Simplicial models for trace spaces

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ATMCS IV 21.6.2010

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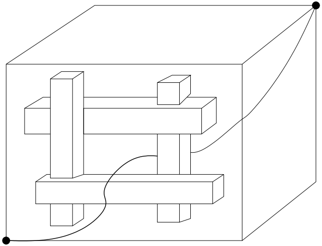
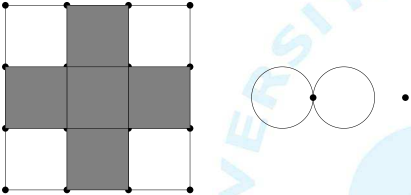
Example:



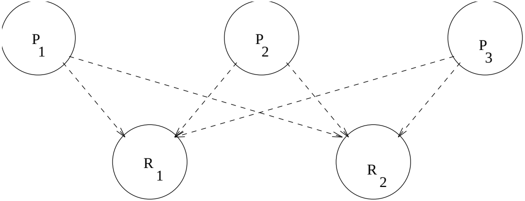
State space =
a cube $\vec{I}^3 \setminus F$
minus 4 box obstructions

$\rightsquigarrow$



Trace space contained in a torus $(\partial\Delta^2)^2$ –
homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

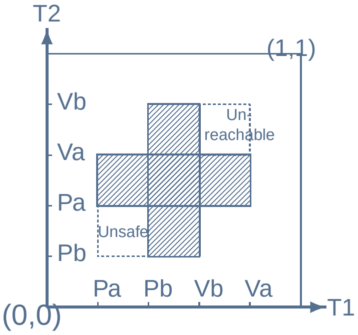
Mutual exclusion occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.
Semaphores!
Semantics: A processor has to lock a resource and to relinquish the lock later on!
Description/abstraction $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)
P: pakken; V: vrijlaten

A geometric model: Schedules in "progress graphs"

The Swiss flag example



PV-diagram from

$P_1 : P_a P_b V_b V_a$

$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

Deadlocks, unsafe and unreachable regions may occur.

A linear PV-program can be modelled as the complement of a forbidden region F consisting of a number of holes in an n-cube I^n:

Hole = isothetic hyperrectangle $R^i, 1 \leq i \leq l$, in an n-cube.

State space

$X = \vec{I}^n \setminus F,\ F = \bigcup_{i=1}^{l} R^i,\ R^i =]a_1^i, b_1^i[\times \cdots \times]a_n^i, b_n^i[.$

with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.

X inherits a partial order from $\vec{I}^n$.

More general PV-programs:

- Replace $\vec{I}^n$ by a product $\Gamma_1 \times \cdots \times \Gamma_n$ of digraphs.
- Holes have then the form $p_1^i((0,1)) \times \cdots \times p_n^i((0,1))$ with $p_j^i : \vec{I} \to \Gamma_j$ a directed injective (d-)path.
- Pre-cubical complexes: like pre-simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks.

Main interest: Spaces of d-paths/traces – up to dihomotopy

- X a d-space, $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in X (continuous and "order-preserving")
- $\vec{P}(X)(a,b) = \{p : \vec{I} \to X \mid p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a,b) = \vec{P}(X)(a,b)$ modulo increasing reparametrizations.
  In most cases: $\vec{P}(X)(a,b) \simeq \vec{T}(X)(a,b)$.
- A dihomotopy on $\vec{P}(X)(a,b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a,b),\ t \in I$; a path in $\vec{P}(X)(a,b)$.

Aim: Description of the homotopy type of $\vec{P}(X)(a,b)$;
in particular of its path components, ie the dihomotopy classes of d-paths.

Covers of X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

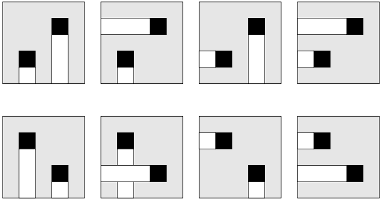
by contractible or empty subspaces

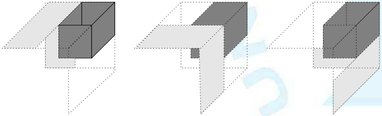
$X = \vec{I}^n \setminus F, F = \bigcup_{i=1}^{l} R^i; R^i = [\mathbf{a}^i, \mathbf{b}^i]; \mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition

$$\begin{aligned} X_{j_1,\dots,j_l} &= \{x \in X \mid \forall i : x_{j_i} \le a^i_{j_i} \vee \exists k : x_k \ge b^i_k\} \\ &= \{x \in X \mid \forall i : x \le \mathbf{b}^i \Rightarrow x_{j_i} \le a^i_{j_i}\}, \qquad 1 \le j_i \le n. \end{aligned}$$

