a d-space, $x_0, x_1 \in X$ and $L \subset X$ a subspace that is achronal and unavoidable from x_0 to x_1.
Then the concatenation map
$c_L : \vec{T}(X)(x_0, L) \times_L \vec{T}(X)(L, x_1) \to \vec{T}(X)(x_0, x_1), \; (p_0, p_1) \mapsto p_0 * p_1$
is a homeomorphism.

Applications to trace spaces I
continued

A pre-cubical complex X is glued together out of a set of hypercubes $\square^n$ along their boundaries – similar to pre-simplicial sets/complexes. Every hypercube defines d-paths $\vec{P}(\square^n)$. Concatenations of these gives rise to $\vec{P}(X)$.

Theorem

Let X be (the geometric realization of) a pre-cubical complex. Let $x_0, x_1 \in X$, $L \subset X$ a subcomplex that is unavoidable from x_0 to x_1.
Then[a] the concatenation map
$c_L : \vec{T}(X)(x_0, L) \times_L \vec{T}(X)(L, x_1) \to \vec{T}(X)(x_0, x_1),\ (p_0, p_1) \mapsto p_0 * p_1$
is a homotopy equivalence.

[a] add an extra technical condition

An important special case

Corollary

If $\vec{T}(X)(x_0, l)$ and $\vec{T}(X)(l, x_1)$ are contractible and locally contractible for every $l \in L \cap [x_0, x_1]$, then $\vec{T}(X)(x_0, x_1)$ is homotopy equivalent to $L \cap [x_0, x_1]$.

Remark: “Huge” trace space identified with “small” space L.

Proof.

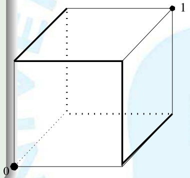
The fibre over $l \in L$ of the “mid point” map $m : \vec{T}(X)(x_0, L) \times_L \vec{T}(X)(L, x_1) \to L \cap [x_0, x_1]$ is $m^{-1}(l) = \vec{T}(X)(x_0, l) \times \vec{T}(X)(l, x_1)$. □

First examples

Example

I^n the unit cube with boundary ∂I^n.
$X = \partial I^n = \{\mathbf{x} \in I^n \mid \exists i : x_i = 0 \vee x_i = 1\} \simeq S^{n-1}$
$L = \partial_\pm I^n = \{\mathbf{x} \in I^n \mid \exists i, j : x_i = 0, x_j = 1\} \simeq S^{n-2}$.

- $\vec{T}(I^n; \mathbf{x}, \mathbf{y})$ is contractible for all $x \preceq y \in I^n$;
- $\vec{T}(\partial I^n; \mathbf{0}, \mathbf{1})$ is homotopy equivalent to S^{n-2}.



Proof.

Choose $L = \partial_\pm I^n$. □

Tool: The Vietoris-Begle mapping theorem

Stephen Smale's version for homotopy groups

What does a surjective map $p : X \to Y$ with highly connected fibres $p^{-1}(y), y \in Y$, tell about invariants of X, Y?

The Vietoris-Begle mapping theorem compares the Alexander-Spanier cohomology groups of X, Y.

Stephen Smale, *A Vietoris Mapping Theorem for Homotopy*, Proc. Amer. Math. Soc. **8** (1957), no. 3, 604 – 610:

Theorem

Let $f : X \to Y$ denote a proper surjective map between connected locally compact separable metric spaces. Let moreover X be locally n-connected, and for each $y \in Y$, let $f^{-1}(y)$ be locally $(n-1)$-connected and $(n-1)$-connected.

1. *Y is locally n-connected, and*
2. *$f_\# : \pi_r(X) \to \pi_r(Y)$ is an isomorphism for all $0 \leq r \leq n-1$ and onto for $r = n$.*

Topology of trace spaces for a pre-cubical complex X

Check conditions; interesting in its own right?

l^1 "arc length" parametrization: On each cube, arc length is the l^1-distance of end-points. Additive continuation $\rightsquigarrow$ Subspace of arc-length parametrized d-paths $\vec{P}_n(X) \subset \vec{P}(X)$.
D-homotopic paths in $\vec{P}_n(X)(x, y)$ have the same arc length!
The spaces $\vec{P}_n(X)$ and $\vec{T}(X)$ are homeomorphic,
$\vec{P}(X)$ is homotopy equivalent to both.

