# Statistical inference for inhomogeneous Cox processes

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# Outline

Inhomogeneous clustered point patterns

Cox processes

Estimating functions for inhomogeneous Cox processes

Maximum likelihood inference for thinned Cox processes

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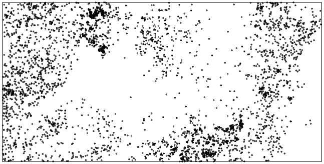
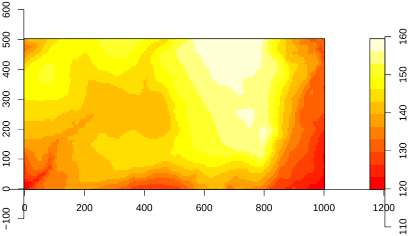
# Tropical rain forests trees

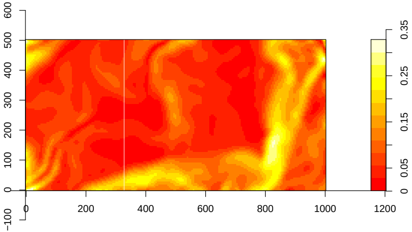
Beilschmiedia



- *observation window* $= 1000 \text{ m} \times 500 \text{ m}$
- seed dispersal $\Rightarrow$ *clustering*
- *covariates* $\Rightarrow$ *inhomogeneity*

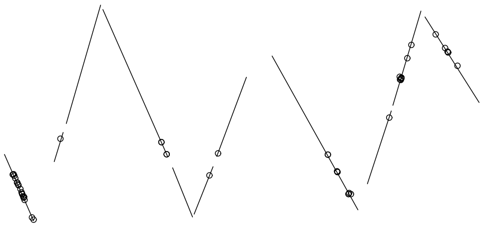



Elevation

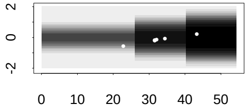



Norm of elevation gradient steepness)

# Whale positions



Close up:




Aim: estimate whale intensity $\lambda$

Observation window $W$ = narrow strips around transect lines

Varying detection probability: inhomogeneity (thinning)

Variation in prey intensity: clustering

# Intensity function and product density

Intensity function of point process $\mathbf{X}$ on $\mathbb{R}^2$:

$$\rho(u)\mathrm{d}A \approx \mathbb{E}N(A) \approx P(\mathbf{X} \text{ has a point in A})$$

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Second order product density

$$\rho^{(2)}(u, v)dAdB \approx P(\mathbf{X} \text{ has a point in each of } A \text{ and } B)$$

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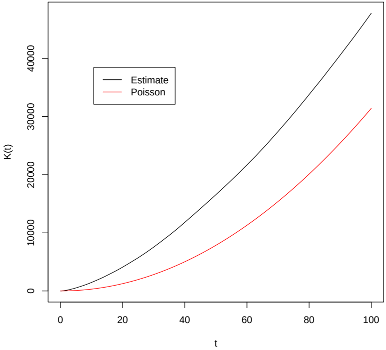
Pair correlation and $K$-function (provided $g(u,v) = g(u-v)$)

$$g(u,v) = \frac{\rho^{(2)}(u,v)}{\rho(u)\rho(v)} \quad \text{and } K(t) = \int_{\mathbb{R}^2} 1[\|u\| \le t]g(u)\mathrm{d}u$$

NB: for Poisson process, $g(u-v) = 1$, clustering: $g(u-v) > 1$.

# K-function for tropical trees (adjusted for inhomogeneity due to covariates)

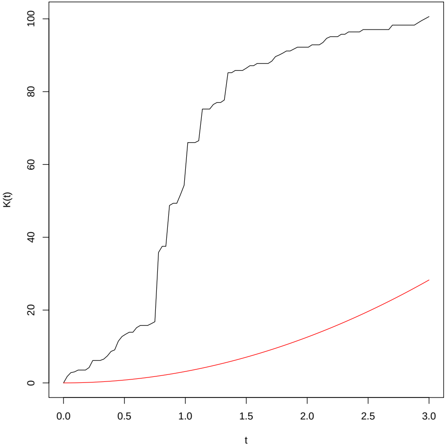
Estimate of $K$ adjusted for inhomogeneous intensity function $\rho(u)$:




(more details later on estimation of $K$-function)

Poisson process not appropriate.

