An introduction to statistics for spatial point processes

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Lectures:

1. Intro to point processes, moment measures and the Poisson process
2. Cox and cluster processes
3. The conditional intensity and Markov point processes
4. Likelihood-based inference and MCMC

Aim: overview of stats for spatial point processes - and spatial point process theory as needed.

Not comprehensive: the most fundamental topics and our favorite things.

1. Intro to point processes, moment measures and the Poisson process

2. Cox and cluster processes

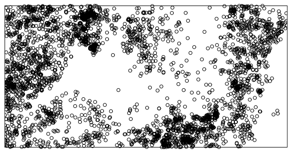
3. The conditional intensity and Markov point processes

4. Likelihood-based inference and MCMC

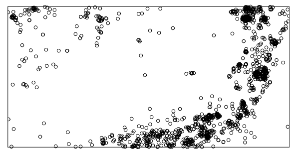
Data example (Barro Colorado Island Plot)

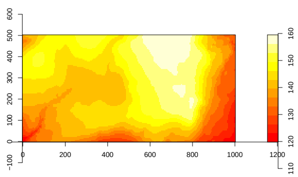
Observation window $W = [0, 1000] \times [0.500]\text{m}^2$

Beilschmiedia

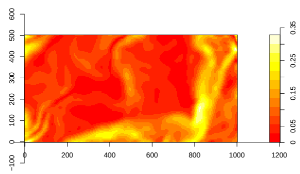


Ocotea





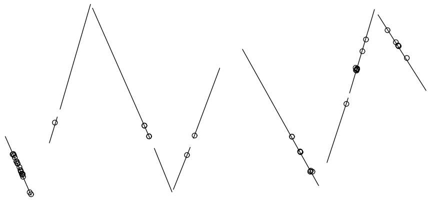
Elevation



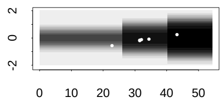
Gradient norm (steepness)

Sources of variation: elevation and gradient covariates *and* clustering due to seed dispersal.

Whale positions



Close up:



Aim: estimate whale intensity λ

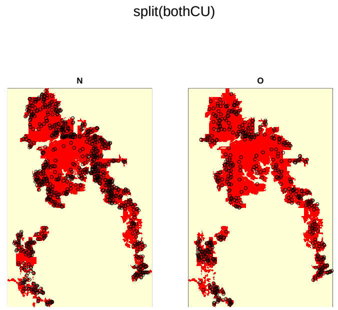
Observation window W = narrow strips around transect lines

Varying detection probability: inhomogeneity (thinning)

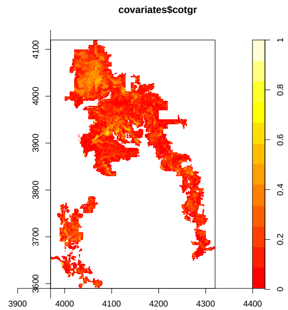
Variation in prey intensity: clustering

Golden plover birds in Peak District

Birds in 1990 and 2005



Cotton grass covariate



Change in spatial distribution of birds between 1990 and 2005 ?

What is a spatial point process ?

Definitions:

1. a locally finite random subset $\mathbf{X}$ of $\mathbb{R}^2$ ($\#(\mathbf{X} \cap A)$ finite for all bounded subsets $A \subset \mathbb{R}^2$)
2. a random counting measure N on $\mathbb{R}^2$

Equivalent provided no multiple points: ($N(A) = \#(\mathbf{X} \cap A)$)

This course: appeal to 1. and skip measure-theoretic details.

In practice distribution specified by an explicit construction (this and second lecture) or in terms of a probability density (third lecture).

Moments of a spatial point process

Fundamental characteristics of point process: mean and covariance of counts $N(A) = \#(\mathbf{X} \cap A)$.

Intensity measure μ:

$$\mu(A) = \mathbb{E}N(A), \quad A \subseteq \mathbb{R}^2$$

In practice often given in terms of *intensity function*

$$\mu(A) = \int_A \rho(u)\mathrm{d}u$$

Infinitesimal interpretation: $N(A)$ binary variable (presence or absence of point in A) when A very small. Hence

$$\rho(u)\mathrm{d}A \approx \mathbb{E}N(A) \approx P(\mathbf{X} \text{ has a point in A})$$

Second-order moments

Second order factorial moment measure:

$$\mu^{(2)}(A \times B) = \mathrm{E} \sum_{u,v \in \mathbf{X}}^{\neq} \mathbf{1}[u \in A,\ v \in B] \qquad A, B \subseteq \mathbb{R}^2$$
$$= \int_A \int_B \rho^{(2)}(u, v)\,\mathrm{d}u\,\mathrm{d}v$$

where $\rho^{(2)}(u, v)$ is the *second order product density*

NB (exercise):

$$\mathbb{C}\mathrm{ov}[N(A), N(B)] = \mu^{(2)}(A \times B) + \mu(A \cap B) - \mu(A)\mu(B)$$

Campbell formula (by standard proof)

$$\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} h(u, v) = \iint h(u, v)\rho^{(2)}(u, v)\mathrm{d}u\mathrm{d}v$$

Pair correlation function and K-function

Infinitesimal interpretation of $\rho^{(2)}$ ($u \in A$,$v \in B$):

$$\rho^{(2)}(u, v) dA dB \approx P(\mathbf{X} \text{ has a point in each of } A \text{ and } B)$$

Pair correlation: tendency to cluster or repel relative to case where points occur independently of each other

$$g(u, v) = \frac{\rho^{(2)}(u, v)}{\rho(u)\rho(v)}$$

Suppose $g(u, v) = g(u - v)$. K-function (cumulative quantity):

$$K(t) := \int_{\mathbb{R}^2} 1[\|u\| \leq t] g(u) \mathrm{d}u = \frac{1}{|B|} \mathbb{E} \sum_{\substack{u \in \mathbf{X} \cap B \\ v \in \mathbf{X}}}^{\neq} \frac{1[\|u - v\| \leq t]}{\rho(u)\rho(v)}$$

($\Rightarrow$ non-parametric estimation if $\rho(u)\rho(v)$ known)

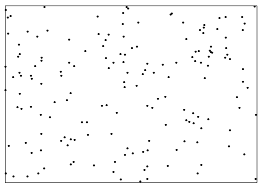
The Poisson process

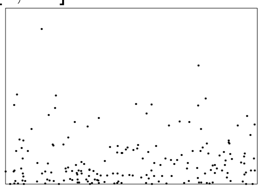
Assume μ locally finite measure on $\mathbb{R}^2$ with density ρ.

$\mathbf{X}$ is a Poisson process with intensity measure μ if for any bounded region B with $\mu(B) > 0$:

1. $N(B) \sim \text{Poisson}(\mu(B))$
2. Given $N(B)$, points in $\mathbf{X} \cap B$ i.i.d. with density $\propto \rho(u)$, $u \in B$

$B = [0, 1] \times [0, 0.7]$:





Homogeneous: $\rho = 150/0.7$ Inhomogeneous: $\rho(x, y) \propto \mathrm{e}^{-10.6y}$

Existence of Poisson process on $\mathbb{R}^2$: use definition on disjoint partitioning $\mathbb{R}^2 = \cup_{i=1}^{\infty} B_i$ of bounded sets B_i.