Theorem

X a pre-cubical set; $x, y \in X$. Then $\vec{T}(X)(x, y)$

- *is metrizable, locally contractible and locally compact.*[a]
- *has the homotopy type of a CW-complex – (using Milnor).*

[a] MR, Trace spaces in a pre-cubical complex, Aalborg preprint

Key points in the proof of Theorem

- Topological conditions ok.
- Check that path components are mapped into each other by bijection.
- Surjectivity of c_L corresponds to unavoidability.
- Order convexity ensures that every fibre $c_L^{-1}(p)$ is an interval, hence contractible.
- The weak homotopy equivalence is a homotopy equivalence since domain and codomain of c_L have the homotopy type of a CW-complex.

Applications to trace spaces II: A generalisation

Definition

pieces and separating **layers**: $x_0, x_1 \in X$. $[x_0, x_1] = \bigcup_{i \in J} X_i, J$ finite; $L_{ij} \subseteq X_i \cap X_j$

L_{**} **order convex** $p \in \vec{P}(X_i) \Rightarrow$

$$\begin{cases} p^{-1}(L_{ji}) = [0, a] & \text{for some} \quad -1 < a < 1 \quad (\emptyset \text{ if } a < 0) \\ p^{-1}(L_{ij}) = [b, 1] & \text{for some} \quad 0 < b < 2 \quad (\emptyset \text{ if } b > 1) \end{cases}$$

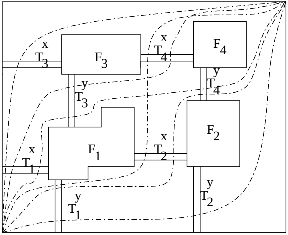
L_{**} **unavoidable**: $\vec{P}(X \setminus \bigcup_k L_{ik})(X_i \setminus \bigcup_k L_{ik}, X \setminus X_i) = \emptyset; \vec{P}(X \setminus \bigcup_k L_{ki})(X \setminus X_i, X_i \setminus \bigcup_k L_{ki}) = \emptyset$

$L_{ij} \leq L_{jk}$ if $\vec{P}(X_j)(L_{ij}, L_{jk}) \neq \emptyset$.

$S = (L_{i_k, i_{k+1}})_{0 \leq k \leq n}$ admissible from x_0 to x_1 if $x_0 \in X_{i_0}, x_1 \in X_{i_n}, L_{i_k, i_{k+1}} \leq L_{i_{k+1}, i_{k+2}}$.

$T_S(X)(x_0, x_1) = \vec{T}(X_{i_0})(x_0, L_{i_0, i_1}) \times_{L_{i_0, i_1}} \cdots \times_{L_{i_{k-1}, i_k}}$
$\vec{T}(X_{i_k})(L_{i_{k-1}, i_k}, L_{i_k, i_{k+1}}) \times_{L_{i_k, i_{k+1}}} \cdots \times_{L_{i_{n-1}, i_n}} \vec{T}(X_{i_n})(L_{i_{n-1}, i_n}, x_1)$

Decomposition of d-path spaces



Theorem

The concatenation map $c : \bigcup_S \vec{T}_S(X)(x_o, x_1) \to \vec{T}(X)(x_o, x_1)$ is

1. *a homeomorphism, if all L_{ij} are achronal.*
2. *a homotopy equivalence if all L_{ij} are subcomplexes of the pre-cubical complex X.*

Proof.

Case (2): Apply Smale's Vietoris theorem.

Surjectivity: Every d-path can be decomposed along an admissible sequence (unavoidability)

Fibres are product of intervals, contractible!

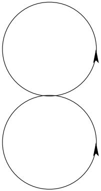
An important special case

Reachability. For a given collection of pieces and layers $\mathcal{L} = \bigcup L_{ij}$ in X that is unavoidable from x_0 to x_1, let $R^{\mathcal{L}}(X)(x_0, x_1) = \{(x_{i_0j_0}, \dots, x_{i_nj_n}) \in L_{i_0j_0} \times \dots \times L_{i_nj_n} \mid \vec{P}(X_{i_k})(x_{i_kj_k}, x_{i_{k+1}j_{k+1}}) \neq \emptyset, n \geq 0\}$. denote the space of mutually reachable points in the given layers.

