Concurrency and directed algebraic topology

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Algebraic Topology and Invariant Theory
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Outline

1. Motivations, mainly from Concurrency Theory
2. Directed topology: algebraic topology with a twist
3. A categorical framework (with examples)
4. “Compression” of d-topological categories: generalized congruences via homotopy flows

Main Collaborators:

- Lisbeth Fajstrup (Aalborg), Éric Goubault, Emmanuel Haucourt (CEA, France)

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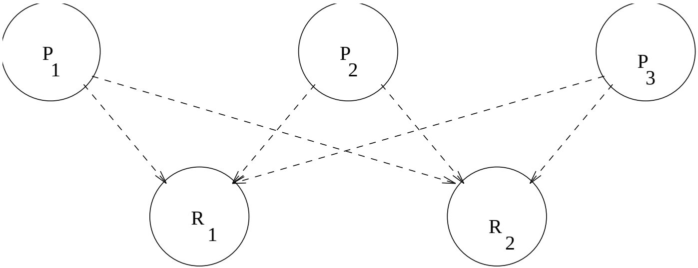
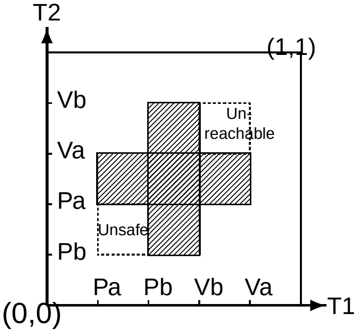
Mutual exclusion occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.
Semaphores!
Semantics: A processor has to lock a resource and relinquish the lock later on!
Description/abstraction $P_i : \dots PR_j \dots VR_j \dots$ (Dijkstra)

Schedules in "progress graphs"

The Swiss flag example



PV-diagram from

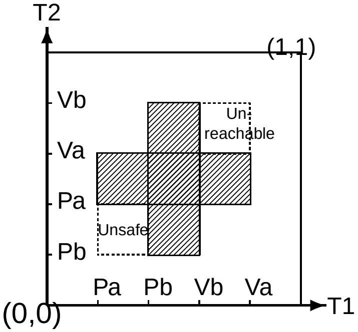
$P_1 : P_a P_b V_b V_a$

$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).

Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

Deadlocks, unsafe and unreachable regions may occur.



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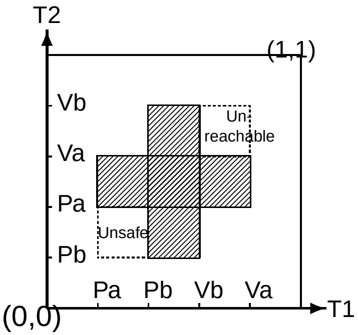

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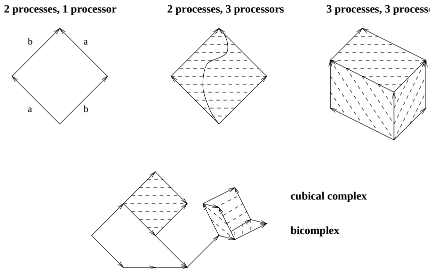
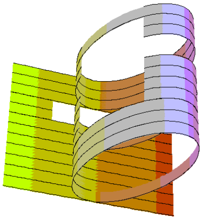
Higher dimensional automata

seen as (geometric realizations of) cubical sets

Vaughan Pratt, Rob van Glabbeek, Eric Goubault...





Squares/cubes/hypercubes are filled in iff actions on boundary are independent.

Higher dimensional automata are (pre)-cubical sets:

- like simplicial sets, but modelled on (hyper)cubes instead of simplices; glueing by face maps
- additionally: preferred directions – not all paths allowable.