Independent scattering:

- $A, B \subseteq \mathbb{R}^2$ disjoint $\Rightarrow$ $\mathbf{X} \cap A$ and $\mathbf{X} \cap B$ independent
- $\rho^{(2)}(u, v) = \rho(u)\rho(v)$ and $g(u, v) = 1$

Characterization in terms of void probabilities

The distribution of $\mathbf{X}$ is uniquely determined by the void probabilities $P(\mathbf{X} \cap B = \emptyset)$, for bounded subsets $B \subseteq \mathbb{R}^2$.

Intuition: consider very fine subdivision of observation window – then at most one point in each cell and probabilities of absence/presence determined by void probabilities.

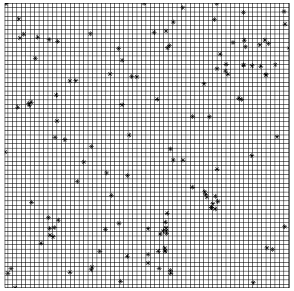
Hence, a point process $\mathbf{X}$ with intensity measure μ is a Poisson process if and only if

$$P(\mathbf{X} \cap B = \emptyset) = \exp(-\mu(B))$$

for any bounded subset B.

Homogeneous Poisson process as limit of Bernouilli trials

Consider disjoint subdivision $W = \cup_{i=1}^{n} C_i$ where $|C_i| = |W|/n$. With probability $\rho|C_i|$ a uniform point is placed in C_i.



Number of points in subset A is $b(n|A|/|W|, \rho|W|/n)$ which converges to a Poisson distribution with mean $\rho|A|$.

Hence, Poisson process default model when points occur independently of each other.

Exercises

1. Show that the covariance between counts $N(A)$ and $N(B)$ is given by

   $$\mathbb{C}\mathrm{ov}[N(A), N(B)] = \mu^{(2)}(A \times B) + \mu(A \cap B) - \mu(A)\mu(B)$$

2. Show that

   $$K(t) := \int_{\mathbb{R}^2} 1[\|u\| \leq t] g(u) \mathrm{d}u = \frac{1}{|B|} \mathbb{E} \sum_{\substack{u \in \mathbf{X} \cap B \\ v \in \mathbf{X}}}^{\neq} \frac{1[\|u - v\| \leq t]}{\rho(u)\rho(v)}$$

   What is $K(t)$ for a Poisson process ?

   (Hint: use the Campbell formula)

3. (Practical `spatstat` exercise) Compute and interpret a non-parametric estimate of the K-function for the spruces data set.

   (Hint: load `spatstat` using `library(spatstat)` and the spruces data using `data(spruces)`. Consider then the `Kest()` function.)

Distribution and moments of Poisson process

$\mathbf{X}$ a Poisson process on S with $\mu(S) = \int_S \rho(u) \mathrm{d}u < \infty$ and F set of finite point configurations in S.

By definition of a Poisson process

$$P(\mathbf{X} \in F) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu(S)}}{n!} \int_{S^n} 1[\{x_1, x_2, \ldots, x_n\} \in F] \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \ldots \mathrm{d}x_n \tag{1}$$

Similarly,

$$\mathbb{E}h(\mathbf{X}) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu(S)}}{n!} \int_{S^n} h(\{x_1, x_2, \ldots, x_n\}) \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

Proof of independent scattering (finite case)

Consider bounded $A, B \subseteq \mathbb{R}^2$.

$\mathbf{X} \cap (A \cup B)$ Poisson process. Hence

$$
\begin{aligned}
& P(\mathbf{X} \cap A \in F, \mathbf{X} \cap B \in G) \quad (\mathbf{x} = \{x_1, \dots, x_n\}) \\
= & \sum_{n=0}^{\infty} \frac{e^{-\mu(A\cup B)}}{n!} \int_{(A\cup B)^n} 1[\mathbf{x} \cap A \in F, \mathbf{x} \cap B \in G] \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \dots \mathrm{d}x_n \\
= & \sum_{n=0}^{\infty} \frac{e^{-\mu(A\cup B)}}{n!} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \int_{A^m} 1[\{x_1, x_2, \dots, x_m\} \in F] \\
& \int_{B^{n-m}} 1[\{x_{m+1}, \dots, x_n\} \in G] \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \dots \mathrm{d}x_n \\
= & \text{(interchange order of summation and sum over } m \text{ and } k = n - m) \\
& P(\mathbf{X} \cap A \in F) P(\mathbf{X} \cap B \in G)
\end{aligned}
$$

Superpositioning and thinning

If $\mathbf{X}_1, \mathbf{X}_2, \dots$ are independent Poisson processes (ρ_i), then *superposition* $\mathbf{X} = \cup_{i=1}^{\infty} \mathbf{X}_i$ is a Poisson process with intensity function $\rho = \sum_{i=1}^{\infty} \rho_i(u)$ (provided ρ integrable on bounded sets).

Conversely: *Independent π-thinning* of Poisson process $\mathbf{X}$: independent retain each point u in $\mathbf{X}$ with probability $\pi(u)$. Thinned process $\mathbf{X}_{\text{thin}}$ and $\mathbf{X} \setminus \mathbf{X}_{\text{thin}}$ are independent Poisson processes with intensity functions $\pi(u)\rho(u)$ and $(1 - \pi(u))\rho(u)$.

(Superpositioning and thinning results most easily verified using void probability characterization of Poisson process, see M & W, 2003)

For general point process $\mathbf{X}$: thinned process $\mathbf{X}_{\text{thin}}$ has product density $\pi(u)\pi(v)\rho^{(2)}(u, v)$ - hence g and K invariant under independent thinning.

Density (likelihood) of a finite Poisson process

$\mathbf{X}_1$ and $\mathbf{X}_2$ Poisson processes on S with intensity functions ρ_1 and ρ_2 where $\int_S \rho_2(u)\mathrm{d}u < \infty$ and $\rho_2(u) = 0 \Rightarrow \rho_1(u) = 0$. Define $0/0 := 0$. Then

$$
\begin{aligned}
& P(\mathbf{X}_1 \in F) \\
= & \sum_{n=0}^{\infty} \frac{e^{-\mu_1(S)}}{n!} \int_{S^n} 1[\mathbf{x} \in F] \prod_{i=1}^{n} \rho_1(x_i) \mathrm{d}x_1 \dots \mathrm{d}x_n \quad (\mathbf{x} = \{x_1, \dots, x_n\}) \\
= & \sum_{n=0}^{\infty} \frac{e^{-\mu_2(S)}}{n!} \int_{S^n} 1[\mathbf{x} \in F] e^{\mu_2(S) - \mu_1(S)} \prod_{i=1}^{n} \frac{\rho_1(x_i)}{\rho_2(x_i)} \prod_{i=1}^{n} \rho_2(x_i) \mathrm{d}x_1 \dots \mathrm{d}x_n \\
= & \mathbb{E}(1[\mathbf{X}_2 \in F] f(\mathbf{X}_2))
\end{aligned}
$$

where

$$f(\mathbf{x}) = e^{\mu_2(S) - \mu_1(S)} \prod_{i=1}^{n} \frac{\rho_1(x_i)}{\rho_2(x_i)}$$

Hence f is a density of $\mathbf{X}_1$ with respect to distribution of $\mathbf{X}_2$.

