Estimating functions for inhomogeneous Cox processes

Rasmus Waagepetersen
Department of Mathematics
Aalborg University
Denmark

Outline

Tropical rain forest data sets

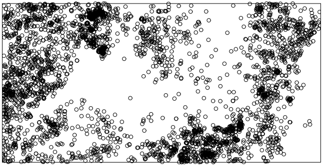
Inhomogeneous Cox processes

Inference based on estimating functions

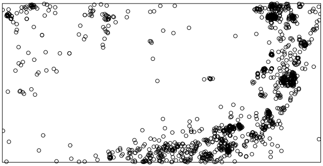
Data (Barro Colorado Island Forest Dynamics Plot)

Observation window: $S = [0, 1000] \times [0, 500]\text{m}^2$

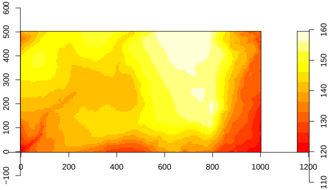
Beilschmiedia



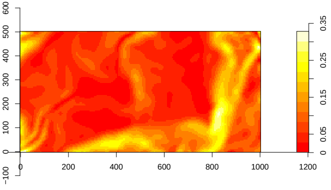
Ocotea



Elevation



Gradient norm (steepness)

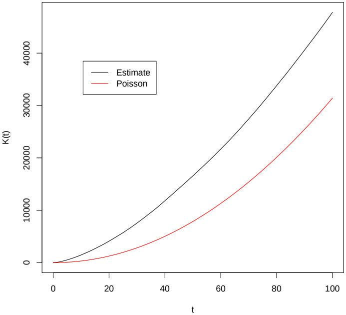


Question: tree intensities related to elevation and gradient ?

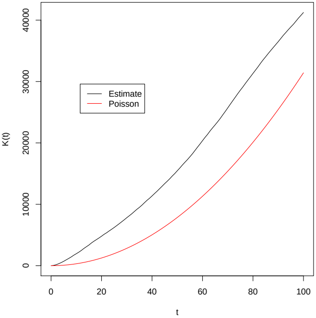
Additional source of variation: clustering due to seed dispersal.

K-functions (adjusted for inhomogeneity due to covariates)

Beilschmidia

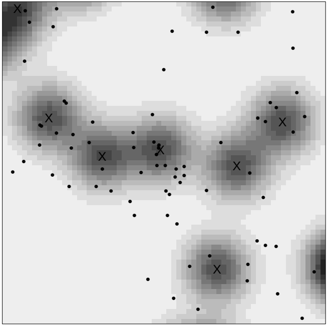


Ocotea



Poisson process not appropriate.

Cluster process (Thomas process)



Mothers (crosses) Poisson point process $\mathbf{\Phi}$ with intensity $\kappa > 0$.

Offspring $\mathbf{X} = \cup_{c \in \mathbf{\Phi}} \mathbf{X}_c$ distributed around mothers c according to bivariate Gaussian density f.

ω: standard deviation of Gaussian density

α: mean of Poisson number of offspring for each mother.

Random intensity function:

$$\Lambda(u) = \alpha \sum_{c \in \mathbf{\Phi}} f(u - c; \omega)$$

Inhomogeneous Cox process

$z_{1:p}(u) = (z_1(u), \ldots, z_p(u))$ vector of p nonconstant covariates.

$\beta_{1:p} = (\beta_1, \ldots \beta_p)$ regression parameter.

