Cox process analysis of clustered point patterns

Rasmus Waagepetersen
Department of Mathematical Sciences
Aalborg University

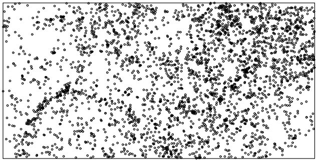
July 30, 2010

Background: Tropical rain forest ecology

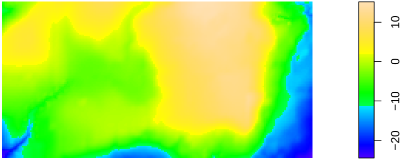
Fundamental questions: which factors influence the spatial distribution of rain forest trees and what is the reason for the high biodiversity of rain forests ?

- environment: topography, soil composition,...
- seed dispersal limitation: by wind, birds or mammals...
- competition between species

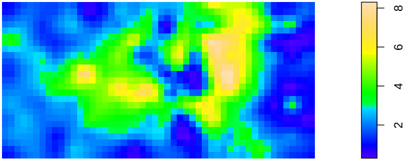
Example: *Capparis Frondosa*



- *observation window W* $= 1000 \text{ m} \times 500 \text{ m}$
- seed dispersal$\Rightarrow$ *clustering*
- environment $\Rightarrow$ *inhomogeneity*



Elevation



Potassium content in soil.

Quantify dependence on environmental variables and seed dispersal using statistics for spatial point processes.

Outline of talk

Outline:

- Background on spatial point processes
  - Summary statistics: intensity, pair correlation,...
  - Cox and cluster point process models
- Estimation
- Application: clustering and seed dispersal
- Work in progress: decomposition of variance for Cox point processes

Intensity function and product density

Point process $\mathbf{X}$: random point pattern

Intensity function and product density

Point process $\mathbf{X}$: random point pattern

Intensity function of point process $\mathbf{X}$

$$\rho(u)|A| \approx P(\mathbf{X} \text{ has a point in A}), \quad u \in A$$

Intensity function and product density

Point process $\mathbf{X}$: random point pattern

Intensity function of point process $\mathbf{X}$

$$\rho(u)|A| \approx P(\mathbf{X} \text{ has a point in A}), \quad u \in A$$

Second order product density

$$\rho^{(2)}(u,v)|A||B| \approx P(\mathbf{X} \text{ has a point in each of } A \text{ and } B) \quad u \in A, \; v \in B$$

Pair correlation and K-function

Pair correlation function

$$g(u, v) = \frac{\rho^{(2)}(u, v)}{\rho(u)\rho(v)}$$

NB: independent points $\Rightarrow \rho^{(2)}(u, v) = \rho(u)\rho(v) \Rightarrow g(u, v) = 1$

Pair correlation and K-function

Pair correlation function

$$g(u,v) = \frac{\rho^{(2)}(u,v)}{\rho(u)\rho(v)}$$

NB: independent points $\Rightarrow \rho^{(2)}(u,v) = \rho(u)\rho(v) \Rightarrow g(u,v) = 1$

K-function

$$K(t) = \int_{\|h\| \le t} g(h) \mathrm{d}h$$

(provided $g(u,v) = g(u-v)$ i.e. $\mathbf{X}$ second-order reweighted stationary, Baddeley, Møller, Waagepetersen, 2000)

Pair correlation and K-function

Pair correlation function

$$g(u,v) = \frac{\rho^{(2)}(u,v)}{\rho(u)\rho(v)}$$

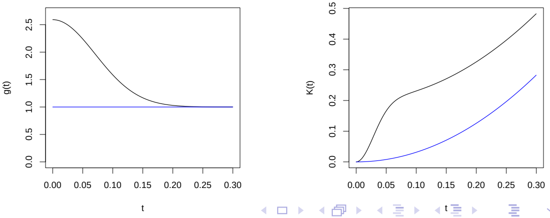
NB: independent points $\Rightarrow \rho^{(2)}(u,v) = \rho(u)\rho(v) \Rightarrow g(u,v) = 1$