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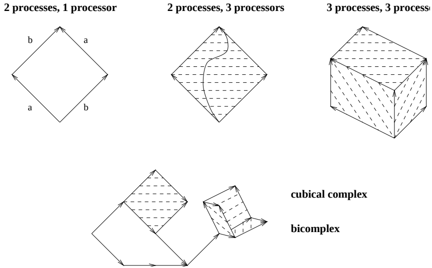

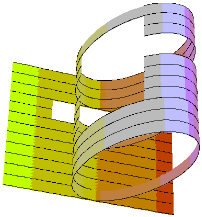

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The state space explosion problem for discrete models for concurrency (transition graph models): The number of states (and the number of possible schedules) grows exponentially in the number of processors and/or the length of programs.

Need clever ways to find out which of the schedules yield equivalent results for general reasons – e.g., to check for correctness.

Alternative: Infinite continuous models allowing for well-known equivalence relations on paths (homotopy = 1-parameter deformations) – but with an important twist!

Analogy: Continuous physics as an approximation to (discrete) quantum physics.

Discrete versus continuous models

How to handle the state-space explosion problem?

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X a topological space. $\vec{P}(X) \subseteq X^I = \{p : I = [0,1] \to X \text{ cont.}\}$ a space of d-paths (CO-topology; "directed" paths $\leftrightarrow$ executions) satisfying

- $\{$ constant paths $\} \subseteq \vec{P}(X)$
- $\varphi \in \vec{P}(X)(x,y), \psi \in \vec{P}(X)(y,z) \Rightarrow \varphi * \psi \in \vec{P}(X)(x,z)$
- $\varphi \in \vec{P}(X), \alpha \in I^I$ a nondecreasing reparametrization $\Rightarrow \varphi \circ \alpha \in \vec{P}(X)$

The pair $(X, \vec{P}(X))$ is called a d-space.

Observe: $\vec{P}(X)$ is in general not closed under reversal:

$$\alpha(t) = 1 - t,\ \varphi \in \vec{P}(X) \not\Rightarrow \varphi \circ \alpha \in \vec{P}(X)!$$

Examples:

- An HDA with directed execution paths.
- A space-time(relativity) with time-like or causal curves.

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D-maps, Dihomotopy, d-homotopy

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special case: $\vec{P}(I) = \{\sigma \in I^I | \sigma$ nondecreasing reparametrization$\}, \vec{I} = (I, \vec{P}(I))$.
Then $\vec{P}(X) =$ space of d-maps from $\vec{I}$ to X.

- Dihomotopy $H : X \times I \to Y$, every H_t a d-map
- elementary d-homotopy = d-map $H : X \times \vec{I} \to Y$ – $H_0 = f \xrightarrow{H} g = H_1$
- d-homotopy: symmetric and transitive closure ("zig-zag")

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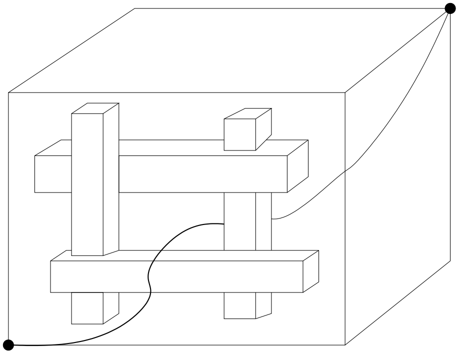
Dihomotopy is finer than homotopy with fixed endpoints

Example: Two wedges in the forbidden region

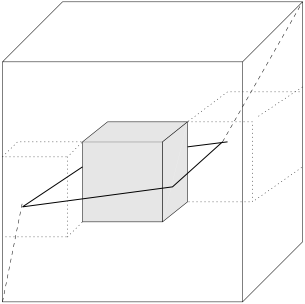


All dipaths from minimum to maximum are homotopic.
A dipath through the “hole” is not dihomotopic to a dipath on the boundary.

The twist has a price

Neither homogeneity nor cancellation nor group structure

Ordinary topology: Path space = loop space (within each path component).
A loop space is an H-space with concatenation, inversion, cancellation.