# K-function for whales (adjusted for inhomogeneity due to thinning)




Poisson process not appropriate.

# Common features

- inference concerning intensity is of main interest
- need to account for clustering when assessing uncertainty of parameter estimates

Inhomogeneous clustered point patterns

Cox processes

Estimating functions for inhomogeneous Cox processes

Maximum likelihood inference for thinned Cox processes

# Cox processes

$\mathbf{X}$ is a *Cox process* driven by the random intensity function $\mathbf{\Lambda}$ if:

conditional on $\mathbf{\Lambda} = \lambda$, $\mathbf{X}$ is a Poisson process with intensity function $\lambda$.

# Log Gaussian Cox process (LGCP)

LGCP:

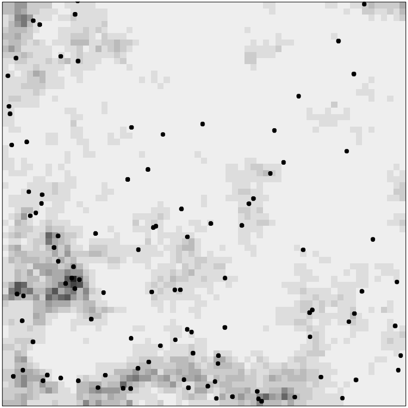
$$\log \Lambda(u) = U(u)$$

where $\mathbf{U} = (U(u))_{u \in \mathbb{R}^2}$ Gaussian process

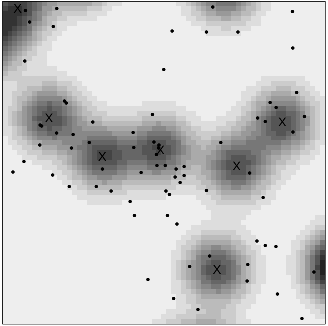
E.g. exponential covariance function:

$$\begin{aligned} c(u, v) &= \mathbb{C}\mathrm{ov}[U(u), U(v)] \\ &= \sigma^2 \exp\left(-\|u - v\|/\psi\right) \end{aligned}$$

$\sigma^2$ variance and $\psi$ correlation scale parameter.



# Example: modified Thomas process



Mothers (crosses) stationary Poisson point process $\mathbf{M}$ with intensity $\kappa > 0$.

Clusters $\mathbf{X}_m, m \in \mathbf{M}$ Poisson processes of offspring dispersed according to $k =$ bivariate isotropic Gaussian density.

$\omega$: standard deviation of Gaussian density

$\alpha$: Expected number of offspring for each mother.

Cox process with random intensity function:

$$\Lambda(u) = \alpha \sum_{m \in \mathbf{M}} k(u - m; \omega)$$

# Inhomogeneous Cox process

$z_{1:p}(u) = (z_1(u), \dots, z_p(u))$ vector of $p$ nonconstant covariates.

$\beta_{1:p} = (\beta_1, \dots \beta_p)$ regression parameter.

Inhomogeneous random intensity function:

$$\Lambda_{\mathrm{inhom}}(u) = \exp(z(u)_{1:p}\beta_{1:p}^{\mathsf{T}})\Lambda(u)$$

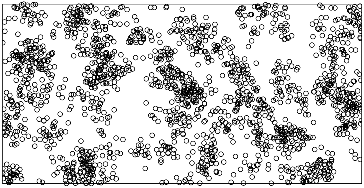
Rain forest example:

$$z_{1:2}(u) = (z_{\mathrm{elev}}(u), z_{\mathrm{grad}}(u))$$

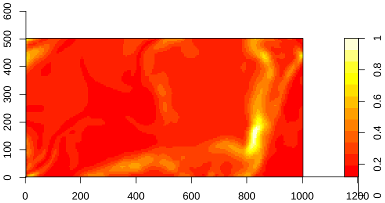
elevation/gradient covariate.