K-function

$$K(t) = \int_{\|h\| \le t} g(h) \mathrm{d}h$$

(provided $g(u,v) = g(u-v)$ i.e. $\mathbf{X}$ second-order reweighted stationary, Baddeley, Møller, Waagepetersen, 2000)

Examples of pair correlation and K-functions:



Parametric models for intensity and pair correlation

Study influence of covariates

$$Z(u) = (Z_1(u), \ldots, Z_p(u))$$

using log-linear model for intensity function:

$$\log \rho(u; \beta) = \beta Z(u)^{\mathsf{T}} \Leftrightarrow \rho(u; \beta) = \exp(\beta Z(u)^{\mathsf{T}})$$

where

$$\beta Z(u)^{\mathsf{T}} = \beta_1 Z_1(u) + \beta_2 Z_2(u) + \ldots + \beta_p Z_p(u)$$

Parametric models for intensity and pair correlation

Study influence of covariates

$$Z(u) = (Z_1(u), \ldots, Z_p(u))$$

using log-linear model for intensity function:

$$\log \rho(u; \beta) = \beta Z(u)^{\mathsf{T}} \Leftrightarrow \rho(u; \beta) = \exp(\beta Z(u)^{\mathsf{T}})$$

where

$$\beta Z(u)^{\mathsf{T}} = \beta_1 Z_1(u) + \beta_2 Z_2(u) + \ldots + \beta_p Z_p(u)$$

Quantify clustering using parameter ψ in parametric model

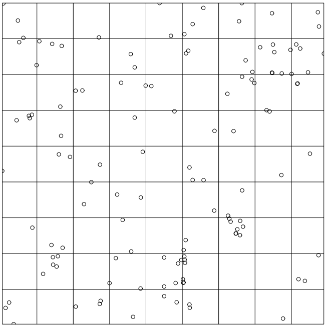
$$K(t; \psi) = \int_{\|h\| \leq t} g(h; \psi) \mathrm{d}h$$

for K/g-function.

Point processes and count variables

Random count variables: $N_i = \#\mathbf{X} \cap C_i$ number of points in C_i.

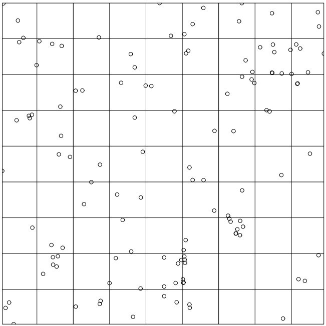
Disjoint subdivision $W = \cup_{i=1}^{n} C_i$ in 'cells' C_i.



Point processes and count variables

Random count variables: $N_i = \#\mathbf{X} \cap C_i$ number of points in C_i.

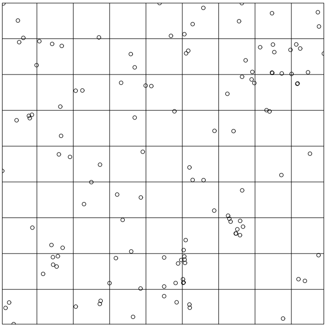
Disjoint subdivision $W = \cup_{i=1}^n C_i$ in 'cells' C_i.



$$\mathbb{E}N_i \approx \rho(u_i)|C_i| \quad \mathbb{E}[N_i N_j] \approx \rho^{(2)}(u_i, u_j)|C_i||C_j|$$

$$\mathbb{C}\mathrm{ov}(N_i, N_j) = (\rho^{(2)}(u_i, u_j) - \rho(u_i)\rho(u_j))|C_i||C_j|$$

Point processes and count variables



Random count variables: $N_i = \#\mathbf{X} \cap C_i$ number of points in C_i.