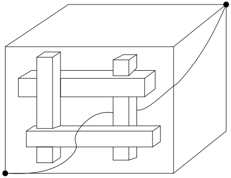
Corollary

If, moreover, all path spaces $\vec{T}(X_k)(x_i, x_j), x_i \in L_{ki}, x_j \in L_{ij}$ are contractible and locally contractible (resp. highly connected), then $\vec{T}(X)(x_0, x_1)$ is homotopy equivalent to $R^{\mathcal{L}}(X)(x_0, x_1)$ (resp. iso on a range of homotopy groups)

Examples

1 

A wedge of two directed circles $X = \vec{S}^1 \vee_{x_0} \vec{S}^1$:
$\vec{T}(X)(x_0, x_0) \simeq \{1,2\}^*$.
(Choose $L_i = \{x_i\}, i = 1,2$ with $x_i \neq x_0$ on the two branches).

2 

Y = cube with two wedges deleted:
$\vec{T}(Y)(\mathbf{0}, \mathbf{1}) \simeq * \sqcup (S^1 \vee S^1)$.
(L_i two vertical cuts through the wedges; product is homotopy equivalent to torus; reachability $\rightsquigarrow$ two components, one of which is contractible, the other a thickening of $S^1 \vee S^1 \subset S^1 \times S^1$.)

Inductive Calculations concerning $T(X)(x_0, x_1)$

In many cases, one can establish the connectivity of $T(X)(x_0, x_1)$ by studying the spaces of mutually reachable pairs $\{(x_{ki}, x_{i_j}) \in L_{ki} \times L_{i_j} \mid x_{ki} \preceq x_{ij}\}$.

Theorem

If all spaces of mutually reachable pairs are k-connected, then $T(X)(x_0, x_1)$ is k-connected.

Piecewise linear traces

Let X denote the geometric realization of a finite pre-cubical complex (□-set) M, i.e., $X = \coprod (M_n \times \vec{I}^n)_{/\simeq}$.
X consists of "cells" e_α homeomorphic to I^{n_α}. A cell is called maximal if it is not in the image of a boundary map $\partial^\pm$.
The d-path structure $\vec{P}(X)$ is inherited from the $\vec{P}(\vec{I}^n)$ by "pasting".

Definition

$p \in \vec{P}(X)$ is called PL if: $p(t) \in e_\alpha$ for $t \in J \subseteq I \Rightarrow p_{|J}$ linear[a].
$\vec{P}_{PL}(X), \vec{T}_{PL}(X)$: subspaces of linear d-paths and traces.

[a] and close-up on boundaries

Theorem

For all $x_0, x_1 \in X$, the inclusion $\vec{T}_{PL}(X)(x_0, x_1) \hookrightarrow \vec{T}(X)(x_0, x_1)$ is a homotopy equivalence.

A prodsimplicial structure on $\vec{T}_{PL}(X)$

Cube paths and the PL-paths in each of them

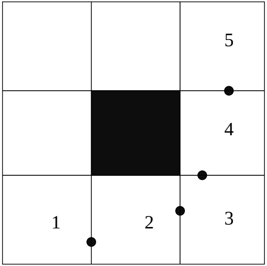
Definition

A maximal cube path in a pre-cubical set is a sequence $(e_{\alpha_1}, \dots, e_{\alpha_k})$ of maximal cells such that $\partial^+ e_{\alpha_i} \cap \partial^- e_{\alpha_{i+1}} \neq \emptyset$.

The *PL*-traces within a given maximal cube path $(e_{\alpha_1}, \dots, e_{\alpha_k})$ correspond to sequences in $\{(y_1, \dots, y_{k-1}) \in \prod_{i=1}^{k-1}(\partial^+ e_{\alpha_i} \cap \partial^- e_{\alpha_{i+1}}) \subset X^k \mid \vec{P}(e_{\alpha_i})(y_{i-1}, y_i) \neq \emptyset, 1 < i < k\}$.
This set carries a natural structure as a product of simplices $\prod \Delta^{j_k}$.
Subsimplices and their products: Some coordinates of d-paths are minimal, maximal or fixed within one or several cells.
The space $\vec{T}_{PL}(X)$ of all PL-d-paths in X is the result of pasting of these products of simplices. It carries thus the structure of a prodsimplicial complex $\rightsquigarrow$ possibilities for inductive calculations.