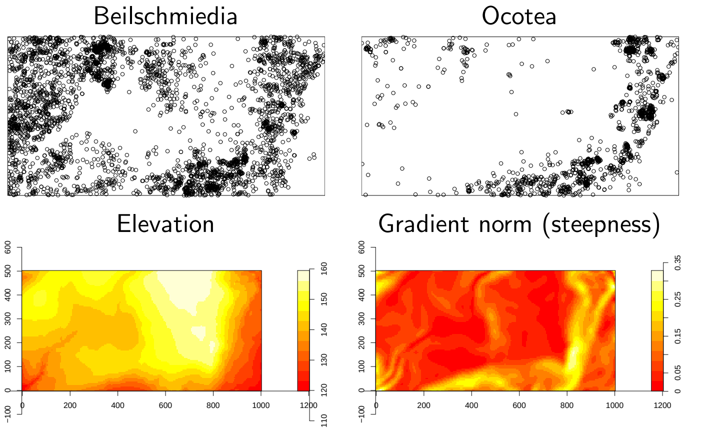
In particular (if S bounded): $\mathbf{X}_1$ has density

$$f(\mathbf{x}) = e^{\int_S (1 - \rho_1(u)) \mathrm{d}u} \prod_{i=1}^{n} \rho_1(x_i)$$

with respect to unit rate Poisson process ($\rho_2 = 1$).

Data example: tropical rain forest trees

Observation window $W = [0, 1000] \times [0, 500]$



Sources of variation: elevation and gradient covariates *and* possible clustering/aggregation due to unobserved covariates and/or seed dispersal.

Inhomogeneous Poisson process

Log linear intensity function

$$\rho(u;\beta) = \exp(z(u)\beta^{\mathsf{T}}), \quad z(u) = (1, z_{\mathrm{elev}}(u), z_{\mathrm{grad}}(u))$$

Estimate β from Poisson log likelihood (`spatstat`)

$$\sum_{u \in \mathbf{X} \cap W} z(u)\beta^{\mathsf{T}} - \int_W \exp(z(u)\beta^{\mathsf{T}})\mathrm{d}u \quad (W = \text{ observation window})$$

Model check using edge-corrected estimate of K-function

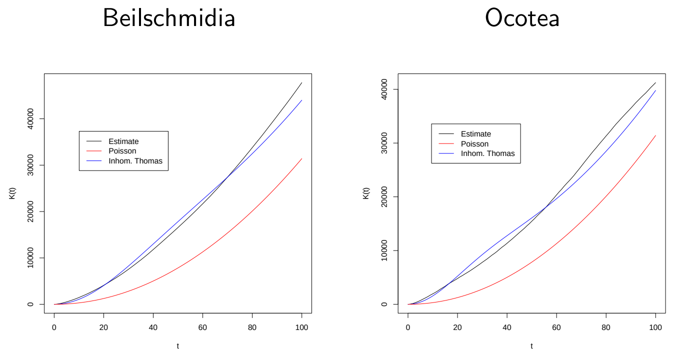
$$\hat{K}(t) = \sum_{u,v \in \mathbf{X} \cap W}^{\neq} \frac{1[\|u-v\| \leq t]}{\rho(u;\hat{\beta})\rho(v;\hat{\beta})|W \cap W_{u-v}|}$$

W_{u-v} translated version of W. $|A|$: area of $A \subset \mathbb{R}^2$.

Implementation in spatstat

```
> bei=ppp(beilpe$X,beilpe$Y,xrange=c(0,1000),yrange=c(0,500))
> beifit=ppm(bei,~elev+grad,covariates=list(elev=elevim,
  grad=gradim))
> coef(beifit) #parameter estimates
(Intercept)        elev        grad
-4.98958664  0.02139856  5.84202684
> asympcov=vcov(beifit) #asymp. covariance matrix
> sqrt(diag(asympcov)) #standard errors
(Intercept)        elev        grad
0.017500262 0.002287773 0.255860860
> rho=predict.ppm(beifit)
> Kbei=Kinhom(bei,rho) #warning: problem with large data sets.
> myKbei=myKest(cbind(bei$x,bei$y),rho,100,3,1000,500,F) #my own
                                                           #procedure
```


K-functions



Poisson process: $K(t) = \pi t^2$ (since $g = 1$) less than K functions for data. Hence Poisson process models not appropriate.

Exercises

1. Check that the Poisson expansion (1) indeed follows from the definition of a Poisson process.
2. Compute the second order product density for a Poisson process $\mathbf{X}$.

   (Hint: compute second order factorial measure using the Poisson expansion for $\mathbf{X} \cap (A \cup B)$ for bounded $A, B \subseteq \mathbb{R}^2$.)
3. (if time) Assume that $\mathbf{X}$ has second order product density $\rho^{(2)}$ and show that g (and hence K) is invariant under independent thinning (note that a heuristic argument follows easy from the infinitesimal interpretation of $\rho^{(2)}$).

   (Hint: introduce random field $\mathbf{R} = \{R(u) : u \in \mathbb{R}^2\}$, of independent uniform random variables on $[0, 1]$, and independent of $\mathbf{X}$, and compute second order factorial measure for thinned process $\mathbf{X}_{\text{thin}} = \{u \in \mathbf{X} | R(u) \leq p(u)\}$.)

Solution: second order product density for Poisson

$$
\begin{aligned}
&\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} 1[u \in A, v \in B] \\
=&\sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu(A \cup B)}}{n!} \int_{(A\cup B)^n} \sum_{u,v \in \mathbf{X}}^{\neq} 1[u \in A, v \in B] \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\
=&\sum_{n=2}^{\infty} \frac{\mathrm{e}^{-\mu(A \cup B)}}{n!} 2\binom{n}{2} \int_{(A\cup B)^n} \int_{(A\cup B)^n} 1[x_1 \in A, x_2 \in B] \prod_{i=1}^{n} \rho(x_i) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\
=&\sum_{n=2}^{\infty} \frac{\mathrm{e}^{-\mu(A \cup B)}}{(n-2)!} \mu(A)\mu(B)\mu(A \cup B)^{n-2} \\
=&\mu(A)\mu(B) = \int_{A \times B} \rho(u)\rho(v) \mathrm{d}u \mathrm{d}v
\end{aligned}
$$

Solution: invariance of g (and K) under thinning

Since $\mathbf{X}_{\text{thin}} = \{u \in \mathbf{X} : R(u) \leq p(u)\}$,

$$
\begin{aligned}
&\mathbb{E} \sum_{u,v \in \mathbf{X}_{\text{thin}}}^{\neq} 1[u \in A, v \in B] \\
=&\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} 1[R(u) \leq p(u), R(v) \leq p(v), u \in A, v \in B] \\
=&\mathbb{E}\, \mathbb{E}\Big[\sum_{u,v \in \mathbf{X}}^{\neq} 1[R(u) \leq p(u), R(v) \leq p(v), u \in A, v \in B] \,\Big|\, \mathbf{X} \Big] \\
=&\mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} p(u)p(v) 1[u \in A, v \in B] \\
=&\int_A \int_B p(u)p(v)\rho^{(2)}(u,v) \mathrm{d}u \mathrm{d}v
\end{aligned}
$$

Cox processes

X is a *Cox process* driven by the random intensity function Λ if, conditional on $\Lambda = \lambda$, **X** is a Poisson process with intensity function λ.

