Concurrency and directed algebraic topology

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Outline

1. Motivations, mainly from Concurrency Theory
2. Directed topology: algebraic topology with a twist
3. A categorical framework (with examples)
4. “Compression” of ditopological categories: generalized congruences via homotopy flows

Main Collaborators:

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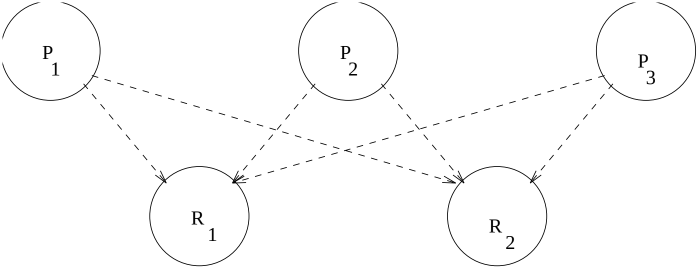


Only k processes can be served at any given time.

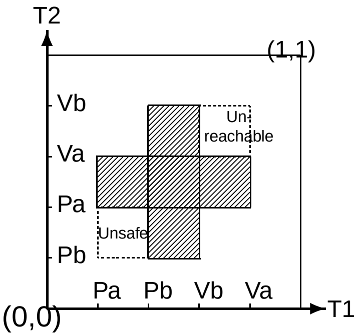
Semaphores!

Semantics: A processor has to lock a resource and relinquish the lock later on!

Description/abstraction $P_i : \dots PR_j \dots VR_j \dots$ (Dijkstra)

Schedules in "progress graphs"

The Swiss flag example



PV-diagram from

$P_1 : P_aP_bV_bV_a$

$P_2 : P_bP_aV_aV_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).

Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

Deadlocks, unsafe and unreachable regions may occur.

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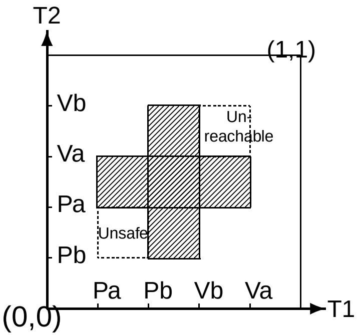

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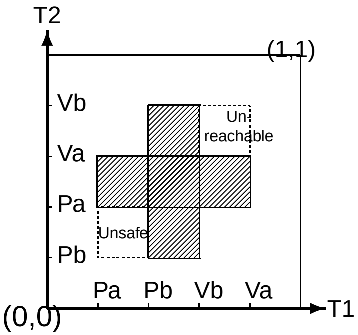

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Higher dimensional automata
seen as (geometric realizations of) cubical sets

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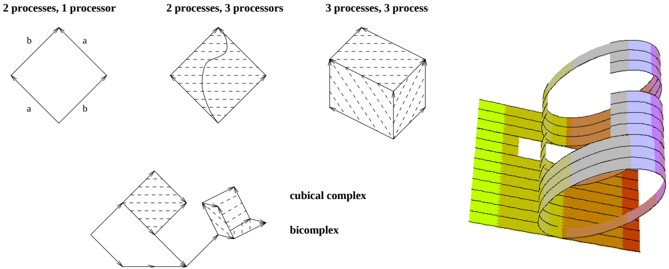

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Higher dimensional automata are cubical sets:

- like simplicial sets, but modelled on (hyper)cubes instead of simplices; glueing by face maps (and degeneracies)
- additionally: preferred directions – not all paths allowable.

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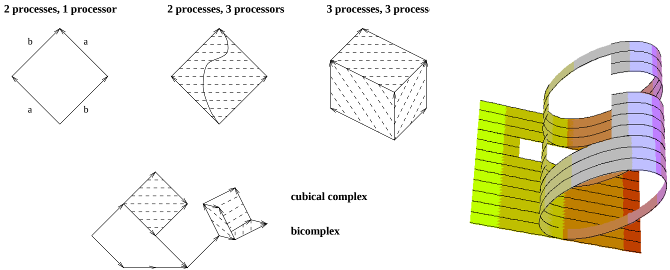

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Discrete versus continuous models

How to handle the state-space explosion problem?

The state space explosion problem for discrete models for concurrency (transition graph models): The number of states (and the number of possible schedules) grows exponentially in the number of processors and/or the length of programs.