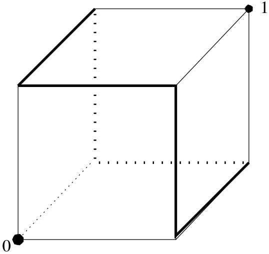
Simple examples

1



Two maximal cube paths from **0** to **1**, each of them contributing $\Delta^2 \times \Delta^2$. Empty intersection.
$\vec{T}_{PL}(X)(\mathbf{0},\mathbf{1}) \simeq (\Delta^2 \times \Delta^2) \sqcup (\Delta^2 \times \Delta^2)$.

2



$X = \partial \vec{I}^n$. Maximal cube paths from **0** to **1** have length 2. Every PL-d-path is determined by an element of $\partial_{\pm} \vec{I}^n \simeq S^{n-2}$.

Future work

on the algebraic topology of trace spaces

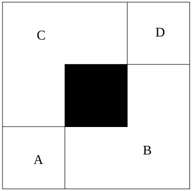
- Is there an automatic way to place consecutive “diagonal cut” layers in complexes corresponding to PV-programs that allow to compare path spaces to subspaces of the products of these layers?
- PL-d-paths come in “rounds” corresponding to the sums of dimensions of the cells they enter. This gives hope for inductive calculations (as in the work of Herlihy, Rajsbaum and others in distributed computing).
- Explore the combinatorial alg. topology of trace spaces
  - with fixed end points and
  - what happens under variations of end points.
- Make this analysis useful for the determination of components (extend the work of Fajstrup, Goubault, Haucourt, MR)
- Geodesics? instead of PL-d-paths?

Categorical organization

First tool: The fundamental category

$\vec{\pi}_1(X)$ of a d-space X [Grandis:03, FGHR:04]:

- Objects: points in X
- Morphisms: d- or dihomotopy classes of d-paths in X
- Composition: from concatenation of d-paths



Property: van Kampen theorem (M. Grandis)
Drawbacks: Infinitely many objects. Calculations?
Question: How much does $\vec{\pi}_1(X)(x, y)$ depend on (x, y)?
Remedy: Localization, component category. [FGHR:04, GH:06]
Problem: This "compression" works only for loopfree categories

Preorder categories

Getting organised with indexing categories

A d-space structure on X induces the preorder $\preceq$:

$$x \preceq y \Leftrightarrow \vec{T}(X)(x,y) \neq \emptyset$$

and an indexing preorder category $\vec{D}(X)$ with

- Objects: (end point) pairs $(x,y), x \preceq y$
- Morphisms: $\vec{D}(X)((x,y),(x',y')) := \vec{T}(X)(x',x) \times \vec{T}(X)(y,y')$:

$$x' \rightrightarrows x \xrightarrow{\preceq} y \rightrightarrows y'$$

- Composition: by pairwise contra-, resp. covariant concatenation.

A d-map $f : X \to Y$ induces a functor $\vec{D}(f) : \vec{D}(X) \to \vec{D}(Y)$.

The trace space functor

Preorder categories organise the trace spaces

The preorder category organises X via the
trace space functor $\vec{T}^X : \vec{D}(X) \to Top$

- $\vec{T}^X(x, y) := \vec{T}(X)(x, y)$
- $\vec{T}^X(\sigma_x, \sigma_y)$:

$$\begin{array}{ccc} \vec{T}(X)(x,y) & \longrightarrow & \vec{T}(X)(x',y') \\ [\sigma] & \longmapsto & [\sigma_x * \sigma * \sigma_y] \end{array}$$

Homotopical variant $\vec{D}_\pi(X)$ with morphisms
$\vec{D}_\pi(X)((x, y), (x', y')) := \vec{\pi}_1(X)(x', x) \times \vec{\pi}_1(X)(y, y')$
and trace space functor $\vec{T}^X_\pi : \vec{D}_\pi(X) \to Ho - Top$ (with homotopy classes as morphisms).

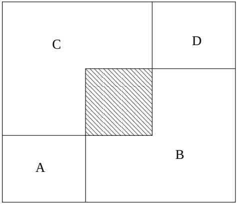
Sensitivity with respect to variations of end points

Questions from a persistence point of view

- How much does (the homotopy type of) $\vec{T}^X(x,y)$ depend on (small) changes of x,y?
- Which concatenation maps $\vec{T}^X(\sigma_x,\sigma_y) : \vec{T}^X(x,y) \to \vec{T}^X(x',y'),\ [\sigma] \mapsto [\sigma_x * \sigma * \sigma_y]$ are homotopy equivalences, induce isos on homotopy, homology groups etc.?
- The persistence point of view: Homology classes etc. are born (at certain branchings/mergings) and may die (analogous to the framework of G. Carlsson etal.)
- Are there "components" with (homotopically/homologically) stable dipath spaces (between them)? Are there borders ("walls") at which changes occur?

