Estimating functions for inhomogeneous spatial point processes with incomplete covariate data

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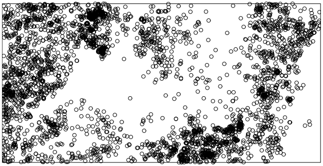
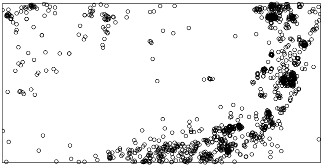
Data (Barro Colorado Island Forest Dynamics Plot)

Observation window: $S = [0, 1000] \times [0, 500]\text{m}^2$

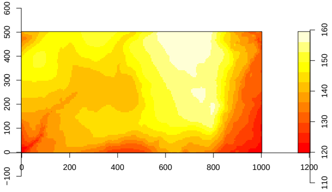
Beilschmiedia



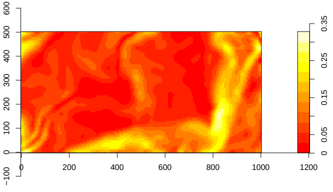
Ocotea



Elevation



Gradient norm (steepness)



Question: tree intensities related to elevation and gradient ?

Outline:

- estimation of log linear parameters for intensity function
- incomplete covariate data: deterministic approx. of score function
- Monte Carlo approx. of score
- distribution of parameter estimates
- numerical examples

Log-linear models for intensity function

$z(u) = \big(z_1(u), \ldots, z_p(u)\big)$ vector of covariates for each location u in observation window W.

E.g. $z(u) = (1, z_{\mathrm{elev}}(u), z_{\mathrm{grad}}(u))$ for rain forest example

Log-linear model for intensity function:

$$\lambda(u; \beta) = \exp(z(u)\beta^{\mathsf{T}})$$

Poisson process case: log likelihood function and derivatives

$\mathbf{x}$ observation of $\mathbf{X}$ Poisson$(W, \lambda(\cdot; \beta))$.

Density wrt. unit rate Poisson process:

$$f(\mathbf{x}; \beta) = \exp(|W| - \int_W \lambda(u; \beta)\mathrm{d}u) \prod_{u \in \mathbf{x}} \lambda(u; \beta)$$

log likelihood function:

$$l(\beta) = \sum_{u \in \mathbf{x}} z(u)\beta^\mathsf{T} - \int_W \lambda(u; \beta)\mathrm{d}u$$

Score function and Fisher information:

$$u(\beta) = \sum_{u \in \mathbf{x}} z(u) - \int_W z(u)\lambda(u; \beta)\mathrm{d}u \quad j(\beta) = \int_W z(u)^\mathsf{T} z(u)\lambda(u; \beta)\mathrm{d}u$$

$$\hat{\beta} \approx N(\beta, V) \quad V = j(\beta)^{-1}$$

Case of second-order reweighted stationary point processes

Solution of

$$u(\beta) = \sum_{u \in \mathbf{x}} z(u) - \int_W z(u)\lambda(u;\beta)\mathrm{d}u = 0$$

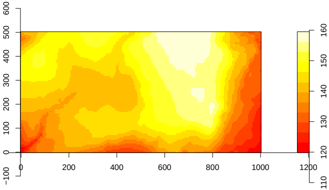
yields asymptotically normal estimate of β for wide class of non-Poisson processes satisfying certain mixing conditions.

Examples: second order reweighted stationary Poisson cluster processes and log Gaussian Cox processes (Waagepetersen, 2007, Guan and Loh, 2007).

(second order reweighted stationary: translation invariant pair correlation function).

Missing covariate data

Elevation covariate



interpolated from elevation observations on grid.

However, evaluating

$$u(\beta) = \sum_{u \in \mathbf{x}} z(u) - \int_W z(u)\lambda(u;\beta)\mathrm{d}u$$

requires $z(u)$ *observed* for any $u \in W$!

Approximations of log likelihood I

Suppose $z(u)$ observed at finite set of locations $\mathbf{Q} \subset W$.

Rathbun (1996) approximate

$$\int_W z(u)\lambda(u;\beta)\mathrm{d}u \approx \int_W \widehat{z(u)\lambda(u;}\beta)\mathrm{d}u$$

where $\widehat{z(u)\lambda(u;}\beta)$ unbiased prediction of $z(u)\lambda(u;\beta)$, $u \in W$ given $z(u)$, $u \in \mathbf{Q}$.

