# Supplementary material for 'Quasi-likelihood for Spatial Point Processes'

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## 1. CONDITIONS AND LEMMAS

To verify the existence of a $|W_n|^{1/2}$ consistent sequence of solutions $\hat{\boldsymbol{\beta}}_n$, we assume that the following conditions are satisfied:

- C1 $\lambda(\mathbf{u};\boldsymbol{\beta}) = \lambda(\mathbf{z}(\mathbf{u})\boldsymbol{\beta}^\mathsf{T})$ where $\lambda(\cdot) > 0$ is twice continuously differentiable and $\sup_{\mathbf{u}\in\mathbb{R}^2} \|\mathbf{z}(\mathbf{u})\| < K_1$ for some $K_1 < \infty$.
- C2 for some $0 < K_2 < \infty$, $\int_{\mathbb{R}^2} \big|g(\mathbf{r};\boldsymbol{\psi}^*) - 1\big|\mathrm{d}\mathbf{r} \le K_2$.
- C3 $\boldsymbol{\phi}_{n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta})$ is differentiable with respect to $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$, and for $|\boldsymbol{\phi}_{n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta})|$, $|\mathrm{d}\boldsymbol{\phi}_{n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta})/\mathrm{d}\boldsymbol{\beta}|$ and $|\mathrm{d}\boldsymbol{\phi}_{n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta})/\mathrm{d}\boldsymbol{\theta}|$, the supremum over $\mathbf{u} \in \mathbb{R}^2, \boldsymbol{\beta} \in b(\boldsymbol{\beta}^*, K_3), \boldsymbol{\theta} \in b(\boldsymbol{\theta}^*, K_3)$ is bounded for some $K_3 > 0$, where $b(\mathbf{x}, r)$ denotes the ball centred at $\mathbf{x}$ with radius $r > 0$.
- C4 $|W_n|^{1/2}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}^*)$ is bounded in probability.
- C5 $l = \liminf_n l_n > 0$, where for each $n$, $l_n$ denotes the minimal eigenvalue of

$$\bar{\mathbf{S}}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*) = |W_n|^{-1}\mathbb{E}\mathbf{J}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*) = |W_n|^{-1}\int_{W_n} \boldsymbol{\phi}_{n,\boldsymbol{\theta}^*}(\mathbf{u})^\mathsf{T}\boldsymbol{\lambda}'(\mathbf{u};\boldsymbol{\beta}^*)\mathrm{d}\mathbf{u}.$$

Condition C1 and C2 imply L1 and L2 below.

- L1 for $\lambda(\mathbf{u};\boldsymbol{\beta})$, $\boldsymbol{\lambda}'(\mathbf{u};\boldsymbol{\beta})$ and $\boldsymbol{\lambda}''(\mathbf{u};\boldsymbol{\beta})$, the supremum over $\mathbf{u} \in \mathbb{R}^2, \boldsymbol{\beta} \in b(\boldsymbol{\beta}^*, K_3), \boldsymbol{\theta} \in b(\boldsymbol{\theta}^*, K_3)$ is bounded.
- L2 for a function $h : \mathbb{R}^2 \to \mathbb{R}$,

$$\mathbb{V}\mathrm{ar} \sum_{\mathbf{u}\in X\cap W_n} h(\mathbf{u}) \le |W_n|\big[1 + \sup_{\mathbf{u}\in W_n} \lambda(\mathbf{u};\boldsymbol{\beta}^*)K_2\big] \sup_{\mathbf{u}\in W_n} h(\mathbf{u})^2 \sup_{\mathbf{u}\in W_n} \lambda(\mathbf{u};\boldsymbol{\beta}^*).$$

  In particular, $|W_n|^{-1}\mathbb{V}\mathrm{ar}\sum_{\mathbf{u}\in X\cap W_n} h(\mathbf{u})$ is bounded when $h$ is bounded.