# Interpretation in terms of thinning

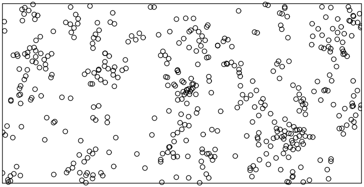
Homogeneous Cox process



Survival probabilities
$p(u) \propto \exp(z_{1:2}(u)\beta_{1:2}^{\mathsf{T}})$




After thinning (inhomogeneous Cox)



# Intensity function and product density for Cox processes

Intensity and product density for inhomogeneous Cox process ($\Lambda_{\text{inhom}}(u) = \exp(z(u)_{1:p}\beta_{1:p}^{\mathsf{T}})\Lambda(u)$):

$$\rho(u) = \mathbb{E}\Lambda_{\text{inhom}}(u), \quad \rho^{(2)}(u, v) = \mathbb{E}[\Lambda_{\text{inhom}}(u)\Lambda_{\text{inhom}}(v)]$$

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For LGCP and inhomogeneous Thomas:

$$\rho(u) = \exp(z(u)\beta^{\mathsf{T}}) \quad z(u) = (1, z_1(u), \ldots, z_p(u)) \quad \beta = (\beta_0, \beta_1, \ldots, \beta_p)$$

and $\beta_0 = \sigma^2/2$ (LGCP) or $\log \kappa\alpha$ (Thomas).

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Pair correlation functions:

$$g(u,v) = \begin{cases} \exp(c(\|u-v\|)) & \text{LGCP} \\ 1 + \exp(-\|u-v\|^2/(4\omega^2))/(4\pi\omega^2\kappa) & \text{Thomas} \end{cases}$$

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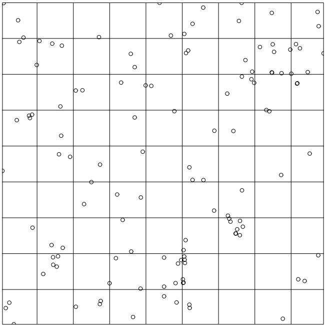
# Composite likelihood for $\beta$ based on intensity function

Consider indicators $N_i = \mathbf{1}[\mathbf{X} \cap C_i \neq \emptyset]$ of occurrence of points in disjoint $C_i$ ($W = \cup C_i$) where

$$P(N_i = 1) \approx \rho_\beta(u_i)\mathrm{d}C_i, \quad u_i \in C_i$$

(quadrat count approach)



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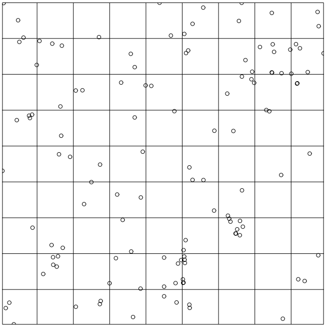

Limit ($\mathrm{d}C_i \to 0$) of composite likelihood

$$\prod_{i=1}^{n} (\rho_\beta(u_i)\mathrm{d}C_i)^{N_i}(1-\rho_\beta(u_i)\mathrm{d}C_i)^{1-N_i} \equiv \prod_{i=1}^{n} \rho_\beta(u_i)^{N_i}(1-\rho_\beta(u_i)\mathrm{d}C_i)^{1-N_i}$$

is

$$L(\beta) = \prod_{u \in \mathbf{X} \cap W} \rho_\beta(u) \exp(-\int_W \rho_\beta(u)\,\mathrm{d}u)$$

# Estimating function

Composite likelihood

$$L(\beta) = \prod_{u \in \mathbf{X} \cap W} \rho_\beta(u) \exp(-\int_W \rho_\beta(u)\, \mathrm{d}u)$$

equivalent with Poisson likelihood.