Calculation of intensity and product density:

$$\rho(u) = \mathbb{E}\Lambda(u), \quad \rho^{(2)}(u, v) = \mathbb{E}[\Lambda(u)\Lambda(v)]$$

$$\mathbb{C}\mathrm{ov}(\Lambda(u), \Lambda(v)) > 0 \Leftrightarrow g(u, v) > 1 \quad \text{(clustering)}$$

Overdispersion for counts:

$$\mathbb{V}\mathrm{ar} N(A) = \mathbb{E}\mathbb{V}\mathrm{ar}[N(A) \mid \Lambda] + \mathbb{V}\mathrm{ar}\mathbb{E}[N(A) \mid \Lambda] = \mathbb{E}N(A) + \mathbb{V}\mathrm{ar}\mathbb{E}[N(A) \mid \Lambda]$$

Log Gaussian Cox process (LGCP)

- Poisson log linear model: $\log \rho(u) = z(u)\beta^{\mathsf{T}}$
- LGCP: in analogy with random effect models, take

$$\log \Lambda(u) = z(u)\beta^{\mathsf{T}} + \Psi(u)$$

  where $\mathbf{\Psi} = (\Psi(u))_{u \in \mathbb{R}^2}$ is a zero-mean Gaussian process
- Often sufficient to use power exponential covariance functions:

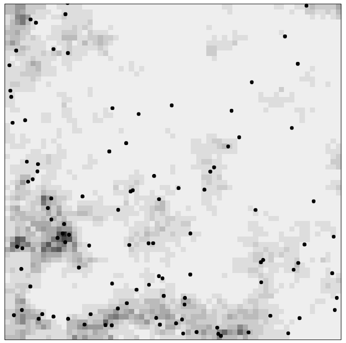
$$c(u, v) \equiv \mathbb{C}\mathrm{ov}[\Psi(u), \Psi(v)] = \sigma^2 \exp\left(-\|u - v\|^{\delta}/\alpha\right),$$

  $\sigma, \alpha > 0, \ 0 \le \delta \le 2$ (or linear combinations)
- Tractable product densities

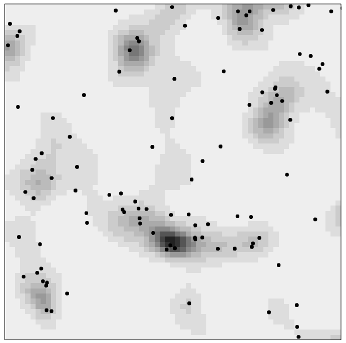
$$\rho(u) = \mathbb{E}\Lambda(u) = \mathrm{e}^{z(u)\beta^{\mathsf{T}}} \mathbb{E}\mathrm{e}^{\Psi(u)} = \exp\left(z(u)\beta^{\mathsf{T}} + c(u, u)/2\right)$$

$$g(u, v) = \frac{\mathbb{E}[\Lambda(u)\Lambda(v)]}{\rho(u)\rho(v)} = \ldots = \exp(c(u, v))$$

Two simulated homogeneous LGCP's



Exponential covariance function



Gaussian covariance function

Cluster processes

M 'mother' point process of cluster centres. Given **M**, $\mathbf{X}_m$, $m \in M$ are 'offspring' point processes (clusters) centered at m.

Intensity function for $\mathbf{X}_m$: $\alpha f(m, u)$ where f probability density and α expected size of cluster.

Cluster process:

$$\mathbf{X} = \cup_{m \in \mathbf{M}} \mathbf{X}_m$$

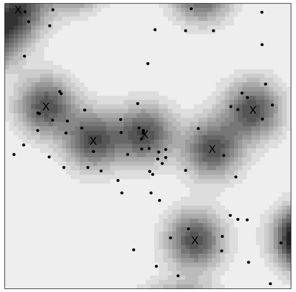
By superpositioning: if cond. on **M**, the $\mathbf{X}_m$ are independent Poisson processes, then **X** Cox process with random intensity function

$$\Lambda(u) = \alpha \sum_{m \in \mathbf{M}} f(m, u)$$

Nice expressions for intensity and product density if **M** Poisson on $\mathbb{R}^2$ with intensity function $\rho(\cdot)$ (Campbell):

$$\mathbb{E}\Lambda(u) = \mathbb{E}\alpha \sum_{m \in \mathbf{M}} f(m, u) = \alpha \int f(m, u)\rho(m)\mathrm{d}m \quad (= \kappa\alpha \text{ if } \rho(\cdot) = \kappa \text{ and } f(m, u) = f(u - m))$$

Example: modified Thomas process



Mothers (crosses) stationary Poisson point process $\mathbf{M}$ with intensity $\kappa > 0$.

Offspring $\mathbf{X} = \cup_m \mathbf{X}_m$ distributed around mothers according to bivariate isotropic Gaussian density f.

ω: standard deviation of Gaussian density
α: Expected number of offspring for each mother.

Cox process with random intensity function:

$$\Lambda(u) = \alpha \sum_{m \in \mathbf{M}} f(u - m; \omega)$$

Inhomogeneous Thomas process

$z_{1:p}(u) = (z_1(u), \ldots, z_p(u))$ vector of p nonconstant covariates.
$\beta_{1:p} = (\beta_1, \ldots \beta_p)$ regression parameter.

Random intensity function:

$$\Lambda(u) = \alpha \exp(z(u)_{1:p} \beta_{1:p}^{\mathsf{T}}) \sum_{m \in \mathbf{M}} f(u - m; \omega)$$

Rain forest example:

$$z_{1:2}(u) = (z_{\text{elev}}(u), z_{\text{grad}}(u))$$

elevation/gradient covariate.