"Birth and death" of d-homotopy classes

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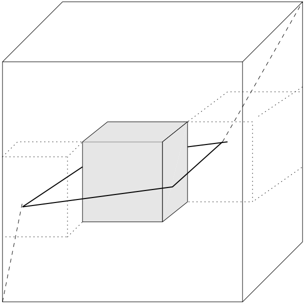

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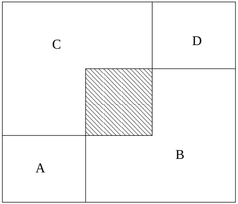
A first remedy: the fundamental category

$\vec{\pi}_1(X)$ of a d-space X [Grandis:03, FGHR:04]:

- Objects: points in X
- Morphisms: d- or dihomotopy classes of d-paths in X
- Composition: from concatenation of d-paths



Property: van Kampen theorem (M. Grandis)
Drawbacks: Infinitely many objects. Calculations?
Question: How much does $\vec{\pi}_1(X)(x, y)$ depend on (x, y)?
Remedy: Localization, component category. [FGHR:04, GH:06]
Problem: This "compression" works only for loopfree categories (d-spaces)

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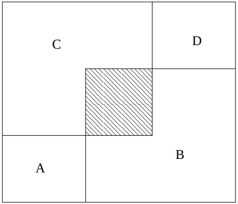

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Outline

- Spaces of d-paths and of traces
- Better bookkeeping: A zoo of categories and functors associated to a directed space – with a lot more animals than just the fundamental category
- Localization of categories with respect to (algebraic topological) functors via automorphic homotopy flows $\rightsquigarrow$ ”components”, compressing information, making calculations feasible.
- Directed homotopy equivalences – more than just the obvious generalization of the classical notion
  Definition? Automorphic homotopy flows! Properties?

X a (saturated) d-space.
$\varphi, \psi \in \vec{P}(X)(x,y)$ are called reparametrization equivalent if there are $\alpha, \beta \in \vec{P}(I)$ such that $\varphi \circ \alpha = \psi \circ \beta$ ("same oriented trace").
(Fahrenberg-R., 07): Reparametrization equivalence is an equivalence relation (transitivity).
$\vec{T}(X)(x,y) = \vec{P}(X)(x,y)/_{\simeq}$ makes $\vec{T}(X)$ into the (topologically enriched) trace category – composition associative.
A d-map $f : X \to Y$ induces a functor $\vec{T}(f) : \vec{T}(X) \to \vec{T}(Y)$.
On a pre-cubical set X, define the space of normalized d-paths $\vec{P}_n(X)$ with "arc length" parametrization (wrt. l^1).
The spaces $\vec{P}_n(X), \vec{P}(X)$ and $\vec{T}(X)$ are all homotopy equivalent.
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Theorem

X a pre-cubical set; $x, y \in X$. Then $\vec{T}(X)(x, y)$ is

- *metrizable and locally contractible.*

Hope: Applications of Vietoris-Begle theorem for "inductive calculations".

Examples

I^n the unit cube, ∂I^n its boundary.

- $\vec{T}(I^n; \mathbf{x}, \mathbf{y})$ is contractible for all $x \preceq y \in I^n$;
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Preorder categories

Getting organised with indexing categories

A d-space structure on X induces the preorder $\preceq$:

$$x \preceq y \Leftrightarrow \vec{T}(X)(x,y) \neq \emptyset$$

and an indexing preorder category $\vec{D}(X)$ with

- Objects: (end point) pairs $(x,y), x \preceq y$
- Morphisms:
  $\vec{D}(X)((x,y),(x',y')) := \vec{T}(X)(x',x) \times \vec{T}(X)(y,y')$:

$$\begin{tikzcd} x' \arrow[r, bend left] \arrow[r, bend right] & x \arrow[r, "\preceq"] & y \arrow[r, bend left] \arrow[r, bend right] & y' \end{tikzcd}$$

- Composition: by pairwise contra-, resp. covariant concatenation.