Estimating function given by Poisson score ($\rho_\beta(u) = \exp(z(u)\beta^\mathsf{T})$):

$$u(\beta) = \frac{\mathrm{d}}{\mathrm{d}\beta} \log L(\beta) = \sum_{u \in \mathbf{X} \cap W} z(u) - \int_W z(u)\rho_\beta(u)\, \mathrm{d}u$$

Solve

$$u(\beta) = 0$$

using `R` package `spatstat` (Baddeley and Turner).

# Parameter Estimation: clustering parameters

Theoretical expression for $K$-function (Thomas process):

$$K(t;\kappa,\omega) = \pi t^2 + \left(1 - \exp(-t^2/(2\omega)^2)\right)/\kappa.$$

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Semi-parametric estimate

$$\hat{K}(t) = \sum_{u,v \in \mathbf{X} \cap W} \frac{1[0 < \|u - v\| \le t]}{\rho_{\hat{\beta}}(u)\rho_{\hat{\beta}}(v)|W \cap W_{u-v}|}$$

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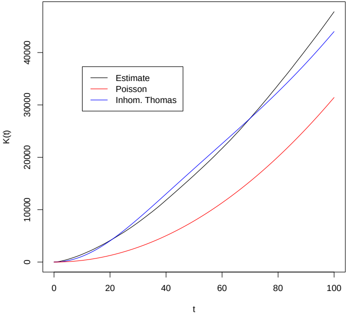
Estimate $\kappa$ and $\omega$ by minimizing contrast

$$\int_0^{100} \left(K(t;\kappa,\omega)^{1/4} - \hat{K}(t)^{1/4}\right)^2 \mathrm{d}t$$



# Asymptotic distribution of parameter estimates

Waagepetersen (2007): asymptotic normality of $\hat{\beta}$ using infinite divisibility of inhomogenous cluster process.

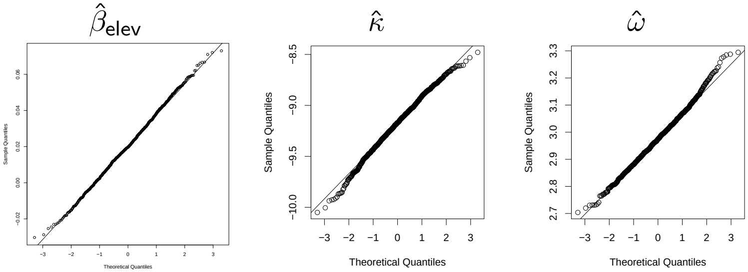
Waagepetersen and Guan (2007): asymptotic normality of $(\hat{\beta}, \hat{\psi})$ ($\psi$ clustering parameter) for mixing Cox processes (including inhom. LGCP and Thomas).

Asymptotic covariance matrix of $\hat{\beta}$: $I^{-1}VI^{-1}$ where $I$ Fisher information for Poisson likelihood and $V$ variance of Poisson score.

(NB: $V = I$ for Poisson process and $V - I$ extra variance due to clustering).

# Simulation study

Quantile plots of $\hat{\beta}_{\text{elev}}, \hat{\kappa}$, and $\hat{\omega}$ (expected numbers 50 for mothers and 800 for offspring)



# Results for Beilschmiedia

Parameter estimates and confidence intervals (Poisson in red).

| Elevation | Gradient | $\kappa$ | $\omega$ |
|---|---|---|---|
| 0.02 [-0.02,0.06] | 5.84 [0.89,10.80] | 8e-5 [4e-5,1.5e-4] | 20.0 [15,26] |
| [0.017,0.026] | [5.340,6.342] | | |

**Clustering:** less information in data and wider confidence intervals than for Poisson process (independence).

Evidence of positive association between gradient and Beilschmiedia intensity.

Same results for LGCP since estimating functions only depend on first and second order properties.