Examples of component categories

Standard example



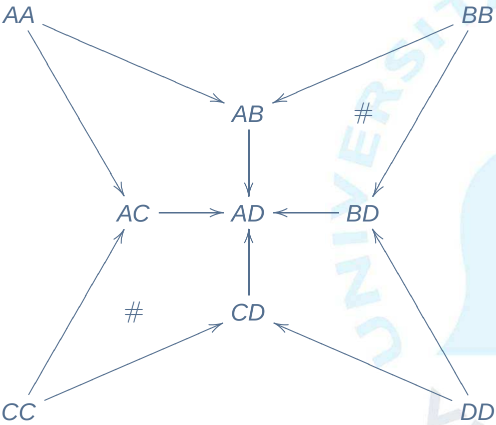


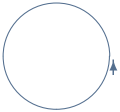
Figure: Standard example ↑ and component category →

Components A, B, C, D – or rather $AA, AB, AC, AD, BB, BD, CC, CD, DD$.

Examples of component categories

Oriented circle – with loops!

$X \quad = \quad \vec{S}^1$



oriented circle

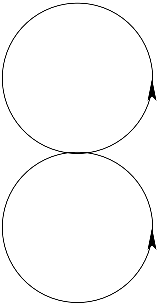
$$\mathcal{C} : \Delta \underset{b}{\overset{a}{\rightleftarrows}} \bar{\Delta}$$

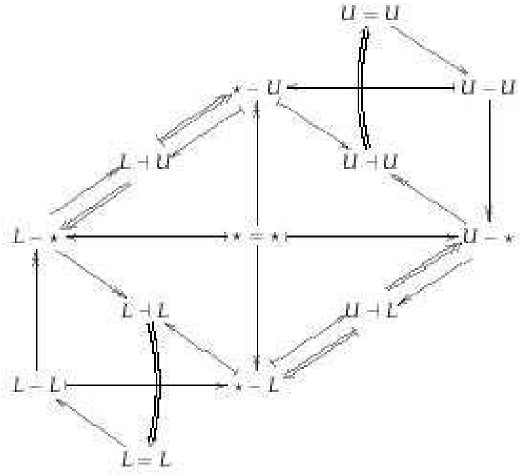
Δ the diagonal, $\bar{\Delta}$ its complement. $\mathcal{C}$ is the free category generated by a, b.

- Remark that the components are no longer products!
- It is essential in order to get a discrete component category to use an indexing category taking care of pairs (source, target).

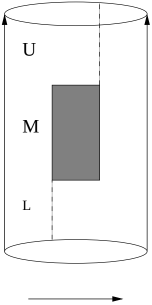
The component category of a wedge of two oriented circles

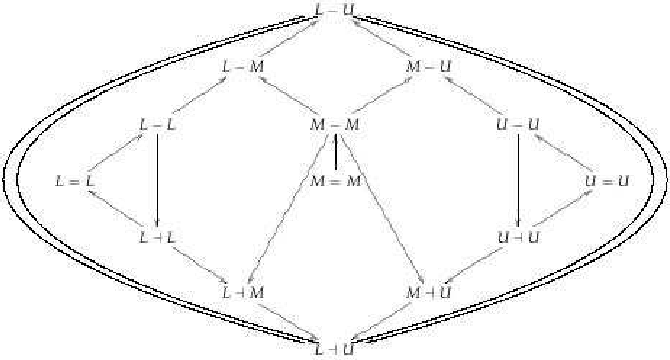
$X = \vec{S}^1 \vee \vec{S}^1$





The component category of an oriented cylinder with a deleted rectangle





Concluding remarks

- Component categories contain the essential information given by (algebraic topological invariants of) d-path spaces
- Compression via component categories is an antidote to the state space explosion problem
- Some of the ideas (for the fundamental category) are implemented and have been tested for huge industrial software from EDF (Éric Goubault & Co., CEA)
- Much more theoretical and practical work remains to be done!

Thanks for your attention!
Questions? Comments?