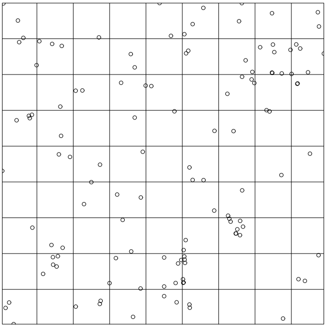
Disjoint subdivision $W = \cup_{i=1}^{n} C_i$ in 'cells' C_i.

$$\mathbb{E}N_i \approx \rho(u_i)|C_i| \quad \mathbb{E}[N_i N_j] \approx \rho^{(2)}(u_i, u_j)|C_i||C_j|$$

$$\mathbb{C}\mathrm{ov}(N_i, N_j) = (\rho^{(2)}(u_i, u_j) - \rho(u_i)\rho(u_j))|C_i||C_j|$$

Poisson process: counts N_i independent and Poisson distributed

Point processes and count variables



Random count variables: $N_i = \#\mathbf{X} \cap C_i$ number of points in C_i.

Disjoint subdivision $W = \cup_{i=1}^n C_i$ in 'cells' C_i.

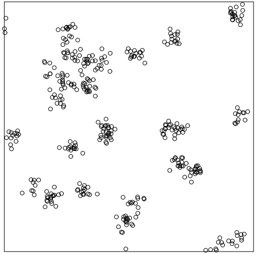
$$\mathbb{E}N_i \approx \rho(u_i)|C_i| \quad \mathbb{E}[N_i N_j] \approx \rho^{(2)}(u_i, u_j)|C_i||C_j|$$

$$\mathbb{C}\mathrm{ov}(N_i, N_j) = (\rho^{(2)}(u_i, u_j) - \rho(u_i)\rho(u_j))|C_i||C_j|$$

Poisson process: counts N_i independent and Poisson distributed

Independence often unrealistic - e.g. clustering due to seed dispersal

Cluster process: Inhomogeneous Thomas process (Waa, 2007)



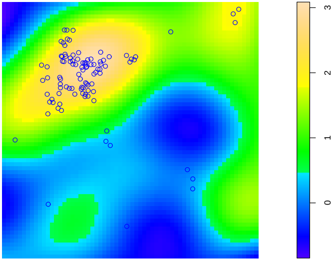
Parents stationary Poisson point process intensity κ

Offspring distributed around mothers according to Gaussian density with standard deviation ω

Inhomogeneity: offspring survive according to probability

$$p(u) \propto \exp(Z(u)\beta^{\mathsf{T}})$$

depending on covariates (independent thinning).



Cox processes

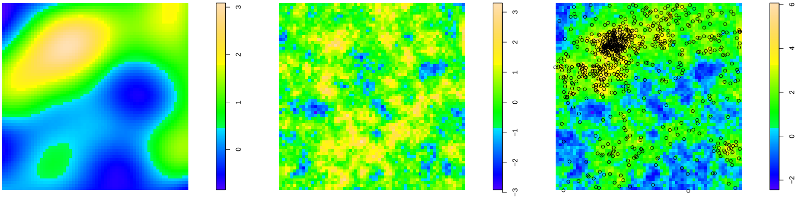
X is a *Cox process* driven by the random intensity function Λ if, conditional on $\Lambda = \lambda$, **X** is a Poisson process with intensity function λ.

Example: log Gaussian Cox process (Møller, Syversveen, Waa, 1998)

$$\log \Lambda(u) = \beta Z(u)^{\mathsf{T}} + Y(u)$$

where $\{Y(u)\}$ Gaussian random field.

$\beta Z(u)^{\mathsf{T}}$ $\quad$ $Y(u)$ $\quad$ $\beta Z(u)^{\mathsf{T}} + Y(u)$



Intensity and product density for a Cox process

Given Λ, Cox process $\mathbf{X}$ becomes Poisson process with intensity function $\Lambda(u)$ and product density $\Lambda(u)\Lambda(v)$.

