A Asymptotic normality of regression parameter estimates

Please recall the notation introduced in Section 2 and 3. In this appendix we derive asymptotic normality of the estimate of the interest parameter $\beta_{1:p}$ when the mother intensity tends to infinity, i.e. we consider an increasing sequence $(\kappa_n)_{n\geq 1}$ of κ values where $\kappa_n = n\tilde{\kappa}$ for some $\tilde{\kappa} > 0$ and $n \to \infty$. The constant $\tilde{\kappa}$ is introduced to allow for non-integer values of κ.

Let $\tilde{\beta}_0 = \log(\alpha)$ and let $u_n(\tilde{\beta}_0, \beta_{1:p}) = u(\log(\kappa_n) + \tilde{\beta}_0, \beta_{1:p})$ be the estimating function for $(\tilde{\beta}_0, \beta_{1:p})$ when κ is known and given by κ_n. Denote by $(\tilde{\beta}_0^n, \hat{\beta}_{1:p}^n)$ the estimate obtained by solving $u_n(\tilde{\beta}_0, \beta_{1:p}) = 0$. The following Theorem 1 is concerned with asymptotic normality of $\sqrt{\kappa_n}(\tilde{\beta}_0^n - \beta_0^*, \hat{\beta}_{1:p}^n - \beta_{1:p}^*)$.

Theorem 1. *Suppose $J(\beta_{1:p}^*)$ is positive definite. Then $\sqrt{\kappa_n}(\tilde{\beta}_0^n - \beta_0^*, \hat{\beta}_{1:p}^n - \beta_{1:p}^*)$ is asymptotically zero mean normal with covariance matrix*

$$(\alpha^* J(\beta_{1:p}^*))^{-1} + J^{-1}(\beta_{1:p}^*) G(\beta_{1:p}^*, \omega^*) J^{-1}(\beta_{1:p}^*). \tag{A.1}$$

Proof. Below we show that $u_n(\tilde{\beta}_0^*, \beta_{1:p}^*)/\sqrt{n}$ is asymptotically normal. Asymptotic normality of $\sqrt{\kappa_n}(\tilde{\beta}_0^n - \beta_0^*, \hat{\beta}_{1:p}^n - \beta_{1:p}^*)$ then follows directly from Theorem 2.8 in Sørensen (1999).

Identify C with $\cup_{i=1}^n C_i$ where the C_i are independent Poisson processes on $\mathbb{R}^2$ with intensity $\tilde{\kappa}$ and let $Z_i = \sum_{c \in C_i} \sum_{\xi \in X_c \cap S} z(\xi)$. By applying twice the Slivnyak-Mecke theorem (see e.g. Theorem 3.1 in Møller and Waagepetersen, 2003),

$$\mathbb{E} Z_i = \tilde{\kappa}\alpha^* \int_S z(\xi) \exp(z_{1:p}(\xi)(\beta_{1:p}^*)^\mathsf{T}) \mathrm{d}\xi$$

and by the extended Slivnyak-Mecke theorem (see e.g. Theorem 3.2 in Møller and Waagepetersen, 2003),

$$\begin{aligned} V = \mathbb{V}\mathrm{ar} Z_i = \mathbb{E}\Big[\sum_{c \in C_i} \Big(\sum_{\xi \in X_c \cap S} z(\xi)\Big)^2\Big] =& \tilde{\kappa}\alpha^* \int_S z(\xi)^\mathsf{T} z(\xi) \exp(z_{1:p}(\xi)(\beta_{1:p}^*)^\mathsf{T}) \mathrm{d}\xi \\ &+ \tilde{\kappa}(\alpha^*)^2 \int_{\mathbb{R}^2} H(\beta_{1:p}^*, \omega^*, c)^\mathsf{T} H(\beta_{1:p}^*, \omega^*, c) \mathrm{d}c. \end{aligned}$$

By the multivariate central limit theorem,

$$u_n(\tilde{\beta}_0^*, \beta_{1:p}^*)/\sqrt{n} = \frac{1}{\sqrt{n}} \sum_{c \in C} \sum_{\xi \in X_c \cap S} z(\xi) - \sqrt{n}\mathbb{E} Z_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Z_i - \mathbb{E} Z_i)$$

converges to a multivariate normal distribution with mean zero and covariance matrix V.

It follows from Theorem 2.8 in Sørensen (1999) (under condition 2.1 and 2.4 with $G_n(\bar{\theta}) = u_n(\tilde{\beta}_0^*, \beta_{1:p}^*)$ and $W(\bar{\theta}) = \tilde{\kappa}\alpha^* J(\beta_{1:p}^*)$) that $\sqrt{\tilde{\kappa} n}(\tilde{\beta}_0^n - \beta_{1:p}^*, \hat{\beta}_{1:p}^n - \beta_{1:p}^*)$ is asymptotically zero mean normal with covariance matrix (A.1). $\square$

For completeness we also include a result on asymptotic normality for the maximum likelihood estimate of $\hat{\beta}_{1:p}$ assuming that X is a Poisson process with intensity function of the form $\kappa_n \alpha \exp(z_{1:p}(\cdot)\beta_{1:p}^{\mathsf{T}})$. The increasing intensity asymptotics considered here is mathematically more simple than the increasing domain asymptotics in Rathbun and Cressie (1994).

Theorem 2. *Let* $(\tilde{\beta}_0^n, \hat{\beta}_{1:p}^n)$ *denote the maximum likelihood estimate obtained by solving* $u(\log(\kappa_n) + \log(\alpha), \beta_{1:p}) = 0$ *assuming that* X *is a Poisson process with intensity function* $\kappa_n \alpha^* \exp(z_{1:p}(\cdot)(\beta_{1:p}^*)^{\mathsf{T}})$. *Suppose* $J(\beta_{1:p}^*)$ *is positive definite. Then* $\sqrt{\kappa_n}(\tilde{\beta}_0^n - \beta_0^*, \hat{\beta}_{1:p}^n - \beta_{1:p}^*)$ *is asymptotically zero mean normal with covariance matrix given by the first term in* (A.1).

Proof. Identify $X \cap S$ with $\cup_{i=1}^n X_i$ where the X_i are independent Poisson processes on S with intensity function $\tilde{\kappa}\alpha^* \exp(z_{1:p}(\xi)(\beta_{1:p}^*)^{\mathsf{T}})$, $\xi \in S$. Then $u_n(\tilde{\beta}_0^*, \beta_{1:p}^*)/\sqrt{n}$ is distributed as

$$\sum_{i=1}^{n} \left(\sum_{\xi \in X_i} z(\xi) - \tilde{\kappa}\alpha^* \int_S z(\xi) \exp(z_{1:p}(\xi)(\beta_{1:p}^*)^{\mathsf{T}}) \mathrm{d}\xi \right) / \sqrt{n}$$

which is asymptotically zero mean normal with covariance matrix $\tilde{\kappa}\alpha^* J(\beta_{1:p}^*)$. Asymptotic normality of $\sqrt{\kappa_n}(\tilde{\beta}_0^n - \beta_0^*, \hat{\beta}_{1:p}^n - \beta_{1:p}^*)$ with covariance matrix $\left(\alpha^* J(\beta_{1:p}^*)\right)^{-1}$ now follows by the same arguments as in the proof of Theorem 1. $\square$

References

Møller, J. & Waagepetersen, R. P. (2003). *Statistical inference and simulation for spatial point processes*. Chapman and Hall/CRC, Boca Raton.

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