Examples:





$$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{1 \le j_1,\dots,j_l \le n} \vec{P}(X_{j_1,\dots,j_l})(\mathbf{0},\mathbf{1}).$$

More intricate subspaces as intersections
either empty or contractible

Definition

$\emptyset \neq J_1, \ldots, J_l \subseteq [1:n]$:

$$X_{J_1,\ldots,J_l} = \bigcap_{j_i \in J_i} X_{j_1,\ldots,j_l}$$
$$= \{x \in X \mid \forall i, j_i \in J_i : x \leq \mathbf{b}^i \Rightarrow x_{j_i} \leq a^i_{j_i}\}$$

Theorem

$\vec{P}(X_{J_1,\ldots,J_l})(\mathbf{0},\mathbf{1})$ is either empty or contractible.

Proof.

relies on: Subspaces $X_{J_1,\ldots,J_l}$ are closed under $\vee$ = l.u.b. $\square$

Question: For which $J_1, \ldots, J_l \subseteq [1:n]$ is

$$\vec{P}(X_{J_1,\ldots,J_l})(\mathbf{0},\mathbf{1}) \neq \emptyset?$$

Combinatorics: Bookkeeping with binary matrices

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices

$M_{l,n}^R$ no row vector is the zero vector

$M_{l,n}^C$ every column vector is a unit vector

Restriction to Index sets	$\leftrightarrow$	Matrix sets
$(\mathcal{P}([1:n]))^l$	$\leftrightarrow$	$M_{l,n}$
$J = (J_1, \dots, J_l)$	$\mapsto$	$M^J = (m_{ij}),\ m_{ij} = 1 \Leftrightarrow j \in J_i$
J^M	$\leftarrow$	$M \quad J_i^M = \{j \mid m_{ij} = 1\}$
l-tuples of subsets$\neq \emptyset$	$\leftrightarrow$	$M_{l,n}^R$
$\{(K_1, \dots, K_l) \mid [1:n] = \bigsqcup K_i\}$	$\leftrightarrow$	$M_{l,n}^C$

$$X_M := X_{J_M}, \qquad \vec{P}(X_M)(\mathbf{0},\mathbf{1}) = \vec{P}(X_{J_M})(\mathbf{0},\mathbf{1}) \neq \emptyset?$$

Poset category – Combinatorics

$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M^R_{l,n} \subseteq M_{l,n}$

$J \leftrightarrow M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$

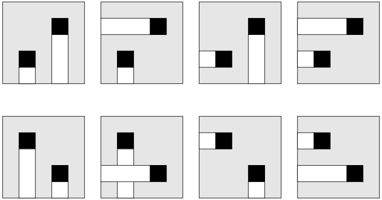
Prodsimplicial complex – Topology

$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$

$\Delta^{|J_1|-1}_{J_1} \times \cdots \times \Delta^{|J_l|-1}_{J_l} \subseteq \mathbf{T}(X)(\mathbf{0},\mathbf{1})$

$$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset.$$

Examples:



- $\mathbf{T}(X_1)(\mathbf{0},\mathbf{1}) = (\partial\Delta^1)^2 = 4*$
- $\mathbf{T}(X_2)(\mathbf{0},\mathbf{1}) = 3*$

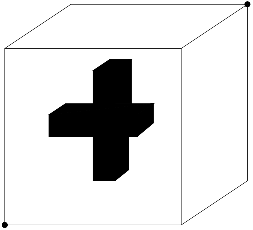
$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \subset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$$

Further examples

(1) $X = \vec{I}^n \setminus \vec{J}^n$

- $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^R_{1,n} \setminus \{[1,\dots 1]\}$.
- $\mathcal{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1} \simeq S^{n-2}$.

(2)



- $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1}) = \left\{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\right\}$.
- $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) =$ 3 diagonal squares $\subset (\partial\Delta^2)^2 = T^2$

  $\simeq S^1$.

Homotopy equivalence between trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ and the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Theorem

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1}).$

Proof.

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1})^{(\mathrm{op})} \to \mathbf{Top}$:
  $\mathcal{D}(J_1,\dots,J_l) = \vec{P}(X_{J_1,\dots,J_l})(\mathbf{0},\mathbf{1})$,
  $\mathcal{E}(J_1,\dots,J_l) = \Delta_{J_1}^{|J_1|-1} \times \cdots \times \Delta_{J_l}^{|J_l|-1}$,
  $\mathcal{T}(J_1,\dots,J_l) = *$
- $\operatorname{colim} \mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, $\operatorname{colim} \mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
  $\operatorname{hocolim} \mathcal{T} = \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
  $\operatorname{hocolim} \mathcal{D} \cong \operatorname{hocolim} \mathcal{T}^* \cong \operatorname{hocolim} \mathcal{T} \cong \operatorname{hocolim} \mathcal{E}$.
- Projection lemma:
  $\operatorname{hocolim} \mathcal{D} \simeq \operatorname{colim} \mathcal{D}$, $\operatorname{hocolim} \mathcal{E} \simeq \operatorname{colim} \mathcal{E}$.