Density of a Cox process

- Restricted to a bounded region W, the density is

$$f(\mathbf{x}) = \mathbb{E}\left[\exp\left(|W| - \int_W \Lambda(u)\,\mathrm{d}u\right) \prod_{u \in \mathbf{X}} \Lambda(u)\right]$$

- Not on closed form
- Fourth lecture: likelihood-based inference (missing data MCMC approach)
- Now: simulation free estimation

Parameter Estimation: regression parameters

Intensity function for inhomogeneous Thomas ($\rho(\cdot) = \kappa$):

$$\rho_\beta(u) = \kappa\alpha \exp(z(u)_{1:p}\beta_{1:p}^{\mathsf{T}}) = \exp(z(u)\beta^{\mathsf{T}})$$
$$z(u) = (1, z_{1:p}(u)) \quad \beta = (\log(\kappa\alpha), \beta_{1:p})$$

Consider indicators $N_i = \mathbf{1}[\mathbf{X} \cap C_i \neq \emptyset]$ of occurrence of points in disjoint C_i ($W = \cup C_i$) where $P(N_i = 1) \approx \rho_\beta(u_i)\mathrm{d}C_i$, $u_i \in C_i$

Limit ($\mathrm{d}C_i \to 0$) of composite log likelihood

$$\prod_{i=1}^n (\rho_\beta(u_i)\mathrm{d}C_i)^{N_i}(1 - \rho_\beta(u_i)\mathrm{d}C_i)^{1-N_i} \equiv \prod_{i=1}^n \rho_\beta(u_i)^{N_i}(1 - \rho_\beta(u_i)\mathrm{d}C_i)^{1-N_i}$$

is

$$l(\beta) = \sum_{u \in \mathbf{X} \cap W} \log \rho(u; \beta) - \int_W \rho(u; \beta)\,\mathrm{d}u$$

Maximize using `spatstat` to obtain $\hat{\beta}$.

Asymptotic distribution of regression parameter estimates

Assume increasing mother intensity: $\kappa = \kappa_n = n\tilde{\kappa} \to \infty$ and $\mathbf{M} = \cup_{i=1}^n \mathbf{M}_i$, $\mathbf{M}_i$ independent Poisson processes of intensity $\tilde{\kappa}$.

Score function asymptotically normal:

$$\frac{1}{\sqrt{n}} \frac{\mathrm{d}l(\beta)}{\mathrm{d}\log\alpha \mathrm{d}\beta_{1:p}} = \frac{1}{\sqrt{n}} \left(\sum_{u \in \mathbf{X} \cap W} z(u) - n\tilde{\kappa}\alpha \int_W z(u) \exp(z(u)_{1:p}\beta_{1:p}^\mathsf{T}) \mathrm{d}u \right)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big[\sum_{m \in \mathbf{M}_i} \sum_{u \in \mathbf{X}_m \cap W} z(u) - \tilde{\kappa}\alpha \int_W \exp(z_{1:p}(u)\beta_{1:p}^\mathsf{T}) \mathrm{d}u \Big] \approx N(0, V)$$

where $V = \mathbb{V}\mathrm{ar} \sum_{m \in \mathbf{M}_i} \sum_{u \in \mathbf{X}_m \cap W} z(u)$ ($\mathbf{X}_m$ offspring for mother m).

By standard results for estimating functions (J observed information for Poisson likelihood):

$$\sqrt{\kappa_n}\big[(\log(\hat{\alpha}), \hat{\beta}_{1:p}) - (\log\alpha, \beta_{1:p})\big] \approx N(0, J^{-1}VJ^{-1})$$

Parameter Estimation: clustering parameters

Theoretical expression for (inhomogeneous) K-function:

$$K(t; \kappa, \omega) = \pi t^2 + \big(1 - \exp(-t^2/(2\omega)^2)\big)/\kappa.$$

Estimate κ and ω by matching theoretical K with semi-parametric estimate (minimum contrast)

$$\hat{K}(t) = \sum_{u,v \in \mathbf{X} \cap W}^{\neq} \frac{1[\|u - v\| \le t]}{\lambda(u; \hat{\beta})\lambda(v; \hat{\beta})|W \cap W_{u-v}|}$$

Results for Beilschmiedia

Parameter estimates and confidence intervals (Poisson in red).

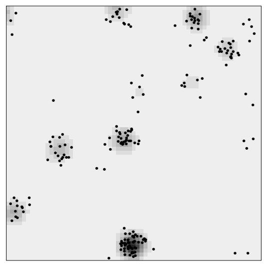
Elevation		Gradient		κ	α	ω
0.02	[-0.02,0.06]	5.84	[0.89,10.80]	8e-05	85.9	20.0
	[0.02,0.03]		[5.34,6.34]			

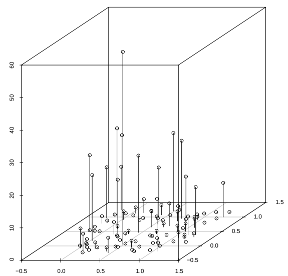
Clustering: less information in data and wider confidence intervals than for Poisson process (independence).

Evidence of positive association between gradient and Beilschmiedia intensity.

Generalisations

- *Shot noise Cox processes* driven by $\Lambda(u) = \sum_{(c,\gamma) \in \mathbf{\Phi}} \gamma k(c, u)$ where $c \in \mathbb{R}^2$, $\gamma > 0$ ($\mathbf{\Phi}$ = marked Poisson process)





- Generalized SNCP's... (Møller & Torrisi, 2005)

Exercises

1. For a Cox process with random intensity function Λ, show that

$$\rho(u) = \mathbb{E}\Lambda(u), \quad \rho^{(2)}(u,v) = \mathbb{E}[\Lambda(u)\Lambda(v)]$$

2. Show that a cluster process with Poisson number of iid offspring is a Cox process with random intensity function

$$\Lambda(u) = \alpha \sum_{m \in \mathbf{M}} f(m,u)$$

(using notation from previous slide on cluster processes. Hint: use void probability characterisation.

3. Compute the intensity and second-order product density for an inhomogeneous Thomas process.

(Hint: interpret the Thomas process as a Cox process and use the Campbell formula)

Density with respect to a Poisson process

$\mathbf{X}$ on bounded S has density f with respect to unit rate Poisson $\mathbf{Y}$ if

$$P(\mathbf{X} \in F) = \mathbb{E}\big(1[\mathbf{Y} \in F] f(\mathbf{Y})\big)$$
$$= \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-|S|}}{n!} \int_{S^n} 1[\mathbf{x} \in F] f(\mathbf{x}) \mathrm{d}x_1 \dots \mathrm{d}x_n \quad (\mathbf{x} = \{x_1, \dots, x_n\})$$

Example: Strauss process

For a point configuration $\mathbf{x}$ on a bounded region S, let $n(\mathbf{x})$ and $s(\mathbf{x})$ denote the number of points and number of (unordered) pairs of R-close points ($R \geq 0$).

A *Strauss process* $\mathbf{X}$ on S has density

$$f(\mathbf{x}) = \frac{1}{c} \exp(\beta n(\mathbf{x}) + \psi s(\mathbf{x}))$$

with respect to a unit rate Poisson process $\mathbf{Y}$ on S and

$$c = \mathbb{E} \exp(\beta n(\mathbf{Y}) + \psi s(\mathbf{Y})) \tag{2}$$

is the normalizing constant (unknown).

Note: only well-defined ($c < \infty$) if $\psi \leq 0$.

Intensity and conditional intensity

Suppose $\mathbf{X}$ has *hereditary* density f with respect to Y:
$f(\mathbf{x}) > 0 \Rightarrow f(\mathbf{y}) > 0, \mathbf{y} \subset \mathbf{x}$.

Intensity function $\rho(u) = \mathbb{E}f(\mathbf{Y} \cup \{u\})$ usually unknown (except for Poisson and Cox/Cluster).

Instead consider *conditional intensity*

$$\lambda(u, \mathbf{x}) = \frac{f(\mathbf{x} \cup \{u\})}{f(\mathbf{x})}$$

(does not depend on normalizing constant !)