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The condition C3 is not so easy to verify in general due to the abstract nature of the function $\boldsymbol{\phi}_{n,\boldsymbol{\theta}}$. However, it can be verified e.g. assuming that $\boldsymbol{\phi}_{n,\boldsymbol{\theta}}$ can be expressed using the Neumann series. Condition C4 holds under conditions specified in Waagepetersen and Guan (2009) (including e.g. C1 and C2). Condition C5 is not unreasonable since

$$\bar{\mathbf{S}}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*) = |W_n|^{-1} \int_{W_n} \Big[\frac{\boldsymbol{\lambda}'(\mathbf{u};\boldsymbol{\beta}^*)}{\lambda(\mathbf{u};\boldsymbol{\beta}^*)^{1/2}}\Big]^{\mathsf{T}} \Big[(\mathbf{I}+\mathbf{T}^s_{n,\boldsymbol{\theta}^*})^{-1}\frac{\boldsymbol{\lambda}'(\cdot;\boldsymbol{\beta}^*)}{\lambda(\cdot;\boldsymbol{\beta}^*)^{1/2}}\Big](\mathbf{u})\mathrm{d}\mathbf{u}$$

and $(\mathbf{I}+\mathbf{T}^s_{n,\boldsymbol{\theta}^*})^{-1}$ is a positive operator (see Section 3.1 in main text). Since $\bar{\boldsymbol{\Sigma}}_n = \bar{\mathbf{S}}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)$, C5 also implies

L3 $l = \liminf_n l_n > 0$ where for each $n$, $l_n$ denotes the minimal eigenvalue of $\bar{\boldsymbol{\Sigma}}_n$.

To prove the asymptotic normality of $|W_n|^{-1/2}\mathbf{e}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}^*)\bar{\boldsymbol{\Sigma}}_n^{-1/2}$, we assume that the following additional conditions are satisfied:

N1 $W_n = nA$ where $A \subset (0,1]\times(0,1]$ is the interior of a simple closed curve with nonempty interior.

N2 $\sup_p \frac{\alpha(p;k)}{p} = \mathrm{O}(k^{-\epsilon})$ for some $\epsilon > 2$, where $\alpha(p;k)$ is the strong mixing coefficient (Rosenblatt, 1956). For each $p$ and $k$, the mixing condition measures the dependence between $X\cap E_1$ and $X\cap E_2$ where $E_1$ and $E_2$ are arbitrary Borel subsets of $\mathbb{R}^2$ each of volume less than $p$ and at distance $k$ apart.

N3 for some $K_4 < \infty$ and $k = 3,4$,

$$\sup_{\mathbf{u}_1\in\mathbb{R}^2} \int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2} \big|Q_k(\mathbf{u}_1,\cdots,\mathbf{u}_k)\big|\mathrm{d}\mathbf{u}_2\cdots\mathrm{d}\mathbf{u}_k < K_4,$$

where $Q_k$ is the $k$-th order cumulant density function of $X$ (e.g. Guan and Loh, 2007).

Conditions N1-N3 correspond to conditions (2), (3) and (6), respectively, in Guan and Loh (2007). See this paper for a discussion of the conditions.

## 2. EXISTENCE OF A $|W_n|^{1/2}$ CONSISTENT $\hat{\boldsymbol{\beta}}_n$

We use Theorem 2 and Remark 1 in Waagepetersen and Guan (2009) to show the existence of a $|W_n|^{1/2}$ consistent sequence of solutions $\hat{\boldsymbol{\beta}}_n$. Let $\|\mathbf{A}\|_M = \sup_{ij}|a_{ij}|$ for a matrix $\mathbf{A} = [a_{ij}]_{ij}$. With $\mathbf{V}_n = |W_n|^{1/2}\bar{\boldsymbol{\Sigma}}_n^{1/2}$ we need to verify the following results:

R1 $\|\mathbf{V}_n^{-1}\|_M \to 0$.

R2 For any $d > 0$,

$$\sup_{\boldsymbol{\beta}:\|(\boldsymbol{\beta}-\boldsymbol{\beta}^*)\mathbf{V}_n\|\le d} \|\mathbf{V}_n^{-1}\big[\mathbf{J}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}) - \mathbf{J}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}^*)\big]\mathbf{V}_n^{-1}\|_M$$

converges to zero in probability.

R3 $\|\mathbf{J}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}^*)/|W_n| - \bar{\mathbf{S}}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\|_M$ converges to zero in probability.

R4 $\mathbf{e}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}^*)\mathbf{V}_n^{-1}$ is bounded in probability.

R5 $\liminf_n l_n > 0$ where

$$l_n = \inf_{\|\mathbf{x}\|=1} \mathbf{x}\bar{\boldsymbol{\Sigma}}_n^{-1/2}\bar{\mathbf{S}}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\bar{\boldsymbol{\Sigma}}_n^{-1/2}\mathbf{x}^{\mathsf{T}}.$$

We now demonstrate that R1-R5 hold under the conditions C1-C5 listed in Appendix 1. For each of the results below the required conditions or previous results are indicated in square brackets.