Hence intensity and product density for $\mathbf{X}$ obtained by averaging with respect to Λ:

$$\rho(u) = \mathbb{E}\Lambda(u)$$

$$\rho^{(2)}(u,v) = \mathbb{E}\Lambda(u)\Lambda(v)$$

$$g(u,v) = \frac{\mathbb{E}\Lambda(u)\Lambda(v)}{\mathbb{E}\Lambda(u)\mathbb{E}\Lambda(v)}$$

Intensity and pair correlation function for specific Cox processes

Log linear intensity (both log Gaussian Cox and inhomogeneous Thomas):

$$\log \rho(u; \beta) = \mu + Z(u)\beta^{\mathsf{T}}$$

Intensity and pair correlation function for specific Cox processes

Log linear intensity (both log Gaussian Cox and inhomogeneous Thomas):

$$\log \rho(u;\beta) = \mu + Z(u)\beta^{\mathsf{T}}$$

Pair correlation function for log Gaussian Cox process:

$$g(u;\psi) = \exp(\sigma^2 c(u;\alpha)), \quad \psi = (\sigma^2, \alpha)$$

where σ^2 variance of Gaussian field and $c(\cdot;\alpha)$ correlation function.

Intensity and pair correlation function for specific Cox processes

Log linear intensity (both log Gaussian Cox and inhomogeneous Thomas):

$$\log \rho(u; \beta) = \mu + Z(u)\beta^{\mathsf{T}}$$

Pair correlation function for log Gaussian Cox process:

$$g(u; \psi) = \exp(\sigma^2 c(u; \alpha)), \quad \psi = (\sigma^2, \alpha)$$

where σ^2 variance of Gaussian field and $c(\cdot; \alpha)$ correlation function.

Pair correlation function for inhomogeneous Thomas:

$$g(t; \psi) = 1 + \exp(-t^2/(4\omega)^2)]/(4\omega^2\kappa\pi), \quad \psi = (\omega, \kappa)$$

Parameter estimation

Possibilities:

1. Maximum likelihood estimation (Monte Carlo computation of likelihood function)
2. Simple estimating functions based on intensity function and pair correlation function - inspired by methods for count variables: least squares, composite likelihood, quasi-likelihood,...

Maximum likelihood estimation for Cox processes

Likelihood (probability density) for Cox process given observed point pattern $\mathbf{x}$:

$$f_\theta(\mathbf{x}) = \mathbb{E}_\theta[\exp(-\int_W \Lambda(u)\mathrm{d}u) \prod_{u \in \mathbf{x}} \Lambda(u)]$$

Problem for Monte Carlo approximation: $\Lambda = \{\Lambda(u)\}_{u \in W}$ infinitely dimensional quantity.

Maximum likelihood estimation for Cox processes

Likelihood (probability density) for Cox process given observed point pattern $\mathbf{x}$:

$$f_\theta(\mathbf{x}) = \mathbb{E}_\theta[\exp(-\int_W \Lambda(u)\mathrm{d}u) \prod_{u \in \mathbf{x}} \Lambda(u)]$$

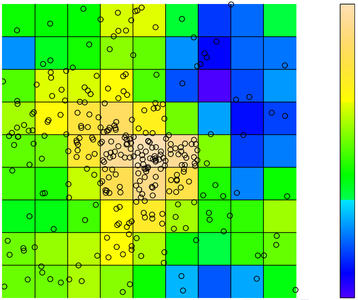
Problem for Monte Carlo approximation: $\Lambda = \{\Lambda(u)\}_{u \in W}$ infinitely dimensional quantity.

LCGP: approximate inference by discretizing random field $\Lambda(u) = \exp(\beta Z(u)^\mathsf{T} + Y(u))$

Counts N_i Poisson with mean

$$\exp(\beta Z(u_i)^\mathsf{T} + Y(u_i))|C_i|$$

(Poisson GLMM, Benes, Bodlak, Møller, Waa, 2005)



Big covariance matrix of highdimensional Gaussian random vector

$$(Y(u_1), \ldots, Y(u_n))$$

($n \approx 100000$) can be handled using FFT (fast Fourier transformation).