Random intensity function:

$$\Lambda(u) = \alpha \exp(z(u)_{1:p}\beta_{1:p}^{\mathsf{T}}) \sum_{c \in \mathbf{\Phi}} f(u - c; \omega)$$

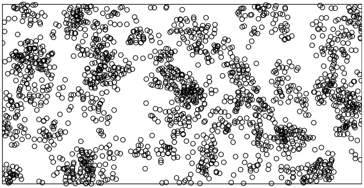
Rain forest example:

$$z_{1:2}(u) = (z_{\mathrm{elev}}(u), z_{\mathrm{grad}}(u))$$

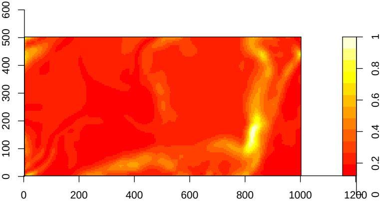
elevation/gradient covariate

Interpretation in terms of thinning

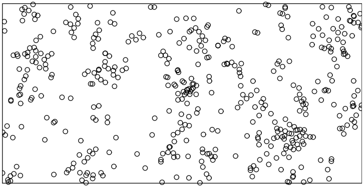
Homogeneous Cox process



Survival probabilities
$p(u) \propto \exp(z_{1:2}(u)\beta_{1:2}^{\mathsf{T}})$



After thinning (inhomogeneous Cox)



Parameter Estimation: regression parameters

Intensity function for inhomogeneous Cox:

$$\rho_{\beta}(u) = \kappa\alpha \exp(z(u)_{1:p}\beta_{1:p}^{\mathsf{T}}) = \exp(z(u)\beta^{\mathsf{T}})$$

$$z(u) = (1, z_{1:p}(u)) \quad \beta = (\log(\kappa\alpha), \beta_{1:p})$$

Parameter Estimation: regression parameters

Intensity function for inhomogeneous Cox:

$$\rho_\beta(u) = \kappa\alpha \exp(z(u)_{1:p}\beta_{1:p}^\mathsf{T}) = \exp(z(u)\beta^\mathsf{T})$$
$$z(u) = (1, z_{1:p}(u)) \quad \beta = (\log(\kappa\alpha), \beta_{1:p})$$

Consider indicators $N_i = \mathbf{1}[\mathbf{X} \cap C_i \neq \emptyset]$ of occurrence of points in disjoint C_i ($W = \cup C_i$) where $P(N_i = 1) \approx \rho_\beta(u_i)\mathrm{d}C_i$, $u_i \in C_i$.

Parameter Estimation: regression parameters

Intensity function for inhomogeneous Cox:

$$\rho_\beta(u) = \kappa\alpha \exp(z(u)_{1:p}\beta_{1:p}^\mathsf{T}) = \exp(z(u)\beta^\mathsf{T})$$
$$z(u) = (1, z_{1:p}(u)) \quad \beta = (\log(\kappa\alpha), \beta_{1:p})$$

Consider indicators $N_i = \mathbf{1}[\mathbf{X} \cap C_i \neq \emptyset]$ of occurrence of points in disjoint C_i ($W = \cup C_i$) where $P(N_i = 1) \approx \rho_\beta(u_i)\mathrm{d}C_i$, $u_i \in C_i$.

Composite likelihood

$$\prod_{i=1}^{n} (\rho_\beta(u_i)\mathrm{d}C_i)^{N_i}(1-\rho_\beta(u_i)\mathrm{d}C_i)^{1-N_i} \equiv \prod_{i=1}^{n} \rho_\beta(u_i)^{N_i}(1-\rho_\beta(u_i)\mathrm{d}C_i)^{1-N_i}$$

Limit ($\mathrm{d}C_i \to 0$) of log composite likelihood

$$l(\beta) = \sum_{u \in \mathbf{X} \cap W} \log \rho_\beta(u) - \int_W \rho_\beta(u)\,\mathrm{d}u$$

Maximize using `spatstat` to obtain $\hat{\beta}$.

Asymptotic distribution of regression parameter estimates

Assume increasing mother intensity: $\kappa_n = n\tilde{\kappa} \to \infty$ and $\mathbf{\Phi} = \cup_{i=1}^{n} \mathbf{\Phi}_i$, $\mathbf{\Phi}_i$ independent Poisson processes of intensity $\tilde{\kappa}$.