Need clever ways to find out which of the schedules yield equivalent results for general reasons – e.g., to check for correctness.

Alternative: Infinite continuous models allowing for well-known equivalence relations on paths (homotopy = 1-parameter deformations) – but with an important twist!

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X a topological space. $\vec{P}(X) \subseteq X^I = \{p : I = [0,1] \to X \text{ cont.}\}$ a set of d-paths ("directed" paths $\leftrightarrow$ executions) satisfying

- { constant paths } $\subseteq \vec{P}(X)$
- $\varphi \in \vec{P}(X)(x,y), \psi \in \vec{P}(X)(y,z) \Rightarrow \varphi * \psi \in \vec{P}(X)(x,z)$
- $\varphi \in \vec{P}(X), \alpha \in I^I$ a nondecreasing reparametrization $\Rightarrow \varphi \circ \alpha \in \vec{P}(X)$

The pair $(X, \vec{P}(X))$ is called a d-space.

Observe: $\vec{P}(X)$ is in general not closed under reversal:

$$\alpha(t) = 1 - t,\ \varphi \in \vec{P}(X) \not\Rightarrow \varphi \circ \alpha \in \vec{P}(X)!$$

Examples:

- An HDA with directed execution paths.
- A space-time(relativity) with time-like or causal curves.

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- Dihomotopy $H : X \times I \to Y$, every H_t a d-map
- elementary d-homotopy = d-map $H : X \times \vec{I} \to Y$ – $H_0 = f \xrightarrow{H} g = H_1$
- d-homotopy: symmetric and transitive closure ("zig-zag")

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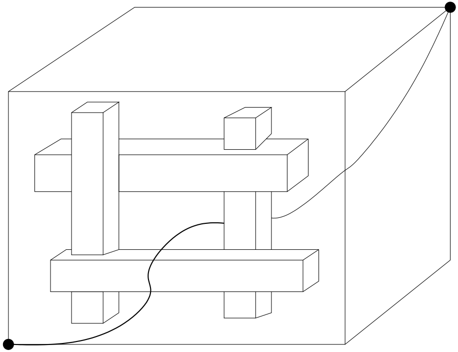
Dihomotopy is finer than homotopy with fixed endpoints

Example: Two wedges in the forbidden region

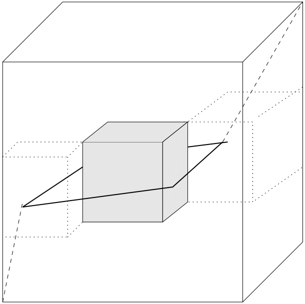


All dipaths from minimum to maximum are homotopic.
A dipath through the "hole" is not dihomotopic to a dipath on the boundary.

The twist has a price

Neither homogeneity nor cancellation nor group structure

Ordinary topology: Path space = loop space (within each path component).
A loop space is an H-space with concatenation, inversion, cancellation.



"Birth and death" of dihomotopy classes

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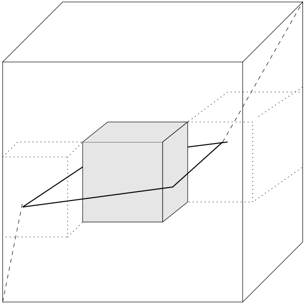

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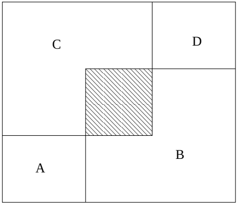
A first remedy: the fundamental category

$\vec{\pi}_1(X)$ of a d-space X [Grandis:03, FGHR:04]:

- Objects: points in X
- Morphisms: d- or dihomotopy classes of d-paths in X
- Composition: from concatenation of d-paths



Property: van Kampen theorem (M. Grandis)
Drawbacks: Infinitely many objects. Calculations?
Question: How much does $\vec{\pi}_1(X)(x,y)$ depend on (x,y)?
Remedy: Localization, component category. [FGHR:04, GH:06]
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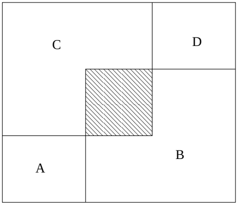