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The trace space functor

Preorder categories organise the trace spaces

The preorder category organises X via the trace space functor $\vec{T}^X : \vec{D}(X) \to Top$

- $\vec{T}^X(x,y) := \vec{T}(X)(x,y)$
- $\vec{T}^X(\sigma_x, \sigma_y)$:

$$\begin{array}{rcl} \vec{T}(X)(x,y) & \longrightarrow & \vec{T}(X)(x',y') \\ [\sigma] & \longmapsto & [\sigma_x * \sigma * \sigma_y] \end{array}$$

Homotopical variant $\vec{D}_\pi(X)$ with morphisms $\vec{D}_\pi(X)((x,y),(x',y')) := \vec{\pi}_1(X)(x',x) \times \vec{\pi}_1(X)(y,y')$ and trace space functor $\vec{T}^X_\pi : \vec{D}_\pi(X) \to Ho-Top$ (with homotopy classes as morphisms).

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D-homology $\vec{H}_*$

For every d-space X, there are homology functors

$$\vec{H}_{*+1}(X) = H_* \circ \vec{T}^X_\pi : \vec{D}_\pi(X) \to Ab, \ (x, y) \mapsto H_*(\vec{T}(X)(x, y))$$

capturing homology of all relevant d-path spaces in X
and the effects of the concatenation structure maps.
A d-map $f : X \to Y$ induces a natural transformation $\vec{H}_{*+1}(f)$ from $\vec{H}_{*+1}(X)$ to $\vec{H}_{*+1}(Y)$.
Properties? Calculations? Not much known in general.
A master's student has studied this topic for X a cubical complex (its geometric realization) by constructing a cubical model for d-path spaces.
Similarly for other algebraic topological functors; a bit more complicated for homotopy groups: base points!

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Similarly for other algebraic topological functors; a bit more complicated for homotopy groups: base points!

D-homology $\vec{H}_*$

For every d-space X, there are homology functors

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Sensitivity with respect to variations of end points

Questions from a persistence point of view

- How much does (the homotopy type of) $\vec{T}^X(x,y)$ depend on (small) changes of x,y?
- Which concatenation maps $\vec{T}^X(\sigma_x,\sigma_y):\vec{T}^X(x,y)\to\vec{T}^X(x',y'),\ [\sigma]\mapsto[\sigma_x*\sigma*\sigma_y]$ are homotopy equivalences, induce isos on homotopy, homology groups etc.?
- The persistence point of view: Homology classes etc. are born (at certain branchings/mergings) and may die (analogous to the framework of G. Carlsson etal.)
- Are there “components” with (homotopically/homologically) stable dipath spaces (between them)? Are there borders (“walls”) at which changes occur?

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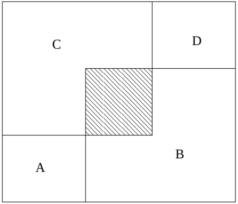
Examples of component categories

Example 1: No nontrivial d-loops



$$
\begin{tikzcd}
AA \arrow[drr] \arrow[ddr] & & & & BB \arrow[dll] \arrow[ddl] \\
& & AB \arrow[d] & \# & \\
& AC \arrow[r] & AD & BD \arrow[l] & \\
& \# & CD \arrow[u] & & \\
CC \arrow[uur] \arrow[urr] & & & & DD \arrow[ull] \arrow[uul]
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Figure: Deleted square with component category

Components A, B, C, D – or rather $AA, AB, AC, AD, BB, BD, CC, CD, DD \subseteq X \times X$.
$\#$: diagram commutes.

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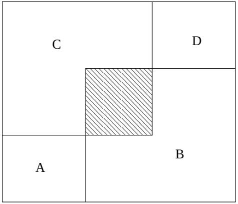

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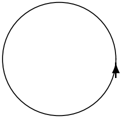

oriented circle

$\vec{T}(\vec{S}^1)(x, y) \simeq \mathbf{N}_0$.

$$\mathcal{C} : \begin{tikzcd} \Delta \arrow[r, bend left, "a"] & \bar{\Delta} \arrow[l, bend left, "b"] \end{tikzcd}$$

Δ the diagonal, $\bar{\Delta}$ its complement. $\mathcal{C}$ is the free category generated by a, b.