From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of path space

Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

- Is $\vec{P}(X)(\mathbf{0},\mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected? Other topological properties?

The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces.
There are fast algorithms to calculate the homology groups of these chain complexes even for very big complexes.
Number of path-components: $rkH_0(\mathbf{T}(X)(\mathbf{0},\mathbf{1}))$.
For path-components, there might be faster "discrete" methods.
Even if "exponential explosion" prevents precise calculations, inductive determination (round by round) of general properties ((simple) connectivity) may be possible.
Implementation in ALCOOL: progress at CEA/LIX-lab.

Deadlocks and unsafe regions determine $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$

A dual view: **extended** hyperrectangles R^i_j
$:= [0,b^i_1[\times\cdots\times[0,b^i_{j-1}[\times]a^i_j,b^i_j[\times[0,b^i_{j+1}[\times\cdots\times[0,b^i_n[\supset R^i$.

$$X_M = X\setminus\bigcup_{m_{ij}=1} R^i_j.$$

Theorem

The following are equivalent:

1. $\vec{P}(X_M)(\mathbf{0},\mathbf{1})=\emptyset \Leftrightarrow M\not\in\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
2. *There is a map $i:[1:n]\to[1:l]$ such that $m_{i(j),j}=1$ and such that $\bigcap_{1\le j\le n} R^{i(j)}_j\neq\emptyset$ – giving rise to a **deadlock unavoidable from $\mathbf{0}$**.*
3. ***Mere combinatorics**: Checking a bunch of **inequalities**: There is a map $i:[1:n]\to[1:l]$ such that $a^{i(j)}_j < b^{i(k)}_j$ for all $1\le j,k\le n$.*

OBS: $\chi(\text{graph}(i)) = M(i)\in M^{C}_{l,n}$!

Which of the l^n matrices in $M_{l,n}^C$ belong to $D(X)(\mathbf{0},\mathbf{1})$?

A matrix $M \in M_{l,n}^C$ is described by a (choice) map

$$i : [1:n] \to [1:l], m_{i(j),j} = 1.$$

Deadlocks $\rightsquigarrow$ inequalities:

$$M \in D(X)(\mathbf{0},\mathbf{1}) \subseteq M_{l,n}^C \Leftrightarrow a_j^{i(j)} < b_j^{i(k)} \text{ for all } 1 \le j,k \le n.$$

Algorithmic organisation: Choice maps with the same image give rise to the same upper bounds b_j^*.

Consider $\Psi : M_{l,n} \to \mathbf{Z}/2$, $\Psi(M) = 1 \Leftrightarrow \vec{P}(X_M)(\mathbf{0}, \mathbf{1}) = \emptyset$.

- Ψ is order preserving, in particular:
  $\Psi^{-1}(0), \Psi^{-1}(1)$ are closed in opposite senses:
  $M \leq N : \Psi(N) = 0 \Rightarrow \Psi(M) = 0; \Psi(M) = 1 \Rightarrow \Psi(N) = 1$
  (thus $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$ prodsimplicial).
- $\Psi(M) = 1 \Leftrightarrow \exists N \in M^C_{l,n}$ such that $N \leq M, \Psi(N) = 1$
  $D(X)(\mathbf{0}, \mathbf{1}) = \{N \in M^C_{l,n} | \Psi(N) = 1\}$ – dead
  $\mathcal{C}(X)(\mathbf{0}, \mathbf{1}) = \{M \in M^R_{l,n} | \Psi(M) = 0\}$ – alive
  $\mathcal{C}_{\max}(X)(\mathbf{0}, \mathbf{1})$ maximal such matrices
  characterized by: $m_{ij} = 1$ apart from:
  $\forall N \in D(X)(\mathbf{0}, \mathbf{1}) \exists ! (i, j) : 0 = m_{ij} < n_{ij} = 1$

Matrices in $\mathcal{C}_{\max}(X)(\mathbf{0}, \mathbf{1})$ correspond to maximal simplex products in $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$.
Example: $X = \vec{I}^n \subset \vec{J}^n, D(X)(\mathbf{0}, \mathbf{1}) =$
$\{[1, \ldots, 1]\}, \mathcal{C}_{\max}(X)(\mathbf{0}, \mathbf{1}) = M^R_{l,n} \setminus \{[1, \ldots, 1]\}$.