R1 [C3, L1-L3]: By C3, L1 and L2 the entries in $\bar{\boldsymbol{\Sigma}}_n$ are bounded from below and above. Moreover, by L3 the determinant of $\bar{\boldsymbol{\Sigma}}_n$ is bounded below by $l^p > 0$.

R2 [R1, C3, L1, L2, C4]: We show that

$$\sup_{(\boldsymbol{\theta},\boldsymbol{\beta}):\|(\boldsymbol{\theta}-\boldsymbol{\theta}^*,\boldsymbol{\beta}-\boldsymbol{\beta}^*)|W_n|^{1/2}\|\le d} \| |W_n|^{-1}\big[\mathbf{J}_{n,\boldsymbol{\theta}}(\boldsymbol{\beta}) - \mathbf{J}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\big]\|_M$$

converges to zero in probability. Note

$$|W_n|^{-1}\mathbf{J}_{n,\boldsymbol{\theta}}(\boldsymbol{\beta}) = \mathbf{L}_{n,\boldsymbol{\theta}}(\boldsymbol{\beta}) + \mathbf{M}_{n,\boldsymbol{\theta}}(\boldsymbol{\beta})$$

where

$$\mathbf{L}_{n,\boldsymbol{\theta}}(\boldsymbol{\beta}) = -\sum_{\mathbf{u}\in X} \mathbf{f}_{1,n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta}) \text{ and } \mathbf{M}_{n,\boldsymbol{\theta}}(\boldsymbol{\beta}) = \int_{\mathbb{R}^2} \mathbf{f}_{2,n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta})$$

with

$$\mathbf{f}_{1,n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta}) = \frac{1[\mathbf{u}\in W_n]}{|W_n|}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\beta}^\mathsf{T}}\boldsymbol{\phi}_{n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta})$$

and

$$\mathbf{f}_{2,n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta}) = \frac{1[\mathbf{u}\in W_n]}{|W_n|}\big[\lambda(\mathbf{u};\boldsymbol{\beta})\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\beta}^\mathsf{T}}\boldsymbol{\phi}_{n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta}) + \boldsymbol{\lambda}'(\mathbf{u};\boldsymbol{\beta})^\mathsf{T}\boldsymbol{\phi}_{n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta})\big].$$

Define

$$h_{i,n}(\mathbf{u}) = \sup_{(\boldsymbol{\theta},\boldsymbol{\beta}):\|(\boldsymbol{\theta}-\boldsymbol{\theta}^*,\boldsymbol{\beta}-\boldsymbol{\beta}^*)|W_n|^{1/2}\|\le d} |\mathbf{f}_{i,n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta}) - \mathbf{f}_{i,n,\boldsymbol{\theta}^*}(\mathbf{u},\boldsymbol{\beta}^*)|,\ i=1,2$$

and note that $h_{i,n}(\mathbf{u})$ converge to zero as $n\to\infty$. Then

$$\sup_{(\boldsymbol{\theta},\boldsymbol{\beta}):\|(\boldsymbol{\theta}-\boldsymbol{\theta}^*,\boldsymbol{\beta}-\boldsymbol{\beta}^*)|W_n|^{1/2}\|\le d} |\mathbf{M}_{n,\boldsymbol{\theta}}(\boldsymbol{\beta}) - \mathbf{M}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)| \le \int_{\mathbb{R}^2} h_{1,n}(\mathbf{u})\mathrm{d}\mathbf{u}$$

where the right hand side converges to zero by dominated convergence. Moreover,

$$\sup_{(\boldsymbol{\theta},\boldsymbol{\beta}):\|(\boldsymbol{\theta}-\boldsymbol{\theta}^*,\boldsymbol{\beta}-\boldsymbol{\beta}^*)|W_n|^{1/2}\|\le d} \big|\mathbf{L}_{n,\boldsymbol{\theta}}(\boldsymbol{\beta}) - \mathbf{L}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\big| \le \sum_{\mathbf{u}\in X} h_{2,n}(\mathbf{u}) \le \\ \Big|\sum_{\mathbf{u}\in X} h_{2,n}(\mathbf{u}) - \mathbb{E}\sum_{\mathbf{u}\in X} h_{2,n}(\mathbf{u})\Big| + \Big|\mathbb{E}\sum_{\mathbf{u}\in X} h_{2,n}(\mathbf{u})\Big|.$$

The first term on the right hand side converges to zero in probability by Chebyshev's inequality and the second term converges to zero by dominated convergence.

