Orthogonal projections

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Orthogonal decomposition

Suppose L subspace of $\mathbb{R}^n$. Let
$L^\perp = \{v \in \mathbb{R}^n | v \bullet w = 0 \text{ for all } w \in L\}$.

Orthogonal decomposition: each $x \in \mathbb{R}^n$ has a unique decomposition

$$x = u + v$$

where $u \in L$ and $v \in L^\perp$.

Orthogonal projection: u and v above are the orthogonal projections $p_L(x)$ and $p_{L^\perp}(x)$ of x on respectively L and $L^\perp$.

Pythagoras:

$$\|x\|^2 = \|u\|^2 + \|v\|^2$$

Orthogonal projections

- the orthogonal projection $p_L : \mathbb{R}^n \to L$ on L is a linear mapping. It is thus given by a unique matrix-transformation $p_L(x) = Px$ where P is an $n \times n$ matrix.
- the projection matrix P is symmetric ($P^\mathsf{T} = P$) and idempotent ($P^2 = P$)
- conversely, if a matrix Q is symmetric, idempotent and $L = \mathrm{col} Q$ then Q is the matrix of the orthogonal projection on L.
- if $L = \mathrm{col} X$ and X full rank then $P = X(X^\mathsf{T} X)^{-1} X^\mathsf{T}$

Example: orthogonal projection on subspace spanned by a single vector v is $\frac{v^\mathsf{T} x}{\|v\|^2} x$

Example: orthogonal projection on subspace spanned by a orthogonal vectors $v_1, \ldots, v_p$ is $\sum_{i=1}^{p} \frac{v_i^\mathsf{T} x}{\|v_i\|^2} x$

Example: If P is matrix of orthogonal projection on L then $I - P$ is matrix of orthogonal projection on $L^\perp$