Algorithmics: Construct $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1})$ incrementally (checking for one matrix $N\in D(X)(\mathbf{0},\mathbf{1})$ at a time), starting with matrix $\mathbf{1}$:

1. $N_{i+1}\not\leq M\in\mathcal{C}^i(X)\Rightarrow M\in\mathcal{C}^{i+1}(X)$;
2. $N_{i+1}\leq M\Rightarrow M$ is replaced by n matrices M^j with one additional 0. Example: $X=\vec{I}^n\setminus\vec{J}^n$.

A matrix in $D(X)(\mathbf{0},\mathbf{1})$ describes a hyperedge on the vertex set $[1:l]\times[1:n]$; $D(X)(\mathbf{0},\mathbf{1})$ describes a hypergraph.
A transversal in a hypergraph is a vertex set that has non-empty intersection with each hyperedge
$\leftrightarrow$ a matrix L such that $\forall N\in D(X)(\mathbf{0},\mathbf{1})\ \exists(i,j): l_{ij}=n_{ij}=1$.
$M=\mathbf{1}-L$: $\forall N\in D(X)(\mathbf{0},\mathbf{1})\exists(i,j): 0=m_{ij}<n_{ij}=1$.
Conclusion: Search for matrices in $A_{max}(\mathbf{0},\mathbf{1})$ corresponds to search for minmal transversals in $D(X)(\mathbf{0},\mathbf{1})$.
In our case: All hyperedges have same cardinality n, include one element per column.

- More general semaphores (intersection with the boundary of I^n allowed)
- Different end points: $\vec{P}(X)(\mathbf{c},\mathbf{d})$ and iterative calculations
- End complexes rather than end points (allowing processes not to respond..., Herlihy & Cie)

Same technique, modification of definition and calculation of $\mathcal{C}(X)(-,-)$, $D(X)(-,-)$ etc.

- New light on definition and determination of components of model space X.

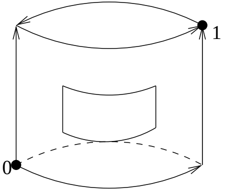
2a. Semaphores corresponding to non-linear programs:

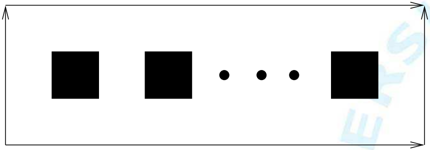
Products of digraphs instead of $\vec{I}^n$:
$\Gamma = \prod_{j=1}^n \Gamma_j$, state space $X = \Gamma \setminus F$,
F a product of generalized hyperrectangles R^i.

- $\vec{P}(\Gamma)(\mathbf{x},\mathbf{y}) = \prod \vec{P}(\Gamma_j)(x_j, y_j)$ – homotopy discrete!

Represent a path component $C \in \vec{P}(\Gamma)(\mathbf{x},\mathbf{y})$ by (regular) d-paths $p_j \in \vec{P}(\Gamma_j)(x_j, y_j)$ – an interleaving.
The map $c : \vec{I}^n \to \Gamma, c(t_1,\dots,t_n) = (c_1(t_1),\dots,c_n(t_n))$ induces a homeomorphism $\circ c : \vec{P}(\vec{I}^n)(\mathbf{0},\mathbf{1}) \to C \subset \vec{P}(\Gamma)(\mathbf{x},\mathbf{y})$.

Pull back F via c:
$\bar{X} = \vec{I}^n \setminus \bar{F}, \bar{F} = \bigcup \bar{R}^i, \bar{R}^i = c^{-1}(R^i)$ – honest hyperrectangles!





$i_X : \vec{P}(X) \hookrightarrow \vec{P}(\Gamma)$.
Given a component $C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.
The d-map $c : \bar{X} \to X$ induces a homeomorphism
$c\circ : \vec{P}(\bar{X})(\mathbf{0}, \mathbf{1}) \to i_X^{-1}(C) \subset \vec{P}(X)(\mathbf{x}, \mathbf{y})$.

- C "lifts to X" $\Leftrightarrow \vec{P}(\bar{X})(\mathbf{0}, \mathbf{1}) \neq \emptyset$; if so:
- Analyse $i_X^{-1}(C)$ via $\vec{P}(\bar{X})(\mathbf{0}, \mathbf{1})$.
- Exploit relations between various components.

- Higher Dimensional Automaton: Pre-cubical complex with preferred diretions. Geometric realization X with d-space structure.
- $P(X)(\mathbf{x},\mathbf{y})$ is ELCX (equi locally convex). D-paths within a specified “cube path” form a contractible subspace.
- $P(X)(\mathbf{x},\mathbf{y})$ has the homotopy type of a simplicial complex: the nerve of an explicit category of cube paths (with inclusions as morphisms).

- Rick Jardine, *Path categories and resolutions*
- forthcoming AGT-paper *Simplicial models of trace spaces*

Thank you for your attention!