Score function asymptotically normal:

$$\frac{1}{\sqrt{n}} \frac{\mathrm{d} l(\beta)}{\mathrm{d} \log \alpha \mathrm{d}\beta_{1:p}} = \frac{1}{\sqrt{n}} \left(\sum_{u \in \mathbf{X} \cap W} z(u) - n\tilde{\kappa}\alpha \int_W z(u) \exp(z(u)_{1:p}\beta_{1:p}^{\mathsf{T}}) \mathrm{d}u \right)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big[\sum_{c \in \mathbf{\Phi}_i} \sum_{u \in \mathbf{X}_c \cap W} z(u) - \tilde{\kappa}\alpha \int_W \exp(z_{1:p}(u)\beta_{1:p}^{\mathsf{T}}) \mathrm{d}u \Big] \approx N(0, V)$$

where $V = \mathbb{V}\mathrm{ar} \sum_{c \in \mathbf{\Phi}_i} \sum_{u \in \mathbf{X}_c \cap W} z(u)$

Asymptotic distribution of regression parameter estimates

Assume increasing mother intensity: $\kappa_n = n\tilde{\kappa} \to \infty$ and $\mathbf{\Phi} = \cup_{i=1}^n \mathbf{\Phi}_i$, $\mathbf{\Phi}_i$ independent Poisson processes of intensity $\tilde{\kappa}$.

Score function asymptotically normal:

$$\frac{1}{\sqrt{n}} \frac{\mathrm{d}l(\beta)}{\mathrm{d}\log\alpha \mathrm{d}\beta_{1:p}} = \frac{1}{\sqrt{n}} \left(\sum_{u \in \mathbf{X} \cap W} z(u) - n\tilde{\kappa}\alpha \int_W z(u) \exp(z(u)_{1:p}\beta_{1:p}^{\mathsf{T}}) \mathrm{d}u \right)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big[\sum_{c \in \mathbf{\Phi}_i} \sum_{u \in \mathbf{X}_c \cap W} z(u) - \tilde{\kappa}\alpha \int_W \exp(z_{1:p}(u)\beta_{1:p}^{\mathsf{T}}) \mathrm{d}u \Big] \approx N(0, V)$$

where $V = \mathbb{V}\mathrm{ar} \sum_{c \in \mathbf{\Phi}_i} \sum_{u \in \mathbf{X}_c \cap W} z(u)$

By standard results for estimating functions (J observed information for Poisson likelihood):

$$\sqrt{\kappa_n}\big[(\log(\hat{\alpha}), \hat{\beta}_{1:p}) - (\log\alpha, \beta_{1:p})\big] \approx N(0, J^{-1} V J^{-1})$$

Parameter Estimation: clustering parameters

Theoretical expression for (inhomogeneous) K-function:

$$K(t;\kappa,\omega) = \pi t^2 + \left(1 - \exp(-t^2/(2\omega)^2)\right)/\kappa.$$

Parameter Estimation: clustering parameters

Theoretical expression for (inhomogeneous) K-function:

$$K(t;\kappa,\omega) = \pi t^2 + \left(1 - \exp(-t^2/(2\omega)^2)\right)/\kappa.$$

Semi-parametric estimate

$$\hat{K}(t) = \sum_{u,v \in \mathbf{X} \cap W} \frac{1[0 < \|u - v\| \leq t]}{\rho_{\hat{\beta}}(u)\rho_{\hat{\beta}}(v)|W \cap W_{u-v}|}$$

Parameter Estimation: clustering parameters

Theoretical expression for (inhomogeneous) K-function:

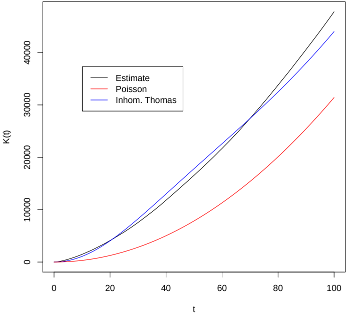
$$K(t;\kappa,\omega) = \pi t^2 + \left(1 - \exp(-t^2/(2\omega)^2)\right)/\kappa.$$