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# Shot-noise Cox process model for whales

Whales: stationary Cox process $\mathbf{Y}$ with random intensity function

$$\Lambda(u) = \sum_{(c,\gamma)\in\mathbf{\Phi}} \gamma k(u-c)$$

$\mathbf{\Phi}$ homogeneous marked Poisson process of marked cluster centres $(c,\gamma)$ where $\gamma \sim \Gamma(\alpha, 1)$.

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$p(u)$ detection probability of observing whale at location $u$.

Observed whales: $\mathbf{X}$ thinning of all whales $\mathbf{Y}$ i.e. inhomogeneous Cox process with random intensity function

$$p(u)\Lambda(u)$$

Note: $\mathbf{X}_{\neg\mathrm{obs}} = \mathbf{Y} \setminus \mathbf{X}$ and $\mathbf{X}$ independent Poisson processes given $\mathbf{\Phi}$.

# Parameters

Assume $p(u)$ known.

Assume $k(\cdot)$ bivariate Gaussian density truncated to have bounded support.

Parameters:

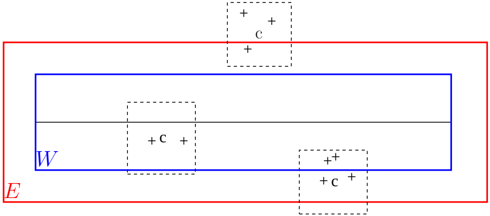
$\kappa$ intensity of cluster centres $c$

$\alpha = \mathbb{E}\gamma$ (expected cluster size)

$\omega$ standard deviation of Gaussian density

# Likelihood function for one transect

A transect:




$W$: support of $p(\cdot)$. $E$: $k(u-c)=0$ if $c \in \mathbb{R}^2 \setminus E$ and $u \in W$.

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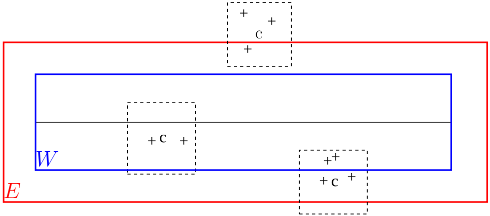


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Likelihood: $\theta = (\kappa, \alpha, \omega)$

1. $\mathbf{x}$ observed whales in $W$ with conditional Poisson density

$$f(\mathbf{x}|\mathbf{\Phi};\omega) = \exp\Big(\int_W (1 - p(u)\Lambda(u))\mathrm{d}u\Big) \prod_{u \in \mathbf{x}} p(u)\Lambda(u)$$

2.

$$L(\theta) = \mathbb{E}_{(\kappa,\alpha)} f_\theta(\mathbf{x}|\mathbf{\Phi};\omega) = \mathbb{E}_{(\kappa,\alpha)} f(\mathbf{x}|\mathbf{\Phi} \cap E;\omega)$$

# Derivatives of likelihood function

$\mathbf{\Phi}_E = \mathbf{\Phi} \cap E$ finite marked Poisson process with density

$$f(\phi; \kappa, \alpha) = \mathrm{e}^{|E|(1-\kappa)} \kappa^{n(\phi)} \prod_{(c,\gamma)\in\phi} \gamma^{\alpha-1} \exp(-\gamma)/\Gamma(\alpha)$$

Joint density of $\mathbf{X}$ and $\mathbf{\Phi}_E$:

$$f(\mathbf{x}, \phi; \kappa, \alpha, \omega) = f(\mathbf{x}|\phi; \omega) f(\phi; \kappa, \alpha)$$

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Joint density of $\mathbf{X}$ and $\mathbf{\Phi}_E$:

$$f(\mathbf{x},\phi;\kappa,\alpha,\omega) = f(\mathbf{x}|\phi;\omega)f(\phi;\kappa,\alpha)$$

Let

$$V_\theta(\mathbf{X},\mathbf{\Phi}_E) = \mathrm{d}\log f(\mathbf{X},\mathbf{\Phi}_E;\theta)/\mathrm{d}\theta$$