Note

$$\rho(u) = \mathbb{E}f(\mathbf{Y} \cup \{u\}) = \mathbb{E}\big[\lambda(u, \mathbf{Y})f(\mathbf{Y})\big] = \mathbb{E}\lambda(u, \mathbf{X})$$

and

$\rho(u)\mathrm{d}A \approx P(\mathbf{X} \text{ has a point in } A) = \mathbb{E}P(\mathbf{X} \text{ has a point in } A|\mathbf{X}\backslash A), u \in A$

Hence, $\lambda(u, \mathbf{X})\mathrm{d}A$ probability that $\mathbf{X}$ has point in very small region A given $\mathbf{X}$ outside A.

Markov point processes

Def: suppose that f hereditary and $\lambda(u, \mathbf{x})$ only depends on $\mathbf{x}$ through $\mathbf{x} \cap b(u, R)$ for some $R > 0$ (*local Markov property*). Then f is *Markov* with respect to the R-close neighbourhood relation.

Thm (Hammersley-Clifford) The following are equivalent.

1. f is Markov.
2. $$f(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi(\mathbf{y}))$$
   where $\phi(\mathbf{y}) = 1$ whenever $\|u - v\| \geq R$ for some $u, v \in \mathbf{y}$.

Pairwise interaction process: $\phi(\mathbf{y}) = 1$ whenever $n(\mathbf{y}) > 2$.

NB: in H-C, R-close neighbourhood relation can be replaced by an arbitrary symmetric relation between pairs of points.

Modelling the conditional intensity function

Suppose we specify a model for the conditional intensity. Two questions:

1. does there exist a density f with the specified conditional intensity ?
2. is f well-defined (integrable) ?

Solution:

1. find f by identifying interaction potentials (Hammersley-Clifford) or guess f.
2. sufficient condition (local stability): $\lambda(u, \mathbf{x}) \leq K$

NB some Markov point processes have interactions of any order in which case H-C theorem is less useful (e.g. area-interaction process).

Some examples

Strauss (pairwise interaction):

$$\lambda(u, \mathbf{x}) = \exp\big(\beta + \psi \sum_{v \in \mathbf{x}} 1[\|u - v\| \leq R]\big), \quad f(\mathbf{x}) = \frac{1}{c}\exp\big(\beta n(\mathbf{x}) + \psi s(\mathbf{x})\big) \quad (\psi \leq 0)$$

Overlap process (pairwise interaction marked point process):

$$\lambda((u, m), \mathbf{x}) = \frac{1}{c}\exp\big(\beta + \psi \sum_{(u', m') \in \mathbf{x}} |b(u, m) \cap b(u', m')|\big) \quad (\psi \leq 0)$$

where $\mathbf{x} = \{(u_1, m_1), \ldots, (u_n, m_n)\}$ and $(u_i, m_i) \in \mathbb{R}^2 \times [a, b]$.

Area-interaction process:

$$f(\mathbf{x}) = \frac{1}{c}\exp\big(\beta n(\mathbf{x}) + \psi V(\mathbf{x})\big), \quad \lambda(u, \mathbf{x}) = \exp\big(\beta + \psi(V(\{u\} \cup \mathbf{x}) - V(\mathbf{x})\big)$$

$V(\mathbf{x}) = |\cup_{u \in \mathbf{x}} b(u, R/2)|$ is area of union of balls $b(u, R/2)$, $u \in \mathbf{x}$.

NB: $\phi(\cdot)$ complicated for area-interaction process.

The Georgii-Nguyen-Zessin formula (‘Law of total probability’)

$$\mathbb{E}\sum_{u\in\mathbf{X}} k(u,\mathbf{X}\setminus\{u\}) = \int_S \mathbb{E}[\lambda(u,\mathbf{X})k(u,\mathbf{X})]\,\mathrm{d}u = \int_S \mathbb{E}^![k(u,\mathbf{X})\,|\,u]\rho(u)\,\mathrm{d}u$$

$\mathbb{E}^![\cdot\,|\,u]$: expectation with respect to the conditional distribution of $\mathbf{X}\setminus\{u\}$ given $u\in\mathbf{X}$ (*reduced Palm distribution*)

Density of reduced Palm distribution:

$$f(\mathbf{x}\,|\,u) = f(\mathbf{x}\cup\{u\})/\rho(u)$$

NB: GNZ formula holds in general setting for point process on $\mathbb{R}^d$.

Useful e.g. for residual analysis.

Statistical inference based on pseudo-likelihood

$\mathbf{x}$ observed within bounded S. Parametric model $\lambda_\theta(u,\mathbf{x})$.

Let $N_i = 1[\mathbf{x}\cap C_i \neq \emptyset]$ where C_i disjoint partitioning of $S = \cup_i C_i$.

$P(N_i = 1\,|\,\mathbf{X}\cap S\setminus C_i) \approx \lambda_\theta(u_i,\mathbf{X})\mathrm{d}C_i$ where $u_i\in C_i$. Hence composite likelihood based on the N_i:

$$\prod_{i=1}^n (\lambda_\theta(u_i,\mathbf{x})\mathrm{d}C_i)^{N_i}(1-\lambda_\theta(u_i,\mathbf{x})\mathrm{d}C_i)^{1-N_i} \equiv \prod_{i=1}^n \lambda_\theta(u_i,\mathbf{x})^{N_i}(1-\lambda_\theta(u_i,\mathbf{x})\mathrm{d}C_i)^{1-N_i}$$

which tends to *pseudo likelihood* function

$$\prod_{u\in\mathbf{x}}\lambda_\theta(u,\mathbf{x})\exp\big(-\int_S \lambda_\theta(u,\mathbf{x})\mathrm{d}u\big)$$

Score of pseudo-likelihood: unbiased estimating function by GNZ.

Pseudo-likelihood estimates asymptotically normal but asymptotic variance must be found by parametric bootstrap.

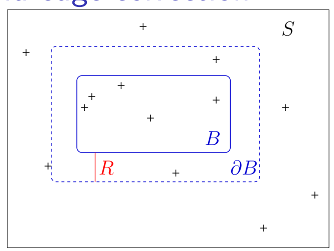
Flexible implementation for log linear conditional intensity (fixed R) in `spatstat`

Estimation of interaction range R: profile likelihood (?)

The spatial Markov property and edge correction

Let $B\subset S$ and assume $\mathbf{X}$ Markov with interaction radius R.

Define: ∂B points in $S\setminus B$ of distance less than R



Factorization (Hammersley-Clifford):

$$f(\mathbf{x}) = \prod_{\mathbf{y}\subseteq\mathbf{x}\cap(B\cup\partial B)}\phi(\mathbf{y})\prod_{\substack{\mathbf{y}\subseteq x\setminus B:\\ \mathbf{y}\cap S\setminus(B\cup\partial B)\neq\emptyset}}\phi(\mathbf{y})$$

Hence, conditional density of $\mathbf{X}\cap B$ given $\mathbf{X}\setminus B$

$$f_B(\mathbf{z}|\mathbf{y}) \propto f(\mathbf{z}\cup\mathbf{y})$$

depends on $\mathbf{y}$ only through $\partial B\cap\mathbf{y}$.