Big covariance matrix of highdimensional Gaussian random vector

$$(Y(u_1), \dots, Y(u_n))$$

($n \approx 100000$) can be handled using FFT (fast Fourier transformation).

Cox-cluster process (e.g. inhomogeneous Thomas):

$$\Lambda(u) = \kappa \exp(\beta Z(u)^T) \sum_{c \in C} k(u - c)$$

Hence need Markov chain Monte Carlo simulation of parent points C given observed points $\mathbf{X} \cap W = \mathbf{x}$.

Monte Carlo estimation of likelihood possible for Cox processes but MCMC computation may be very time-consuming.

Example: composite likelihood I (Schoenberg, 2005; Waa, 2007)

Consider indicators $X_i = 1[N_i > 0]$ for presence of points in cells C_i. $P(X_i = 1) = \rho_\beta(u_i)|C_i|$.

Example: composite likelihood I (Schoenberg, 2005; Waa, 2007)

Consider indicators $X_i = 1[N_i > 0]$ for presence of points in cells C_i. $P(X_i = 1) = \rho_\beta(u_i)|C_i|$.

Composite Bernouilli likelihood

$$\prod_{i=1}^{n}(P(X_i = 1))^{X_i}(1-P(X_i = 1))^{1-X_i} \equiv \prod_{i=1}^{n}\rho_\beta(u_i)^{X_i}(1-\rho_\beta(u_i)|C_i|)^{1-X_i}$$

has limit ($|C_i| \to 0$)

$$L(\beta) = [\prod_{u\in\mathbf{X}\cap W}\rho(u;\beta)]\exp(-\int_W \rho(u;\beta)\,\mathrm{d}u)$$

Example: composite likelihood I (Schoenberg, 2005; Waa, 2007)

Consider indicators $X_i = 1[N_i > 0]$ for presence of points in cells C_i. $P(X_i = 1) = \rho_\beta(u_i)|C_i|$.

Composite Bernouilli likelihood

$$\prod_{i=1}^{n}(P(X_i = 1))^{X_i}(1-P(X_i = 1))^{1-X_i} \equiv \prod_{i=1}^{n}\rho_\beta(u_i)^{X_i}(1-\rho_\beta(u_i)|C_i|)^{1-X_i}$$

has limit ($|C_i| \to 0$)

$$L(\beta) = [\prod_{u \in \mathbf{X}\cap W}\rho(u;\beta)]\exp(-\int_W \rho(u;\beta)\,\mathrm{d}u)$$

Estimate $\hat{\beta}$ maximizes $L(\beta)$.

NB: $L(\beta)$ formally equivalent to likelihood function of a Poisson process with intensity function $\rho_\beta(\cdot)$.

Composite likelihood II (Waa, 2007)

Composite likelihood based on $\rho_\beta(\cdot)$ only involves β - what about ψ ?

Proceed as before but for indicators

$$X_{ij} = 1[N_i > 0, N_j > 0] \quad (= N_i N_j \text{ when } C_i \text{ and } C_j \text{ small })$$

for joint presence of points in pairs of cells C_i and C_j.

Composite likelihood II (Waa, 2007)

Composite likelihood based on $\rho_\beta(\cdot)$ only involves β - what about ψ ?

Proceed as before but for indicators

$$X_{ij} = 1[N_i > 0, N_j > 0] \quad (= N_i N_j \text{ when } C_i \text{ and } C_j \text{ small })$$

for joint presence of points in pairs of cells C_i and C_j.