Score function and observed information

$$u(\theta) = \frac{\mathrm{d}\log L(\theta)}{\mathrm{d}\theta} = \mathbb{E}_\theta[V_\theta(\mathbf{X},\mathbf{\Phi}_E)|\mathbf{X}=\mathbf{x}] \quad \text{and}$$

$$j(\theta) = -\mathbb{E}_\theta[\frac{\mathrm{d}V_\theta(\mathbf{X},\mathbf{\Phi}_E)}{\mathrm{d}\theta^{\mathsf{T}}}|\mathbf{X}=\mathbf{x}] - \mathbb{V}\mathrm{ar}_\theta[V_\theta(\mathbf{X},\mathbf{\Phi}_E)|\mathbf{X}=\mathbf{x}]$$

# Importance sampling

$\theta = (\kappa, \alpha, \omega)$

$\mathbf{\Phi}_0, \mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_{n-1}$ sample from $f(\phi|\mathbf{x};\theta_0) = f(\mathbf{x},\phi;\theta_0)/f(\mathbf{x};\theta_0)$ for fixed $\theta_0 = (\kappa_0, \alpha_0, \omega_0)$

$$\begin{aligned}
\mathbb{E}_\theta[k(\mathbf{\Phi})|\mathbf{X}=\mathbf{x}] &= \frac{f(\mathbf{x};\theta_0)}{f(\mathbf{x},\theta)}\mathbb{E}_{\theta_0}\left[k(\mathbf{\Phi})\frac{f(\mathbf{x},\mathbf{\Phi};\theta)}{f(\mathbf{x};\mathbf{\Phi};\theta_0)}|\mathbf{X}=\mathbf{x}\right] \\
&\approx \frac{f(\mathbf{x};\theta_0)}{f(\mathbf{x},\theta)}\frac{1}{n}\sum_{m=0}^{n-1} k(\mathbf{\Phi}_m)\frac{f(\mathbf{x},\mathbf{\Phi}_m;\theta)}{f(\mathbf{x};\mathbf{\Phi}_m;\theta_0)}
\end{aligned}$$

$$\frac{f(\mathbf{x};\theta)}{f(\mathbf{x},\theta_0)} = \frac{L(\theta)}{L(\theta_0)} \approx \frac{1}{n}\sum_{m=0}^{n-1}\frac{f(\mathbf{x},\mathbf{\Phi}_m;\theta)}{f(\mathbf{x};\mathbf{\Phi}_m;\theta_0)}$$

Hence Monte Carlo approximations of likelihood ratios, score, and observed information.

# Markov chain Monte Carlo

Conditional density of $\mathbf{\Phi}_E$ given $\mathbf{X} = \mathbf{x}$:

$$f(\phi|\mathbf{x}) \propto f(\phi)f(\mathbf{x}|\phi) = f(\phi)\,\mathrm{e}^{-\int_W p(u)\Lambda(u|\phi)\mathrm{d}u} \prod_{u\in\mathbf{x}} p(u)\Lambda(u|\phi)$$

Computation of $\int_W p(u)\Lambda(u|\phi)\mathrm{d}u$ not straightforward.

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**Demarginalisation** impute $\mathbf{X}_{\neg\mathrm{obs}} = (\mathbf{Y} \cap W) \setminus \mathbf{X}$:

Full conditional distributions for $(\mathbf{\Phi}, X_{\neg\mathrm{obs}})$:

$$\mathbf{X}_{\neg\mathrm{obs}} | \mathbf{\Phi}_E, \mathbf{X} : \quad \mathrm{Poisson}\big((1 - p(\cdot))\Lambda(\cdot|\phi)\big)$$

$$\mathbf{\Phi}_E | \mathbf{X}_{\neg\mathrm{obs}}, \mathbf{X} : \quad f(\phi|\mathbf{x}, \mathbf{x}_{\neg\mathrm{obs}}) \propto f(\phi)\, \mathrm{e}^{-\int_W \Lambda(u|\phi)\mathrm{d}u} \prod_{u \in \mathbf{x} \cup \mathbf{x}_{\neg\mathrm{obs}}} \Lambda(u|\phi)$$

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MCMC (Metropolis-within-Gibbs):

- $\mathbf{X}_{\neg\mathrm{obs}}|\mathbf{\Phi}_E, \mathbf{X}$: straightforward.
- $\mathbf{\Phi}_E|\mathbf{X}_{\neg\mathrm{obs}}, \mathbf{X}$: birth/death MCMC updates (Geyer & Møller 1994).