Edge correction using the border method

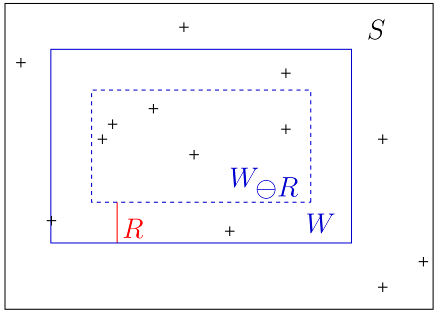
Suppose we observe $\mathbf{x}$ realization of $\mathbf{X} \cap W$ where $W \subset S$.

Problem: density (likelihood) $f_W(\mathbf{x}) = \mathbb{E} f(\mathbf{x} \cup Y_{S \setminus W})$ unknown.

Border method: base inference on

$$f_{W_{\ominus R}}(\mathbf{x} \cap W_{\ominus R} | \mathbf{x} \cap (W \setminus W_{\ominus R}))$$

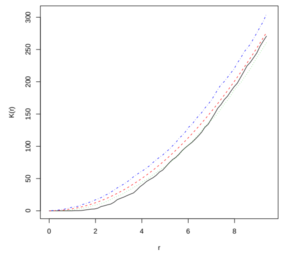
i.e. conditional density of $\mathbf{X} \cap W_{\ominus R}$ given $\mathbf{X}$ outside $W_{\ominus R}$.



Example: spruces

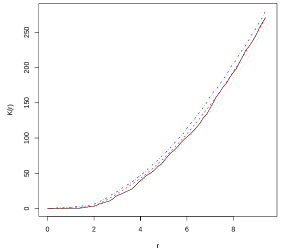
Check fit of a homogeneous Poisson process using K-function and simulations:

```
> library(spatstat)
> data(spruces)
> plot(Kest(spruces)) #estimate K function
> Kenve=envelope(spruces,nrank=2)# envelopes "alpha"=4 %
 Generating 99 simulations of CSR ...
 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,
 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,
 ......
```



Strauss model for spruces

```
> fit=ppm(unmark(spruces),~1,Strauss(r=2),rbord=2)
> coef(fit)
 (Intercept) Interaction
-1.987940   -1.625994
> summary(fit)#details of model fitting
> simpoints=rmh(fit)#simulate point pattern from fitted model
> Kenvestrauss=envelope(fit,nrank=2)
```



Exercises

1. Suppose that S contains a disc of radius $\epsilon \le R/2$. Show that (2) is not finite, and hence the Strauss process not well-defined, when ψ is positive.

   (Hint: $\sum_{n=0}^{\infty} \frac{(\pi\epsilon^2)^n}{n!} \exp(n\beta + \psi n(n-1)/2) = \infty$ if $\psi > 0$.)
2. Show that local stability for a spatial point process density ensures integrability. Verify that the area-interaction process is locally stable.
3. (`spatstat`) The multiscale process is an extension of the Strauss process where the density is given by

   $$f(\mathbf{x}) \propto \exp(\beta n(\mathbf{x}) + \sum_{m=1}^{k} \psi_m s_m(\mathbf{x}))$$

   where $s_m(\mathbf{x})$ is the number of pairs of points u_i, u_j with $\|u_i - u_j\| \in]r_{m-1}, r_m]$ where $0 = r_0 < r_1 < r_2 < \cdots < r_k$. Fit a multiscale process with $k = 4$ and of interaction range $r_k = 5$ to the spruces data. Check the model using the K-function.

   (Hint: use the `spatstat` function `ppm` with the `PairPiece` potential. The function `envelope` can be used to compute envelopes for the K-function under the fitted model.)

Exercises

4. (if time) Verify the Georgii-Nguyen-Zessin formula for a finite point process.

   (Hint: consider first the case of a finite Poisson-process $\mathbf{Y}$ in which case the identity is known as the Slivnyak-Mecke theorem, next apply $\mathbb{E}g(\mathbf{X}) = \mathbb{E}\big[g(\mathbf{Y})f(\mathbf{Y})\big]$.)
5. (if time) Check using the GNZ formula, that the score of the pseudo-likelihood is an unbiased estimating function.

Maximum likelihood inference for point processes

Concentrate on point processes specified by unnormalized density $h_\theta(\mathbf{x})$,

$$f_\theta(\mathbf{x}) = \frac{1}{c(\theta)} h_\theta(\mathbf{x})$$

Problem: $c(\theta)$ in general unknown $\Rightarrow$ unknown log likelihood

$$l(\theta) = \log h_\theta(\mathbf{x}) - \log c(\theta)$$

Importance sampling

Importance sampling: θ_0 fixed reference parameter:

$$l(\theta) \equiv \log h_\theta(\mathbf{x}) - \log \frac{c(\theta)}{c(\theta_0)}$$

and

$$\frac{c(\theta)}{c(\theta_0)} = \mathbb{E}_{\theta_0} \frac{h_\theta(\mathbf{X})}{h_{\theta_0}(\mathbf{X})}$$

Hence

$$\frac{c(\theta)}{c(\theta_0)} \approx \frac{1}{m} \sum_{i=0}^{m-1} \frac{h_\theta(\mathbf{X}^i)}{h_{\theta_0}(\mathbf{X}^i)}$$

where $\mathbf{X}^0, \mathbf{X}^1, \ldots,$ sample from f_{θ_0} (later).

Exponential family case

$$h_\theta(\mathbf{x}) = exp(t(\mathbf{x})\theta^\mathsf{T})$$

$$l(\theta) = t(\mathbf{x})\theta^\mathsf{T} - \log c(\theta)$$

$$\frac{c(\theta)}{c(\theta_0)} = \mathbb{E}_{\theta_0} \exp(t(\mathbf{X})(\theta - \theta_0)^\mathsf{T})$$

Caveat: unless $\theta - \theta_0$ 'small', $\exp(t(\mathbf{X})(\theta - \theta_0)^\mathsf{T})$ has very large variance in many cases (e.g. Strauss).

Path sampling (exp. family case)

Derivative of cumulant transform:

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log \frac{c(\theta)}{c(\theta_0)} = \mathbb{E}_\theta t(\mathbf{X})$$

Hence, by integrating over differentiable path $\theta(t)$ (e.g. line) linking θ_0 and θ_1:

$$\log \frac{c(\theta_1)}{c(\theta_0)} = \int_0^1 \mathrm{E}_{\theta(s)}[t(\mathbf{X})] \frac{\mathrm{d}\theta(s)^\mathsf{T}}{\mathrm{d}s} \mathrm{d}s$$

Approximate $E_{\theta(s)} t(\mathbf{X})$ by Monte Carlo and $\int_0^1$ by numerical quadrature (e.g. trapezoidal rule).

NB Monte Carlo approximation on log scale more stable.