Then

$$P(X_{ij} = 1) = \mathbb{E} N_i N_j = \rho^{(2)}(u, v; \beta, \psi)|C_i||C_j|$$

and Bernouilli composite likelihood based on X_{ij} converges to

$$L_2(\beta, \psi) = [\prod_{u,v \in \mathbf{X} \cap W}^{\neq} \rho^{(2)}(u, v; \beta, \psi)] \exp(-\int_{W^2} \rho^{(2)}(u, v; \beta, \psi) \, \mathrm{d}u \mathrm{d}v)$$

Minimum contrast estimation for ψ I

Computationally easy alternative if **X** second-order reweighted stationary so that K-function well-defined:

$$K(t) = \int_{\|h\| \leq t} g(h) \mathrm{d} h = \frac{1}{|W|} \mathbb{E} \sum_{u \in \mathbf{X} \cap W, v \in \mathbf{X}} \frac{1[0 < \|u - v\| \leq t]}{\rho(u; \beta) \rho(v; \beta)}$$

Minimum contrast estimation for ψ I

Computationally easy alternative if $\mathbf{X}$ second-order reweighted stationary so that K-function well-defined:

$$K(t) = \int_{\|h\| \leq t} g(h) \mathrm{d}h = \frac{1}{|W|} \mathbb{E} \sum_{u \in \mathbf{X} \cap W, v \in \mathbf{X}} \frac{1[0 < \|u - v\| \leq t]}{\rho(u; \beta)\rho(v; \beta)}$$

Estimate of K-function (Baddeley, Møller and Waa, 2000):

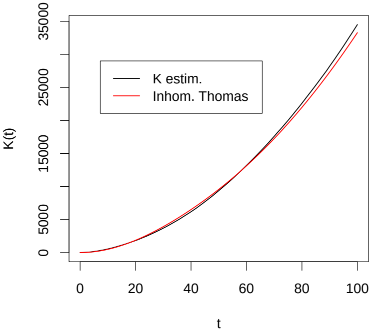
$$\hat{K}_\beta(t) = \frac{1}{|W|} \sum_{u,v \in \mathbf{X} \cap W} \frac{1[0 < \|u - v\| \leq t]}{\rho(u; \beta)\rho(v; \beta)} e_{u,v}$$

Unbiased if β 'true' regression parameter

Minimum contrast estimation for ψ II

Estimate ψ by minimizing squared distance between theoretical K and $\hat{K}$:

$$\hat{\psi} = \underset{\psi}{\mathrm{argmin}} \int_0^r \left(\hat{K}_{\hat{\beta}}(t) - K(t;\psi)\right)^2 \mathrm{d}t$$



Two-step estimation

Obtain estimates $(\hat{\beta}, \hat{\psi})$ in two steps

1. obtain $\hat{\beta}$ using composite likelihood I
2. obtain $\hat{\psi}$ using minimum contrast or composite likelihood II $L_2(\hat{\beta}, \psi)$

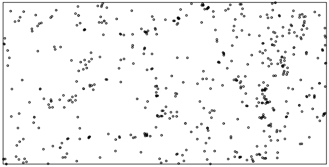
Waa (2007), Waa and Guan (2009): consistency and asymptotic normality of $(\hat{\beta}, \hat{\psi})$ for *infinitely divisible* or *mixing* point processes (e.g. Poisson cluster processes):

$$|W|^{-1/2}((\hat{\beta}, \hat{\psi}) - (\hat{\beta}, \hat{\psi})) \xrightarrow{d} N(0, V)$$

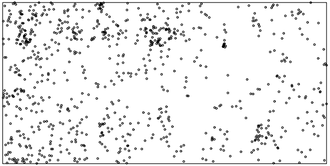
Example: modes of seed dispersal and clustering

Three species with different modes of seed dispersal:

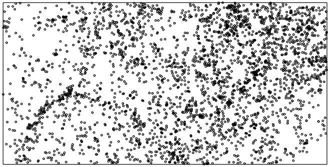
Acalypha Diversifolia explosive capsules



Loncocharpus Heptaphyllus wind



Capparis Frondosa bird/mammal



Is degree of clustering related to mode of seed dispersal ?