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X a (saturated) d-space.
$\varphi, \psi \in \vec{P}(X)(x,y)$ are called reparametrization equivalent if there are $\alpha, \beta \in \vec{P}(I)$ such that $\varphi \circ \alpha = \psi \circ \beta$ ("same oriented trace").
(Fahrenberg-R., 07): Reparametrization equivalence is an equivalence relation (transitivity).
$\vec{T}(X)(x,y) = \vec{P}(X)(x,y)/_{\simeq}$ makes $\vec{T}(X)$ into the (topologically enriched) trace category – composition associative.
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Variant: $\vec{R}(X)(x,y)$ consists of regular d-paths (not constant on any non-trivial interval $J \subset I$). The contractible group $Homeo_+(I)$ of increasing homeomorphisms acts on these – freely if $x \neq y$.

Theorem (FR:07)

- *$\vec{R}(X)(x,y)/_{\simeq} \to \vec{P}(X)(x,y)/_{\simeq}$ is a homeomorphism.*
- *$\vec{R}(X)(x,y) \to \vec{R}(X)(x,y)/_{\simeq}$ is a (weak) homotopy equivalence.*

For X the geometric realisation of a cubical complex, all trace spaces $\vec{T}(X)(x,y)$ are locally contractible.

Examples I^n the unit cube, ∂I^n its boundary.

- $\vec{T}(I^n;\mathbf{x},\mathbf{y})$ is contractible for all $x \preceq y \in I^n$;
- (Conjecture) $\vec{T}(\partial I^n;\mathbf{0},\mathbf{1})$ is (weakly) homotopy equivalent to S^{n-2}.

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Preorder categories
Getting organised with indexing categories

A d-space structure on X induces the preorder $\preceq$:

$$x \preceq y \Leftrightarrow \vec{T}(X)(x,y) \neq \emptyset$$

and an indexing preorder category $\vec{D}(X)$ with

- Objects: (end point) pairs $(x,y), x \preceq y$
- Morphisms:
  $\vec{D}(X)((x,y),(x',y')) := \vec{T}(X)(x',x) \times \vec{T}(X)(y,y')$:

$$\begin{tikzcd} x' \arrow[r, shift left] \arrow[r, shift right] & x \arrow[r, "\preceq"] & y \arrow[r, shift left] \arrow[r, shift right] & y' \end{tikzcd}$$

- Composition: by pairwise contra-, resp. covariant concatenation.

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- $\vec{T}^X(x,y) := \vec{T}(X)(x,y)$
- $\vec{T}^X(\sigma_x,\sigma_y)$: $\vec{T}(X)(x,y) \longrightarrow \vec{T}(X)(x',y')$

$$[\sigma] \longmapsto [\sigma_x * \sigma * \sigma_y]$$

Homotopical variant $\vec{D}_\pi(X)$ with morphisms
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Sensitivity with respect to variations of end points
A persistence point of view

- How much does (the homotopy type of) $\vec{T}^X(x,y)$ depend on (small) changes of x,y?
- Which concatenation maps $\vec{T}^X(\sigma_x,\sigma_y):\vec{T}^X(x,y)\to\vec{T}^X(x',y'),\ [\sigma]\mapsto[\sigma_x*\sigma*\sigma_y]$ are homotopy equivalences, induce isos on homotopy, homology groups etc.?
- The persistence point of view: Homology classes etc. are born (at certain branchings/mergings) and may die (analogous to the framework of G. Carlsson etal.)
- Are there "components" with (homotopically/homologically) stable dipath spaces (between them)? Are there borders ("walls") at which changes occur?

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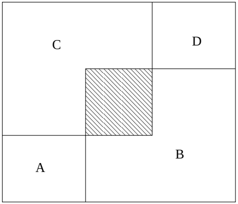
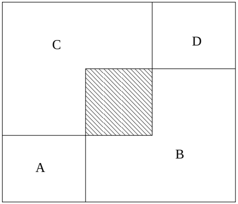


$$
\begin{tikzcd}
AA \arrow[rrd] \arrow[rdd] & & & & BB \arrow[lld] \arrow[ldd] \\
& & AB \arrow[d] & \# & \\
& AC \arrow[r] & AD & BD \arrow[l] & \\
& \# & CD \arrow[u] & & \\
CC \arrow[ruu] \arrow[rru] & & & & DD \arrow[llu] \arrow[luu]
\end{tikzcd}
$$

Figur: Standard example and component category

Components A, B, C, D – or rather $AA, AB, AC, AD, BB, BD, CC, CD, DD \subseteq X \times X$.