Maximisation of likelihood (exp. family case)

Score and observed information:

$$u(\theta) = t(\mathbf{x}) - \mathrm{E}_\theta t(\mathbf{X}), \quad j(\theta) = \mathrm{Var}_\theta t(\mathbf{X}),$$

Newton-Rahpson iterations:

$$\theta^{m+1} = \theta^m + u(\theta^m) j(\theta^m)^{-1}$$

Monte Carlo approximation of score and observed information: use importance sampling formula

$$\mathrm{E}_\theta k(\mathbf{X}) = \mathrm{E}_{\theta_0} \left[k(\mathbf{X}) \exp\left(t(\mathbf{X})(\theta - \theta_0)^\mathsf{T} \right) \right] / (c_\theta / c_{\theta_0})$$

with $k(\mathbf{X})$ given by $t(\mathbf{X})$ or $t(\mathbf{X})^\mathsf{T} t(\mathbf{X})$.

MCMC simulation of spatial point processes

Birth-death Metropolis-Hastings algorithm for generating ergodic sample $\mathbf{X}^0, \mathbf{X}^1, \ldots$ from locally stable density f on S:

Suppose current state is $\mathbf{X}^i$, $i \geq 0$.

1. Either: with probability $1/2$
   - (birth) generate new point u uniformly on S and accept $\mathbf{X}^{\text{prop}} = \mathbf{X}^i \cup \{u\}$ with probability
   $$\min\left\{1, \frac{f(\mathbf{X}^i \cup \{u\})|S|}{f(\mathbf{X}^i)(n+1)}\right\}$$
   or
   - (death) select uniformly a point $u \in \mathbf{X}^i$ and accept $\mathbf{X}^{\text{prop}} = \mathbf{X}^i \setminus \{u\}$ with probability
   $$\min\left\{1, \frac{f(\mathbf{X}^i \setminus \{u\})n}{f(\mathbf{X}^i)|S|}\right\}$$
   (if $\mathbf{X}^i = \emptyset$ do nothing)
2. if accept $\mathbf{X}^{i+1} = \mathbf{X}^{\text{prop}}$; otherwise $\mathbf{X}^{i+1} = \mathbf{X}^i$.

Initial state $\mathbf{X}_0$: arbitrary (e.g. empty or simulation from Poisson process).

Note: Metropolis-Hastings ratio does not depend on normalizing constant:

$$\frac{f(\mathbf{X}^i \cup \{u\})|S|}{f(\mathbf{X}^i)(n+1)} = \lambda(u, \mathbf{X}^i)\frac{|S|}{(n+1)}$$

Generated Markov chain $\mathbf{X}_0, \mathbf{X}_1, \ldots$ irreducible and aperiodic and hence ergodic: $\frac{1}{m}\sum_{i=0}^{m-1} k(\mathbf{X}^i) \to \mathbb{E}k(\mathbf{X}))$

Moreover, geometrically ergodic and CLT:

$$\sqrt{m}\Big(\frac{1}{m}\sum_{i=0}^{m-1} k(\mathbf{X}^i) - \mathbb{E}k(\mathbf{X})\Big) \to N(0, \sigma_k^2)$$

Missing data

Suppose we observe $\mathbf{x}$ realization of $\mathbf{X} \cap W$ where $W \subset S$.
Problem: likelihood (density of $\mathbf{X} \cap W$)

$$f_{W,\theta}(\mathbf{x}) = \mathbb{E}f_\theta(\mathbf{x} \cap \mathbf{Y}_{S\setminus W})$$

not known - not even up to proportionality ! ($\mathbf{Y}$ unit rate Poisson on S)

Possibilities:

- Monte Carlo methods for missing data.
- Conditional likelihood

$$f_{W_{\ominus R},\theta}(\mathbf{x} \cap W_{\ominus R} | \mathbf{x} \cap (W \setminus W_{\ominus R})) \propto \exp(t(\mathbf{x})\theta^{\mathsf{T}})$$

(note: $\mathbf{x} \cap (W \setminus W_{\ominus R})$ fixed in $t(\mathbf{x})$)

Likelihood-based inference for Cox/Cluster processes

Consider Cox/cluster process $\mathbf{X}$ with random intensity function

$$\Lambda(u) = \alpha \sum_{m \in \mathbf{M}} f(m, u)$$

observed within W ($\mathbf{M}$ Poisson with intensity κ).

Assume $f(m, \cdot)$ of bounded support and choose bounded $\tilde{W}$ so that

$$\Lambda(u) = \alpha \sum_{m \in \mathbf{M} \cap \tilde{W}} f(m, u) \quad \text{for } u \in W$$

$(\mathbf{X} \cap W, \mathbf{M} \cap \tilde{W})$ finite point process with density:

$$f(\mathbf{x}, \mathbf{m}; \theta) = f(\mathbf{m}; \theta) f(\mathbf{x}|\mathbf{m}; \theta) = \mathrm{e}^{|\tilde{W}|(1-\kappa)} \kappa^{n(\mathbf{m})} \mathrm{e}^{|W| - \int_W \Lambda(u)\mathrm{d}u} \prod_{u \in \mathbf{x}} \Lambda(u)$$

Likelihood

$$L(\theta) = \mathbb{E}_\theta f(\mathbf{x}|\mathbf{M}) = L(\theta_0)\mathbb{E}_{\theta_0}\Big[\frac{f(\mathbf{x}, \mathbf{M} \cap \tilde{W}; \theta)}{f(\mathbf{x}, \mathbf{M} \cap \tilde{W}; \theta_0)} \Big| \mathbf{X} \cap W = \mathbf{x}\Big]$$

+ derivatives can be estimated using importance sampling/MCMC
- however more difficult than for Markov point processes.

Bayesian inference: introduce prior $p(\theta)$ and sample posterior

$$p(\theta, \mathbf{m}|\mathbf{x}) \propto f(\mathbf{x}, \mathbf{m}; \theta) p(\theta)$$

(data augmentation) using birth-death MCMC.

Exercises

1. Check the importance sampling formulas

$$\mathrm{E}_{\theta} k(\mathbf{X}) = \mathrm{E}_{\theta_0}\left[k(\mathbf{X})\frac{h_\theta(\mathbf{X})}{h_{\theta_0}(\mathbf{X})}\right] / (c_\theta / c_{\theta_0})$$

   and

$$\frac{c(\theta)}{c(\theta_0)} = \mathbb{E}_{\theta_0}\frac{h_\theta(\mathbf{X})}{h_{\theta_0}(\mathbf{X})} \tag{3}$$

2. Show that the formula

$$L(\theta)/L(\theta_0) = \mathbb{E}_{\theta_0}\Big[\frac{f(\mathbf{x}, \mathbf{M} \cap \tilde{W}; \theta)}{f(\mathbf{x}, \mathbf{M} \cap \tilde{W}; \theta_0)} \,\Big|\, \mathbf{X} \cap W = \mathbf{x}\Big]$$

   follows from (3) by interpreting $L(\theta)$ as the normalizing constant of $f(\mathbf{m}|\mathbf{x}; \theta) \propto f(\mathbf{x}, \mathbf{m}; \theta)$.

3. (practical exercise) Compute MLEs for a multiscale process applied to the spruces data. Use the `newtonraphson.mpp()` procedure in the package `MppMLE`.