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Riemann approximation:

$$\int_W z(u)\lambda(u;\beta)\mathrm{d}u \approx \sum_{u\in\mathbf{Q}} w(u)z(u)\lambda(u;\beta)$$

where $w(u)$ quadrature weight for $u \in \mathbf{Q}$.

Approximations of log likelihood II: spatstat

Approximation of score function used in R package spatstat (Baddeley and Turner)

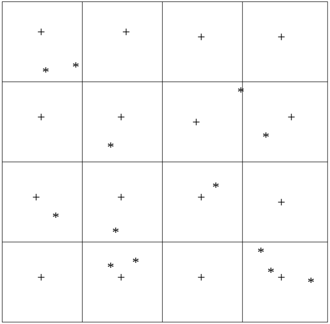
$$u(\beta) \approx u^{\mathsf{spat}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{Q}} z(u)\lambda(u;\beta)w(u)$$

but now $\mathbf{Q} = \mathbf{X} \cup \mathbf{D}$ *includes observed points* in addition to 'dummy' points $\mathbf{D}$.

Two types of weights: `grid` or `dirichlet`

Quadrature schemes in spatstat

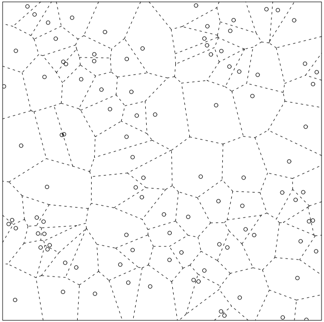
Grid



$w(u) = \frac{|C_v|}{\#(\mathbf{X} \cap C_v)+1}, \; u \in C_v$

where $W = \cup_{v \in \mathbf{D}} C_v$

Dirichlet



$w(u)$ area of *Dirichlet cell* for u in Dirichlet tesselation generated by $\mathbf{Q}$.

spatstat: relation to generalized linear models and iterative weighted least squares

Z: matrix with rows $z(u)$, $u \in \mathbf{X} \cup \mathbf{D}$

A: diagonal matrix with diagonal entries $w(u)\lambda(u;\beta)$, $u \in \mathbf{X} \cup \mathbf{D}$.

$$u^{\text{spat}}(\beta) = Z^{\mathsf{T}}\big(1[u \in x] - w(u)\lambda(u;\beta)\big)_{u \in \mathbf{X} \cup \mathbf{D}} = Z^{T} A y$$

where $y = A^{-1}\big(1[u \in x] - w(u)\lambda(u;\beta)\big)_{u \in \mathbf{X} \cup \mathbf{D}}$.

Derivative

$$\frac{\mathrm{d}}{\mathrm{d}\beta} u^{\text{spat}}(\beta) = Z^{\mathsf{T}} A Z$$

Newton-Raphson iterations

$$Z^{\mathsf{T}} A Z(\beta^{m+1} - \beta^{m}) = Z^{T} A y$$

equivalent to iterative weighted least squares (Berman and Turner, 1992) - hence implementation straightforward using e.g. `glm()` in `R`.

Distribution of parameter estimates from approximate score functions

?

Problem: hard to obtain distribution of parameter estimates from approximate score functions

Monte Carlo approximation of integral

Consider M *random* uniform dummy points on W (binomial point process $\mathbf{D}$ of intensity $M/|W|$).

Rathbun et al. (2006): Monte Carlo approx. of integral:

$$u^{rath}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{D}} \frac{z(u)\lambda(u;\beta)}{M/|W|}$$

CLT for Monte Carlo approximation:

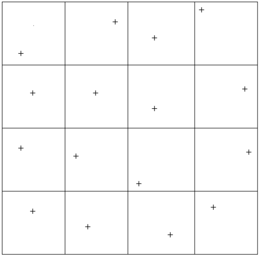
$$n^{1/2}\Big[\sum_{u \in \mathbf{D}_n} \frac{f(u)}{n\rho} - \int_W f(u)\mathrm{d}u\Big] \xrightarrow{d} N(0, G_f/\rho)$$

where

$$G_f = \int_W f(u)^{\mathsf{T}} f(u)\mathrm{d}u - \frac{1}{|W|}\int_W f(u)^{\mathsf{T}}\mathrm{d}u \int_W f(u)\mathrm{d}u.$$