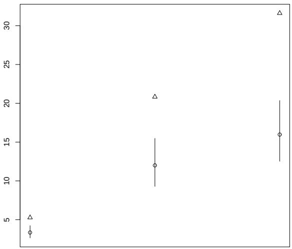
Fit Thomas cluster process with log linear model for intensity function.

Acalypha and *Capparis*: positive dependence on elevation and potassium (significant positive values of $\hat{\beta}$).

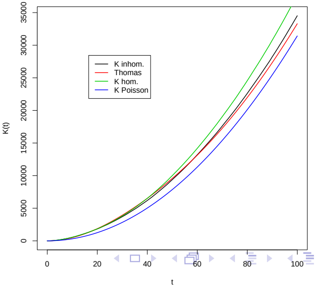
Loncocharpus: negative dependence on nitrogen and phosphorous.

Recall $\omega =$ 'width' of clusters.

Estimates of ω for explosive, wind and bird/mammal:



Estimates of K-functions for bird/mammal dispersed species



Decomposition of variance for rain forest tree point patterns (Shen, Jalilian, Waa, in progress)

Question: how much of the spatial variation for rain forest trees is due to environment ?

Variance of a count N_i (number of points in cell C_i) for a stationary Cox process:

Decomposition of variance for rain forest tree point patterns (Shen, Jalilian, Waa, in progress)

Question: how much of the spatial variation for rain forest trees is due to environment ?

Variance of a count N_i (number of points in cell C_i) for a stationary Cox process:

$$\mathbb{V}\mathrm{ar} N_i = \int_{C_i} \rho \mathrm{d}u$$

Variance=Poisson variance

Decomposition of variance for rain forest tree point patterns (Shen, Jalilian, Waa, in progress)

Question: how much of the spatial variation for rain forest trees is due to environment ?

Variance of a count N_i (number of points in cell C_i) for a stationary Cox process:

$$\mathbb{V}\mathrm{ar} N_i = \int_{C_i} \rho \mathrm{d}u + \iint_{C_i^2} \rho^2 [g(u,v) - 1] \mathrm{d}u \mathrm{d}v$$

Variance=Poisson variance+Extra variance due to random intensity

Decomposition of variance for rain forest tree point patterns (Shen, Jalilian, Waa, in progress)

Question: how much of the spatial variation for rain forest trees is due to environment ?

Variance of a count N_i (number of points in cell C_i) for a stationary Cox process:

$$\mathbb{V}\mathrm{ar} N_i = \int_{C_i} \rho \mathrm{d}u + \iint_{C_i^2} \rho^2 [g(u,v) - 1] \mathrm{d}u \mathrm{d}v$$

Variance=Poisson variance+Extra variance due to random intensity

Summary of extra Poisson variation:

$$\frac{\rho^2[g(u,v)-1]}{\rho} = \rho[g(u,v)-1]$$

Variance of log linear random intensity:

Variance of log linear random intensity:

$$\mathbb{V}\mathrm{ar} \log \Lambda(u) = \mathbb{V}\mathrm{ar} \beta Z(u)^\mathsf{T}$$

Variance=Environment

Variance of log linear random intensity:

$$\mathbb{V}\mathrm{ar} \log \Lambda(u) = \mathbb{V}\mathrm{ar} \beta Z(u)^{\mathsf{T}} + \mathbb{V}\mathrm{ar}\, Y(u) = \sigma_Z^2 + \sigma_Y^2$$

Variance=Environment+Seed dispersal

Variance of log linear random intensity:

$$\mathbb{V}\mathrm{ar} \log \Lambda(u) = \mathbb{V}\mathrm{ar} \beta Z(u)^{\mathsf{T}} + \mathbb{V}\mathrm{ar}\, Y(u) = \sigma_Z^2 + \sigma_Y^2$$

Variance=Environment+Seed dispersal

Note $\tilde{Z}(u) = \beta Z(u)^{\mathsf{T}}$ regarded as stationary random process.