Semi-parametric estimate

$$\hat{K}(t) = \sum_{u,v \in \mathbf{X} \cap W} \frac{1[0 < \|u-v\| \le t]}{\rho_{\hat{\beta}}(u)\rho_{\hat{\beta}}(v)|W \cap W_{u-v}|}$$

Estimate κ and ω by minimizing contrast

$$\int_0^{100} \left(K(t;\kappa,\omega)^{1/4} - \hat{K}(t)^{1/4}\right)^2 \mathrm{d}t$$



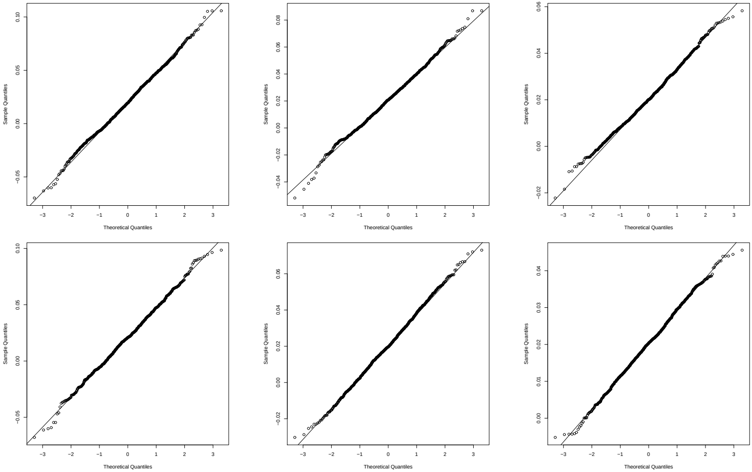
Simulation study

Quantile plots of $\hat{\beta}_{elev}$ (varying expected numbers 25, 50 and 250 of mothers and offspring, 200 or 800)

	25	50	250

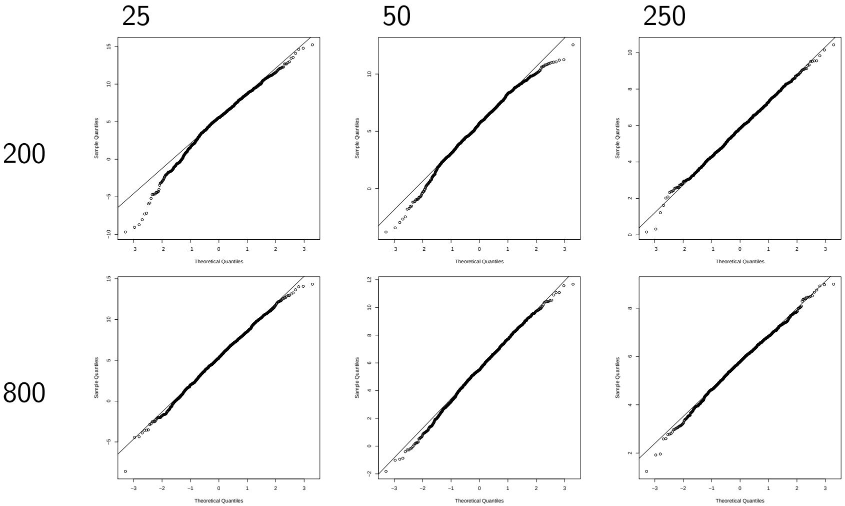
200

800



Simulation study II

Quantile plots of $\hat{\beta}_{\text{grad}}$ (varying expected numbers 25, 50 and 250 of mothers and offspring, 200 or 800)



Results for Beilschmiedia

Parameter estimates and confidence intervals (Poisson in red).

Elevation		Gradient		κ	α	ω
0.021	[-0.018,0.061]	5.842	[0.885,10.797]	8e-05	85.9	20.0
	[0.017,0.026]		[5.340,6.342]			

Clustering: less information in data and wider confidence intervals than for Poisson process (independence).

Evidence of positive association between gradient and Beilschmiedia intensity.