#: diagram commutes.



$$
\begin{tikzcd}
AA \arrow[drr] \arrow[ddr] & & & & BB \arrow[dll] \arrow[ddl] \\
& & AB \arrow[d] & \# & \\
& AC \arrow[r] & AD & BD \arrow[l] & \\
& \# & CD \arrow[u] & & \\
CC \arrow[uur] \arrow[urr] & & & & DD \arrow[ull] \arrow[uul]
\end{tikzcd}
$$

Figur: Standard example and component category

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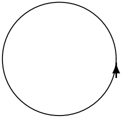

$$\mathcal{C} : \Delta \underset{b}{\overset{a}{\rightleftarrows}} \bar{\Delta}$$

Δ the diagonal, $\bar{\Delta}$ its complement. $\mathcal{C}$ is the free category generated by a, b.

oriented circle

- Remark that the components are no longer products!
- In order to get a discrete component category, it is essential to use an indexing category taking care of pairs (source, target).

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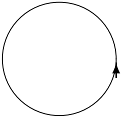

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Component categories
via generalized congruences and homotopy flows

- How to identify morphisms in a category between different objects in an organised manner?
  Generalized congruence (Bednarczyk, Borzyszkowski, Pawlowski, TAC 1999) $\rightsquigarrow$ quotient category.
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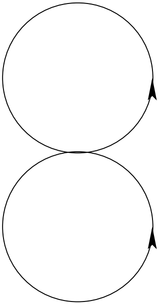
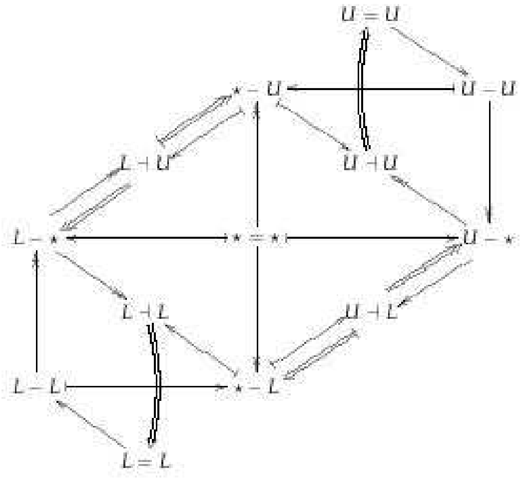
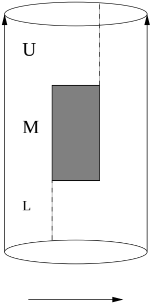
The component category of a wedge of two oriented circles

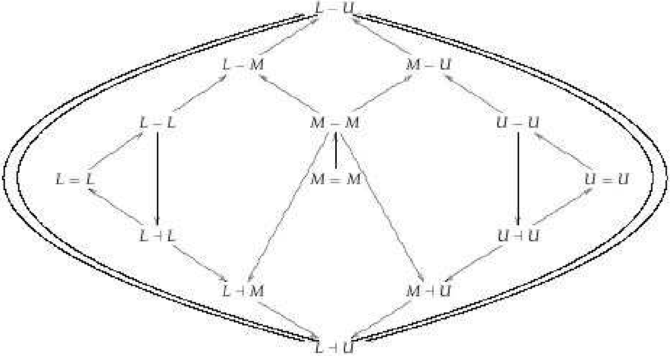
$X = \vec{S}^1 \vee \vec{S}^1$





The component category of an oriented cylinder with a deleted rectangle





Concluding remarks

- Component categories contain the essential information given by (algebraic topological invariants of) path spaces
- Compression via component categories is an antidote to the state space explosion problem
- Some of the ideas (for the fundamental category) are implemented and have been tested for huge industrial software from EDF (Éric Goubault & Co., CEA)
- Dihomotopy equivalence: Definition uses automorphic homotopy flows to ensure homotopy equivalences

$$\vec{T}(f)(x,y) : \vec{T}(X)(x,y) \to \vec{T}(Y)(fx,fy) \text{ for all } x \preceq y.$$

- Much more theoretical and practical work remains to be done!

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