Variance of log linear random intensity:

$$\mathbb{V}\mathrm{ar} \log \Lambda(u) = \mathbb{V}\mathrm{ar} \beta Z(u)^\mathsf{T} + \mathbb{V}\mathrm{ar}\, Y(u) = \sigma_Z^2 + \sigma_Y^2$$

Variance=Environment+Seed dispersal

Note $\tilde{Z}(u) = \beta Z(u)^\mathsf{T}$ regarded as stationary random process.

Estimate β and σ_Y^2 using two-step approach.

Simple empirical estimate of σ_Z^2 :

$$\hat{\sigma}_Z^2 = \frac{1}{n_G} \sum_{u \in G} (\tilde{Z}(u) - \bar{\tilde{Z}})^2$$

Compute

$$\frac{\sigma_Z^2}{\sigma_Z^2 + \sigma_Y^2} \quad \text{and} \quad \frac{\sigma_Y^2}{\sigma_Z^2 + \sigma_Y^2}$$

Example (*Capparis Frondosa*): $\hat{\sigma}_Z^2 = 0.10$ and $\hat{\sigma}_Y^2 = 0.69$. Hence 12.5% of variance due to environment.

Additive model for random intensity function (Jalilian, Guan, Waa, in progress)

Alternative to log linear model:

$$\Lambda(u) = \beta Z(u)^{\mathsf{T}} + Y(u)$$

Cox process superposition of point processes with (random) intensity functions $\tilde{Z}(u) = \beta Z(u)^{\mathsf{T}}$ and $Y(u)$

Intensity function:

$$\rho(u) = \rho_Y + \beta Z(u)^{\mathsf{T}} \quad (\rho_Y = \mathbb{E}Y(u))$$

Least squares estimation:

$$(\hat{\rho}_Y, \hat{\beta}) = \underset{(\rho_Y,\beta)}{\operatorname{argmin}} \sum_i (N_i - \rho(u_i)|C_i|)^2 =$$

$$\underset{(\rho_Y,\beta)}{\operatorname{argmin}} \sum_i (N_i/|C_i| - \rho_Y - \beta Z(u_i)^{\mathsf{T}})^2$$

NB: for $|C_i| \to 0$ least squares estimate solves estimating equation

$$\sum_{u \in \mathbf{X}} Z(u) - \int_W Z(u)(\rho_Y + \beta Z(u)^\mathsf{T})\mathrm{d}u = 0$$

Further possibilities: weighted least squares using covariance matrix of counts.

$$\mathbb{C}\mathrm{ov}[N_i, N_j] = \rho_Y^2(g_Y(u_i, u_j; \psi) - 1)$$

NB: estimated $\hat{\beta} Z(u)^\mathsf{T}$ not necessarily positive.

Finally estimate parameter ψ for Y using composite likelihood II since

$$\rho^{(2)}(u, v) = \rho(u)\rho(v) + \rho_Y^2(g_Y(u, v; \psi) - 1)$$

References

Møller, J., Syversveen, A.-R., and Waagepetersen (1998). Log Gaussian Cox processes, *Scandinavian Journal of Statistics*, **25**, 451-482.

Baddeley, A., Møller, J., and Waagepetersen, R. (2000). Non- and semi-parametric estimation of interaction in inhomogeneous point processes, *Statistica Neerlandica*, **54**, 329-350.

Waagepetersen, R. (2004). Convergence of posteriors for discretized LGCPs, *Statistics and Probability Letters*, **66**, 229-235.

Benes, V., Bodlak, K., Møller, J. and Waagepetersen, R. (2005). A case study on point process modelling in disease mapping, *Image Analysis and Stereology*, **24**, 159-168.

Waagepetersen, R. (2007). An estimating function approach to inference for inhomogeneous Neyman-Scott processes, *Biometrics*, **63**, 252-258.

Waagepetersen, R. and Guan, Y. (2009). Two-step estimation for inhomogeneous spatial point processes, *Journal of the Royal Statistical Society, Series B*, **71**, 685-702.

Thanks for your attention