Propriety of posteriors for Poisson processes

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Posterior propriety

Lemma. *Consider a Poisson process on a bounded region W with intensity function*

$$\lambda(\xi;\theta) = \exp(h(\xi)\theta^{\mathsf{T}}),\ \xi \in W,$$

where $\theta \in \mathbb{R}^p$ and $h(\xi) = (h_1(\xi), \ldots, h_p(\xi))$ are row vectors with $h_i : W \to \mathbb{R}$. Let $y = \{\xi_1, \ldots, \xi_n\}$ denote an observed point pattern from this Poisson process and let H be the matrix with rows $h(\xi_i)$, $i = 1 \ldots, n$. Assume that H has full rank $p \le n$, that $h(\cdot)$ is continuous at the observed points $\xi_1, \ldots, \xi_n$, and that the prior density p for θ is bounded but not necessarily proper. Then the posterior for θ given y is proper.

Proof. The posterior is proper provided

$$\int L(\theta)\mathrm{d}\theta < \infty$$

where

$$L(\theta) = \exp\Big(- \int_W \lambda(\eta;\theta)\mathrm{d}\eta\Big) \prod_{\xi \in y} \lambda(\xi;\theta)$$

is the likelihood function. Consider a partition $\mathcal{P} = \{C_l\}_{l=1}^{L}$ of W into disjoint cells C_l so that $\mathrm{card}(C_l \cap y) \le 1$, and for $\xi \in y$, denote by C^{ξ} the cell which contains ξ. Then

$$\int_W \exp(h(\eta)\theta^\mathsf{T})\mathrm{d}\eta \geq \sum_{\xi\in y} |C^\xi| \exp\big(\inf_{\eta\in C^\xi} h(\eta)\theta^\mathsf{T}\big)$$

$$\geq \sum_{\xi\in y} |C^\xi| \exp\Big(\sum_{i=1}^{p} \theta_i\big(\inf_{\eta\in C^\xi} h_i(\eta)1[\theta_i > 0] + \sup_{\eta\in C^\xi} h_i(\eta)1[\theta_i \leq 0]\big)\Big).$$

Let $\epsilon = \max_{\xi\in y, i=1,\dots,p} \sup_{\eta\in C^\xi} |h_i(\eta) - h_i(\xi)|$, $\tilde{\epsilon} = \epsilon\big(\mathrm{sign}(\theta_1),\dots,\mathrm{sign}(\theta_p)\big)$, and $\tilde{h} = h - \tilde{\epsilon}$. Then

$$\sum_{i=1}^{p} \theta_i\big(\inf_{\eta\in C^\xi} h_i(\eta)1[\theta_i > 0] + \sup_{\eta\in C^\xi} h_i(\eta)1[\theta_i \leq 0]\big) \geq \tilde{h}(\xi)\theta^\mathsf{T},$$

so

$$\int_W \lambda(\eta;\theta)\mathrm{d}\eta \geq \sum_{\xi\in y} |C^\xi| \exp(\tilde{h}(\xi)\theta^\mathsf{T}).$$

Let $\tilde{H}$ have rows $\tilde{h}(\xi_i)$, $i = 1,\dots,n$. By continuity of h, we can choose the partition $\mathcal{P}$ so that ϵ becomes arbitrarily small. Hence we choose the partition so that $\tilde{H}$ has full rank p and denote by $\tilde{H}^{-1}$ a left inverse of $\tilde{H}$. Let $\eta^\mathsf{T} = \tilde{H}\theta^\mathsf{T}$. Then $\tilde{\epsilon}\theta^\mathsf{T} = k\eta^\mathsf{T}$ where $k = \tilde{\epsilon}\tilde{H}^{-1}$, and so

$$\log(L(\theta)) \leq$$

$$\sum_{y\in\xi} [\tilde{\epsilon}\theta^\mathsf{T} + \tilde{h}(\xi)\theta^\mathsf{T} - |C^\xi| \exp(\tilde{h}(\xi)\theta^\mathsf{T})] = \sum_{i=1}^{n} [\eta_i(nk_i + 1) - |C^{\xi_i}| \exp(\eta_i)].$$

If ϵ is small enough, each of the terms $\exp\big(\eta_i(nk_i + 1) - |C^{\xi_i}| \exp(\eta_i)\big)$ is both bounded by a constant K and integrable as a function of η_i. We can assume that the partition is chosen so that this is the case, and without loss of generality we can assume that the first p rows in $\tilde{H}$ are linearly independent. Then

$$\int L(\theta)\mathrm{d}\theta \leq$$

$$\frac{K^{(n-p)}}{\det \tilde{H}_1} \int \exp\Big(\sum_{i=1}^{p} [\eta_i(nk_i + 1) - |C^{\xi_i}| \exp(\eta_i)]\Big)\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_p < \infty$$

where $\tilde{H}_1$ consists of the first p rows in $\tilde{H}$. $\square$