- Remark that the components are no longer products!
- In order to get a discrete component category, it is essential to use an indexing category taking care of pairs (source, target).

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Example 2: Oriented circle

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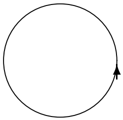

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Tool: Homotopy flows

in particular: Automorphic homotopy flows

A d-map $H : X \times \vec{I} \to X$ is called a (f/p) homotopy flow if

future $H_0 = id_X \xrightarrow{H} f = H_1$

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H_t is not a homeomorphism, in general; the flow is irreversible.

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automorphic if $\vec{T}(H_t) : \vec{T}(X)(x, y) \to \vec{T}(X)(H_t x, H_t y)$ is a homotopy equivalence for all $x \preceq y, t \in I$.

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Compression: Generalized congruences and quotient categories

Bednarczyk, Borzyszkowski, Pawlowski, TAC 1999

How to identify morphisms in a category between different objects in an organised manner?

Start with an equivalence relation $\simeq$ on the objects.

A generalized congruence is an equivalence relation on non-empty sequences $\varphi = (f_1 \dots f_n)$ of morphisms with $cod(f_i) \simeq dom(f_{i+1})$ ($\simeq$-paths) satisfying

1. $\varphi \simeq \psi \Rightarrow dom(\varphi) \simeq dom(\psi), codom(\varphi) \simeq codom(\psi)$
2. $a \simeq b \Rightarrow id_a \simeq id_b$
3. $\varphi_1 \simeq \psi_1, \varphi_2 \simeq \psi_2, cod(\varphi_1) \simeq dom(\varphi_2) \Rightarrow \varphi_2\varphi_1 \simeq \psi_2\psi_1$
4. $cod(f) = dom(g) \Rightarrow f \circ g \simeq (fg)$

Quotient category $\mathcal{C}/\simeq$: Equivalence classes of objects and of $\simeq$-paths; composition: $[\varphi] \circ [\psi] = [\varphi\psi]$.

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Automorphic homotopy flows give rise to generalized congruences

Let X be a d-space and $Aut_{\pm}(X)$ the monoid of all (future/past) automorphisms.
"Flow lines" are used to identify objects (pairs of points) and morphisms (classes of d-paths) in an organized manner.
$Aut_{\pm}(X)$ gives rise to a generalized congruence on the (homotopy) preorder category $\vec{D}_{\pi}(X)$ as the symmetric and transitive congruence closure of:

- $f_+ : (x,y) \overset{\simeq}{\leftrightarrow} (x',y') : f_-, \qquad f_\pm \in Aut_\pm(X)$
-
$$(x,y) \xrightarrow{(\sigma_1,\sigma_2)} (u,v) \simeq (x',y') \xrightarrow{(\tau_1,\tau_2)} (u',v'),$$
$f_+ : (x,y,u,v) \leftrightarrow (x',y',u',v') : f_-, \qquad f_\pm \in Aut_\pm(X)$, and
$$\begin{tikzcd} \vec{T}(X)(x',y') \arrow[r, "(\tau_1,\tau_2)"] \arrow[d, shift left, "\vec{T}(f_-)"] & \vec{T}(X)(u',v') \arrow[d, shift left, "\vec{T}(f_-)"] \\ \vec{T}(X)(x,y) \arrow[r, "(\sigma_1,\sigma_2)"] \arrow[u, shift left, "\vec{T}(f_+)"] & \vec{T}(X)(u,v) \arrow[u, shift left, "\vec{T}(f_+)"] \end{tikzcd}$$
commutes (up to ...).
- $(x,y) \xrightarrow{(c_x,H_y)} (x,fy) \simeq (fx,fy) \xrightarrow{(H_x,c_{fy})} (x,fy), \; H : id_X \to f.$
Likewise for $H : g \to id_X$.

The component category $\vec{D}_\pi(X)/\simeq$ identifies pairs of points on the same “homotopy flow line” and (chains of) morphisms.