Alternative methods of parameter estimation

1. MLE based on birth-death MCMC algorithm for mother points computationally difficult:
   - need to evaluate

     $$f(\mathbf{x}|\Lambda) = \mathrm{e}^{\int_S (1-\Lambda(u))\mathrm{d}u} \prod_{u \in \mathbf{x}} \Lambda(u)$$

     in each MCMC iteration (birth or death of mother point): numerical integration
   - birth or death of mother point: big change in

     $$\Lambda(u) = \alpha \exp(z(u)_{1:p} \beta_{1:p}^{\mathsf{T}}) \sum_{c \in \mathbf{\Phi}} f(u - c; \omega)$$

     hence low acceptance rates.

Alternative methods of parameter estimation

1. MLE based on birth-death MCMC algorithm for mother points computationally difficult:
   - need to evaluate

     $$f(\mathbf{x}|\Lambda) = \mathrm{e}^{\int_S (1-\Lambda(u))\mathrm{d}u} \prod_{u \in \mathbf{x}} \Lambda(u)$$

     in each MCMC iteration (birth or death of mother point): numerical integration
   - birth or death of mother point: big change in

     $$\Lambda(u) = \alpha \exp(z(u)_{1:p} \beta_{1:p}^{\mathsf{T}}) \sum_{c \in \mathbf{\Phi}} f(u - c; \omega)$$

     hence low acceptance rates.

2. Second-order estimating function: instead of intensity use second-order product density

   $$\lambda^{(2)}(u, v; \beta, \kappa, \omega) = \rho_\beta(u)\rho_\beta(v) g(\|u - v\|; \kappa, \omega)$$

   where $g(r; \kappa, \omega) = 1 + \exp(-r^2/(2\omega)^2)/(4\pi\kappa\omega^2)$ (pair correlation)

Second order estimating function

Consider composite likelihood for indicators

$$N_{ij} = 1[\mathbf{X} \cap C_i \neq \emptyset \text{ and } \mathbf{X} \cap C_j \neq \emptyset]$$

of simultaneous occurrence of points in disjoint C_i and C_j where

$$P(N_{ij} = 1) \approx \rho_{\beta}^{(2)}(u, v; \kappa, \omega)\mathrm{d}C_i\mathrm{d}C_j = \rho_{\beta}(u)\rho_{\beta}(v)g(\|u-v\|; \kappa, \omega)\mathrm{d}C_i\mathrm{d}C_j$$

Second order estimating function

Consider composite likelihood for indicators

$$N_{ij} = 1[\mathbf{X} \cap C_i \neq \emptyset \text{ and } \mathbf{X} \cap C_j \neq \emptyset]$$

of simultaneous occurrence of points in disjoint C_i and C_j where

$$P(N_{ij} = 1) \approx \rho_\beta^{(2)}(u, v; \kappa, \omega)\mathrm{d}C_i\mathrm{d}C_j = \rho_\beta(u)\rho_\beta(v)g(\|u-v\|; \kappa, \omega)\mathrm{d}C_i\mathrm{d}C_j$$

Log composite likelihood converges ($\mathrm{d}C_i \to 0$) to

$$l_2(\beta, \kappa, \omega) = \sum_{u,v \in \mathbf{X}}^{\neq} \log \rho_\beta^{(2)}(u, v; \kappa, \omega) - \iint_{W^2} \rho_\beta^{(2)}(u, v; \kappa, \omega)\mathrm{d}u\mathrm{d}v$$

Maximize to obtain joint estimate of (κ, ω, β)

Computationally involved (double integrals), similar efficiency as two-step method in preliminary simulation study.

References

Waagepetersen, R. (2006) An estimating function approach to inference for inhomogeneous Neyman-Scott processes, *Biometrics*, to appear.

Møller, J. and Waagepetersen, R. (2003) *Statistical inference and simulation for spatial point processes*, Chapman & Hall/CRC Press.

Software: `R` packages `spatstat` (Baddeley & Turner) and `InhomCluster`