Stratified dummy points

Alternative: one uniformly sampled dummy point in each cell (stratified dummy points)



Suppose covariates continuously differentiable. Then CLT

$$n^{1/2}\left[\sum_{u\in\mathbf{D}_n}\frac{f(u)}{n^{1/2}\rho}-\int_W f(u)\mathrm{d}u\right]\xrightarrow{d} N(0,G_f/\rho^2)$$

where $n^{1/2}\rho$ increasing intensity of dummy point process $\mathbf{D}_n$ and

$$G_f=\frac{1}{12}\int_W A_f(u)\mathrm{d}u \quad A_f(u_1,u_2)=\left[\frac{\partial f_i}{\partial u_1}\frac{\partial f_j}{\partial u_1}+\frac{\partial f_i}{\partial u_2}\frac{\partial f_j}{\partial u_2}\right]$$

(faster rate of convergence).

Distribution of parameter estimate

Consider increasing intensity asymptotics: intensities

$$\lambda_n(u;\beta) = n\lambda(u;\beta),\ \beta \in \mathbb{R}^p \quad \text{and} \quad \rho_n = n^k \rho$$

for observed $\mathbf{X}_n$ and dummy $\mathbf{D}_n$ ($k = 1$ (bin.) or $1/2$ (strat.))
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$$u_n(\beta) + n\Big[\int_W z(u)\lambda(u;\beta)\mathrm{d}u - \sum_{u \in \mathbf{D}_n} \frac{z(u)\lambda(u;\beta)}{n^k \rho}\Big]$$

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Note $\mathbf{X}_n \sim \cup_{i=1}^n \mathbf{X}^i$ where $\mathbf{X}^i$ *iid* Poisson processes $\lambda(u;\beta) \Rightarrow$ CLT.

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Note $\mathbf{X}_n \sim \cup_{i=1}^n \mathbf{X}^i$ where $\mathbf{X}^i$ *iid* Poisson processes $\lambda(u;\beta) \Rightarrow$ CLT.
Hence,

$$n^{-1/2}u_n^{rath}(\beta) \xrightarrow{d} N(0, j(\beta) + G_k/\rho^{1/k}),$$

$$n^{1/2}(\hat{\beta}_n - \beta) \xrightarrow{d} N(0,V + VG_kV/\rho^{1/k}) \quad V = j(\beta)^{-1}$$

Monte Carlo versions of `spatstat` (Waagepetersen, 2007)

D point process of dummy points of intensity ρ. Monte Carlo version of `dirichlet`

$$u^{\mathsf{dir}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{X} \cup \mathbf{D}} z(u)\lambda(u;\beta)\frac{1}{\lambda(u;\beta)+\rho}$$

(either binomial or stratified dummy points)

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Monte Carlo version of `grid`:

$$u^{\text{grid}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{u \in \mathbf{X} \cup \mathbf{D}} z(u)\lambda(u;\beta)\frac{1}{\rho(\#(\mathbf{X} \cap C_{v(u)}) + 1)}$$

($v(u) = v$ if $u \in C_v$, only stratified dummy points)

Advantage: implementation (IWLS) just as for usual `spatstat` except that weights depend on β for `dirichlet`.

Asymptotic distribution of parameter estimates I (Poisson process case)

Grid version

$$u^{\text{grid}}(\beta) = \sum_{u \in \mathbf{X}} z(u) - \sum_{v \in \mathbf{D}} \frac{z(u)\lambda(u;\beta) + \sum_{u \in \mathbf{X} \cap C_v} z(v)\lambda(v;\beta)}{\rho(\#(\mathbf{X} \cap C_v) + 1)}$$

Hence assuming continuously differentiable covariates

$$n^{-1/2} u_n^{\text{grid}}(\beta) \sim n^{-1/2} u_n^{\text{rath}}(\beta) \quad n \to \infty$$

and asymptotic covariance matrix becomes

$$V + V G_{1/2} V / \rho^2$$

where $g(u) = z(u)\lambda(u;\beta)$. Tends to MLE asymp. cov. V if $\rho \to \infty$.