Congruences and component categories

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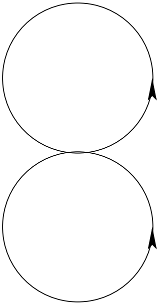
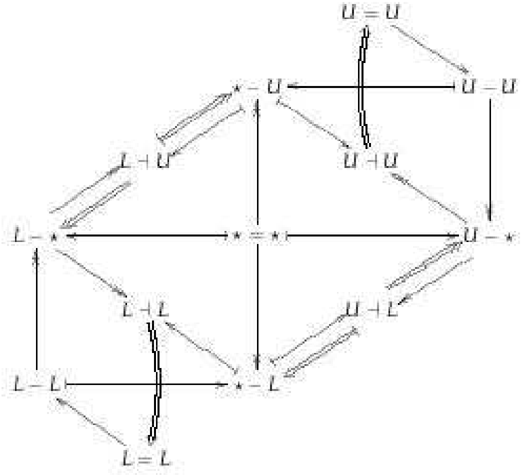
Examples of component categories

Example 3: The component category of a wedge of two oriented circles

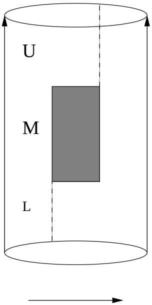
$$X = \vec{S}^1 \vee \vec{S}^1$$
$$\vec{T}(X)(x,y) \qquad \simeq$$
$$\mathbf{N}_0 * \mathbf{N}_0$$

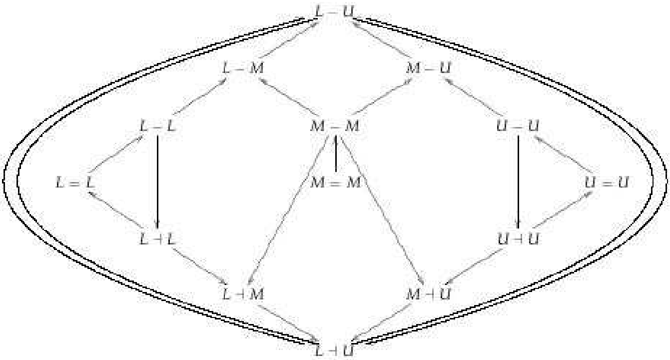




Examples of component categories

Example 4: The component category of an oriented cylinder with a deleted rectangle





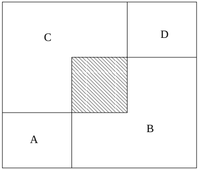
Dihomotopy equivalence – a naive definition

Definition

A d-map $f : X \to Y$ is a dihomotopy equivalence if there exists a d-map $g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

But this does not imply an obvious property wanted for:
A dihomotopy equivalence $f : X \to Y$ should induce (ordinary) homotopy equivalences

$$\vec{T}(f) : \vec{T}(X)(x,y) \to \vec{T}(Y)(fx,fy)!$$



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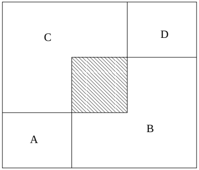

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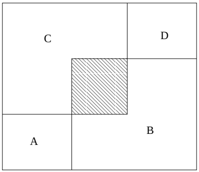

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Dihomotopy equivalences

using automorphic homotopy flows

Definition

A d-map $f: X \to Y$ is called a future dihomotopy equivalence if there are maps $f_+: X \to Y, g_+: Y \to X$ with $f \to f_+$ and automorphic homotopy flows $id_X \to g_+ \circ f_+, id_Y \to f_+ \circ g_+$.

Property of dihomotopy class!

likewise: past dihomotopy equivalence $f_- \to f, g_- \to g$

dihomotopy equivalence = both future and past dhe

(g_-, g_+ are then d-homotopic).

Theorem

A (future/past) d-homotopy equivalence $f: X \to Y$ induces homotopy equivalences

$$\vec{T}(f)(x,y): \vec{T}(X)(x,y) \to \vec{T}(Y)(fx,fy) \text{ for all } x \preceq y.$$

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Thanks!

The end!

Thanks to

Larry Smith for the invitation!

you all for listening to this talk!