R3 [R1, L1, L2, C4]:

$$|W_n|^{-1}\mathbf{J}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}^*) - \bar{\mathbf{S}}_n(\boldsymbol{\beta}^*) = \\ |W_n|^{-1}\big[\mathbf{J}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}^*) - \mathbf{J}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\big] + \big[|W_n|^{-1}\mathbf{J}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*) - \bar{\mathbf{S}}_n(\boldsymbol{\beta}^*)\big]$$

It follows from the proof of R2 that the first term on the right hand side converges to zero in probability. The last term converges to zero in probability by Chebyshev's inequality.

R4 [C3, L1, L2, C4]: Since $\mathbb{V}\mathrm{ar}\mathbf{e}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\mathbf{V}_n^{-1}$ is the identity matrix, $\mathbf{e}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\mathbf{V}_n^{-1}$ is bounded in probability by Chebyshev's inequality. The result then follows by showing that $|W_n|^{-1/2}\big[\mathbf{e}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}^*) - \mathbf{e}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\big]$ converges to zero in probability. Let

$$\mathbf{f}_n(\boldsymbol{\theta}) = |W_n|^{-1}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}^\mathsf{T}}\mathbf{e}_{n,\boldsymbol{\theta}}(\boldsymbol{\beta}^*) = \\ |W_n|^{-1}\Big[\sum_{\mathbf{u}\in X\cap W_n}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}^\mathsf{T}}\boldsymbol{\phi}_{n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta}^*) - \int_{W_n}\lambda(\mathbf{u};\boldsymbol{\beta}^*)\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}^\mathsf{T}}\boldsymbol{\phi}_{n,\boldsymbol{\theta}}(\mathbf{u},\boldsymbol{\beta}^*)\mathrm{d}\mathbf{u}\Big].$$

Then

$$|W_n|^{-1/2}\big[\mathbf{e}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}^*) - \mathbf{e}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\big] = |W_n|^{1/2}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}^*)\mathbf{f}_n(\mathbf{t}_n)$$

where $\|\mathbf{t}_n - \boldsymbol{\theta}^*\| \le \|\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}^*\|$ and the factor $|W_n|^{1/2}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}^*)$ is bounded in probability. Further,

$$\mathbf{f}_n(\mathbf{t}_n) = \mathbf{f}_n(\mathbf{t}_n) - \mathbf{f}_n(\boldsymbol{\theta}^*) + \mathbf{f}_n(\boldsymbol{\theta}^*)$$

where $\mathbf{f}_n(\boldsymbol{\theta}^*)$ converges to zero in probability by Chebyshev's inequality and $\mathbf{f}_n(\mathbf{t}_n) - \mathbf{f}_n(\boldsymbol{\theta}^*)$ converges to zero in probability along the lines of the proof of R2.

R5 [C5, L3]: Follows directly from C5 and L3.

## 3. ASYMPTOTIC NORMALITY OF $|W_n|^{-1/2}\mathbf{e}_{n,\tilde{\boldsymbol{\theta}}_n}(\boldsymbol{\beta}^*)\boldsymbol{\Sigma}_n^{-1/2}$

By the proof of R4 it suffices to show that $|W_n|^{-1/2}\mathbf{e}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\bar{\boldsymbol{\Sigma}}_n^{-1/2}$ is asymptotically normal. To do so we use the blocking technique used in Guan and Loh (2007). Specifically, Condition N1 implies that there is a sequence of windows $W_n^B = \cup_{i=1}^{k_n} W_n^i$ given for each $n$ by a union of $m_n \times m_n$ sub squares $W_n^i$, $i = 1, \cdots, k_n$, such that $|W_n^B|/|W_n| \to 1$, $m_n = \mathrm{O}(n^\alpha)$ and the inter-distance between any two neighbouring sub squares is of order $n^\eta$ for some $4/(2+\epsilon) < \eta < \alpha < 1$. Let

$$\mathbf{e}^B_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}) = \sum_{\mathbf{u}\in X\cap W_n^B}\boldsymbol{\phi}_{n,\boldsymbol{\theta}^*}(\mathbf{u};\boldsymbol{\beta}) - \int_{W_n^B}\boldsymbol{\phi}_{n,\boldsymbol{\theta}^*}(\mathbf{u};\boldsymbol{\beta})\lambda(\mathbf{u};\boldsymbol{\beta})\mathrm{d}\mathbf{u} \equiv \sum_{i=1}^{k_n}\mathbf{e}^{B,i}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}),$$

where

$$\mathbf{e}^{B,i}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}) = \sum_{\mathbf{u}\in X\cap W_n^i}\boldsymbol{\phi}_{n,\theta^*}(\mathbf{u};\boldsymbol{\beta}) - \int_{W_n^i}\boldsymbol{\phi}_{n,\boldsymbol{\theta}^*}(\mathbf{u};\boldsymbol{\beta})\lambda(\mathbf{u};\boldsymbol{\beta})\mathrm{d}\mathbf{u}.$$