Asymptotic distribution of parameter estimates II

Dirichlet estimating function:

$$u_n^{\text{dir}}(\beta) = \sum_{u \in \mathbf{X}_n} z(u) - \sum_{u \in \mathbf{X}_n \cup \mathbf{D}_n} z(u) \frac{\lambda(u;\beta)}{\lambda(u;\beta) + n^{k-1}\rho}$$

$$\sum_{u \in \mathbf{X}_n} z(u) \left(\frac{n^{k-1}\rho}{\lambda(u;\beta) + n^{k-1}\rho} \right) - \sum_{u \in \mathbf{D}_n} z(u) \frac{\lambda(u;\beta)}{\lambda(u;\beta) + n^{k-1}\rho}$$

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$$n^{-k+1/2} u_n^{\mathrm{dir}}(\beta) \xrightarrow{d} N(0, \rho^2 C_k + \rho^{2-1/k} H_k)$$

$$n^{-k} j_n^{\mathrm{dir}}(\beta) = -n^{-k} \frac{\mathrm{d}}{\mathrm{d}\beta} u_n^{\mathrm{dir}}(\beta) \xrightarrow{p} \rho F_k$$

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Hence

$$n^{1/2}(\hat{\beta}_n - \beta) \xrightarrow{d} N(0, F_k^{-1} C_k F_k^{-1} + F_k^{-1} H_k F_k^{-1} / \rho^{1/k})$$

Case $k = 1/2$: $F_k^{-1} C_k F^{-1}$ differs from V even when $\rho \to \infty$!

Cluster processes

Consider cluster process $\mathbf{X} = \mathbf{X}_{c \in \mathbf{Y}}$ where $\mathbf{Y}$ stationary Poisson point process of intensity $\kappa > 0$.

Given $\mathbf{Y}$, clusters $\mathbf{X}_c$ are independent Poisson processes with intensity functions

$$\lambda_c(u) = \alpha \exp(z_{2:p}(u)\beta_{2:p}^{\mathsf{T}})h(u - c)$$

Intensity function of $\mathbf{X}$ is then of log-linear form $\exp(z(u)\beta^{\mathsf{T}})$ where $\beta_1 = \log(\kappa\alpha)$ and $z_1(u) = 1$.

Estimates $\hat{\beta}_n$ using Monte Carlo `spatstat` still asymptotically normal (consider increasing 'mother intensity' $\kappa_n = n\kappa$).

Asymptotic covariance=Asymptotic covariance for Poisson + extra term due to clustering.

Numerical example: Poisson process

Case of Poisson process with covariate vector $(1, z_{\text{elev}})$ $\beta_{\text{elev}} = 0.1$.

Ratios of asymptotic standard errors for estm. funct. estimate $\hat{\beta}_{\text{elev}}$ relative to asymptotic standard error for MLE.

	bin.				str.			
Est. fct. \ q	0.25	1	10	100	0.25	1	10	100
u^{rath} (u^{grid})	2.47	1.51	1.06	1.01	1.08	1.01	1.00	1.00
u^{dir}	2.12	1.43	1.06	1.01	1.56	1.53	1.53	1.53

$q = \#\mathbf{D}/\#(\mathbf{X} \cup \mathbf{D})$ proportion of dummy points.

u^{dir} slightly better than Rathbuns Monte Carlo approximation u^{rath} but not useful in case of stratified dummy points.

Numerical example: Clustered point process

Ratios of standard errors for $\hat{\beta}_{\text{elev}}$ (relative to complete covariate data case) in case of clustered point process with varying values of 'mother' intensity and varying numbers M of dummy points.

	κ	8e-5			8e-4			8e-3	
	M	450	800	1800	450	800	1800	450	800
u^{rath}	(bin.)	1.06	1.03	1.01	1.44	1.26	1.12	2.49	1.98
u^{dir}	(bin.)	1.01	0.99	0.97	1.35	1.20	1.08	2.32	1.86
u^{grid}	(str.)	1.00	1.00	1.00	1.04	1.01	1.00	1.17	1.06

In highly clustered case $\kappa = 8\text{e-}5$ not big loss of efficiency due to missing covariate data: term due to clustering V_{clust} dominating in asymptotic covariance matrix

$$V + V_{\text{incompl}} + V_{\text{clust}}$$

Perspective

- methodology available for handling missing covariate
- implementation straightforward
- random sampling schemes required
- need to take this into account in future experiments

References:

Waagepetersen, R. (2007). An estimating function approach to inference for inhomogeneous Neyman-Scott processes, *Biometrics*, **63**, 252-258.

Waagepetersen, R. (2007) Estimating functions for inhomogeneous spatial point processes with incomplete covariate data, submitted.