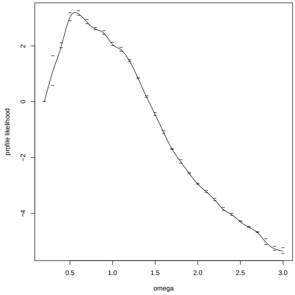
# Maximization of likelihood

Likelihood based on all transects: multiply likelihoods for the different transects (approximately independent)

Maximize with respect to $(\kappa, \alpha)$ for finite set of $\omega$ values (Newton-Raphson)

Profile log likelihood function

$l_p(\omega) = \max_{\kappa,\alpha} \log L(\kappa, \alpha, \omega)$:

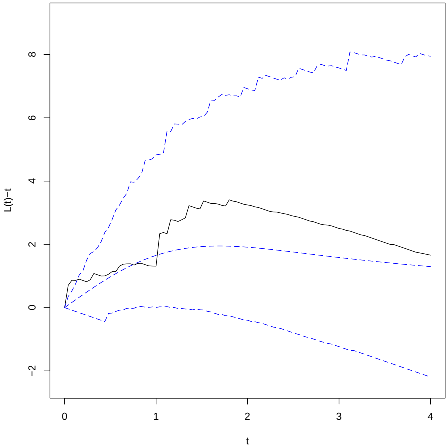



MLE: $\hat{\kappa} = 0.025$ $\hat{\alpha} = 2.4$ $\hat{\omega} = 0.6$.

95 % Confidence interval for whale intensity $\lambda = \kappa\alpha$: $[0.03; 0.08]$ (parametric bootstrap)

# Model check using $K$-function

Plot based on $L(t) - t = \sqrt{K(t)/\pi} - t$



# Bayesian approach

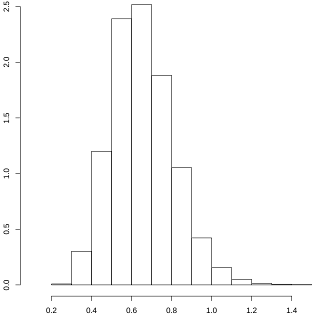
Use MCMC to sample posterior joint posterior of $(\kappa, \alpha, \omega, \Phi_E)$:

$$f(\kappa, \alpha, \omega, \phi | \mathbf{x}) \propto f(\mathbf{x}|\phi; \omega) f(\phi; \kappa, \alpha) p(\kappa, \alpha, \omega)$$

where $p(\kappa, \alpha, \omega)$ prior distribution.

(algorithm as before but extended with updates of parameters)

Posterior for $\omega$:




Bayesian 95% credibility interval for $\lambda$: $[0.04; 0.08]$

# Summary

Estimating functions:

- computationally fast
- `R` packages available: `spatstat` and `InhomCluster`

Likelihood-based inference:

- statistically more efficient
- long computations for MLE (Bayesian easier)
- no standard software

# References

Waagepetersen, R. and Guan, Y. (2007) Two-step estimation for inhomogeneous spatial point processes, submitted.

Waagepetersen, R. (2007) An estimating function approach to inference for inhomogeneous Neyman-Scott processes, *Biometrics*, **63**, 252-258.

Waagepetersen, R. and Schweder, T. (2006) Likelihood-based inference for clustered line transect data, *Journal of Agricultural, Biological, and Environmental Statistics*, **11**, 264-279.

Møller, J. and Waagepetersen, R. (2003) *Statistical inference and simulation for spatial point processes*, Chapman & Hall/CRC Press.