Define

$$\tilde{\mathbf{e}}^B_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}) = \sum_{i=1}^{k_n}\tilde{\mathbf{e}}^{B,i}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}),$$

where the $\tilde{\mathbf{e}}^{B,i}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta})$'s are independent and for each $i$ and $n$, $\tilde{\mathbf{e}}^{B,i}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta})$ is distributed as $\mathbf{e}^{B,i}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta})$. Let $\bar{\boldsymbol{\Sigma}}_n^B = |W_n^B|^{-1}\mathbb{V}\mathrm{ar}\mathbf{e}^B_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)$ and $\tilde{\boldsymbol{\Sigma}}_n^B = |W_n^B|^{-1}\mathbb{V}\mathrm{ar}\tilde{\mathbf{e}}^B_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)$. We need to verify the following results:

S1 $||\tilde{\mathbf{\Sigma}}_n^B - \bar{\mathbf{\Sigma}}_n^B||_M \to 0$ and $||\bar{\mathbf{\Sigma}}_n^B - \bar{\mathbf{\Sigma}}_n||_M \to 0$ as $n \to \infty$,

S2 $|W_n^B|^{-1/2}\tilde{\mathbf{e}}_{n,\theta^*}^B(\boldsymbol{\beta}^*)\left(\tilde{\mathbf{\Sigma}}_n^B\right)^{-1/2}$ is asymptotically standard normal,

S3 $|W_n^B|^{-1/2}\mathbf{e}_{n,\boldsymbol{\theta}^*}^B(\boldsymbol{\beta}^*)\left(\tilde{\mathbf{\Sigma}}_n^B\right)^{-1/2}$ has the same asymptotic distribution as $|W_n^B|^{-1/2}\tilde{\mathbf{e}}_{n,\boldsymbol{\theta}^*}^B(\boldsymbol{\beta}^*)\left(\bar{\mathbf{\Sigma}}_n^B\right)^{-1/2}$,

S4 $\||W_n^B|^{-1/2}\mathbf{e}_{n,\boldsymbol{\theta}^*}^B(\boldsymbol{\beta}^*) - |W_n|^{-1/2}\mathbf{e}_{n,\boldsymbol{\theta}^*}(\boldsymbol{\beta}^*)\|$ converges to zero in probability.

S1 [C2, C3, N1]: This follows from the proof of Theorem 2 in Guan and Loh (2007).

S2 [C2, C3, N3]: Conditions C2, C3 and N3 imply $\mathbb{E}[\tilde{\mathbf{e}}_{n,\boldsymbol{\theta}^*}^i(\boldsymbol{\beta})^4]$ is bounded (see the proof of Lemma 1 in Guan and Loh, 2007). Thus, S2 follows from an application of Lyapunov's central limit theorem.

S3 [N2]: this follows by bounding the difference between the characteristic functions of $|W_n^B|^{-1/2}\mathbf{e}_{n,\boldsymbol{\theta}^*}^B(\boldsymbol{\beta}^*)$ and $|W_n^B|^{-1/2}\tilde{\mathbf{e}}_{n,\boldsymbol{\theta}^*}^B(\boldsymbol{\beta}^*)$ using techniques in Ibramigov and Linnik (1971) and secondly applying the mixing condition N2, see also Guan et al. (2004).

S4 [C1-C3, C5, N1]: Recall that $|W_n^B|/|W_n| \to 1$ due to N1. By C5 we only need to show $\mathbb{V}\mathrm{ar}\big[\mathbf{e}_{n,\theta^*}(\boldsymbol{\beta}^*) - \mathbf{e}_{n,\boldsymbol{\theta}^*}^B(\boldsymbol{\beta}^*)\big]/|W_n| \to 0$. This is implied by conditions C1-C3 and $|W_n^B|/|W_n| \to 1$.

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