GLM - tætheder

$$f(y|\mu) = \exp\left(\frac{y\xi - b(\xi)}{a(\phi)} + c(y, \phi)\right) \qquad \mu = EY$$

$$g(\mu) = \eta = x^T\beta \ (+ Z^T\alpha \text{ hvis GLMM}) \qquad g \text{ injektiv link-funktion}$$

$$\mu = b'(\xi) \quad \text{hvor} \quad b' \text{ injektiv}$$

$$\eta \Leftrightarrow \mu \Leftrightarrow \xi$$

Kanonisk link: $\quad g = (b')^{-1} \quad \Rightarrow \quad \xi = \eta$

Varians $\quad \operatorname{Var} Y = b''(\xi)\, a(\phi) = V(\mu)\, a(\phi).$

NB: hvis kanonisk link fås

$$g' = \frac{1}{b''} = V(\mu)^{-1}.$$

<u>Ex</u> (Poisson, log link)

$g = \log \qquad b' = \exp \qquad b'' = \exp \qquad a(\phi) = 1$

$EY = \mu = \exp(\eta) = \operatorname{Var} Y.$

$f(y|\mu) = \exp(y \log\mu - \exp(\mu))$

Laplace-transform

$$L(t) = E\exp(tY) = \int \exp\left(\frac{y\,t\,a(\varphi) + y\xi - b(\xi)}{a(\varphi)} + c(y,\varphi)\right) dy$$

$$= \exp\left(\frac{-b(\xi)}{a(\varphi)} + \frac{b(t\,a(\varphi) + \xi)}{a(\varphi)}\right)$$

$$EY = \frac{d}{dt}\log L(t)\Big|_{t=0} = b'(\xi)$$

$$\mathrm{Var}\,Y = \frac{d^2}{dt^2}\log L(t)\Big|_{t=0} = a(\varphi)\,b''(\xi).$$

Laplace og PQL

Likelihood

$$L(\beta) = \int \prod_{i=1}^{n} f(y_i|\mu_i)\, f(\alpha)\, d\alpha$$

$$\equiv |G|^{-\frac{1}{2}} \int \exp\left(-\frac{1}{2}\sum_{i=1}^{n} d_i - \frac{1}{2}\alpha^T G^{-1}\alpha\right) d\alpha$$

hvor

$$d_i = -2\left[l(y_i|\mu_i) - l(y_i|y_i)\right]$$

$$= -2\int_{y_i}^{\mu_i} \frac{d}{d\mu}\, l(y_i|\mu)\, d\mu$$

$$\left(l(y_i|\mu_i) = \log f(y_i|\mu_i)\right)$$

d_i : deviance for i'te observation

$$\frac{d}{d\mu}\, l(y_i|\mu) = \frac{1}{a(\varphi)}\left[y_i - \mu\right]\frac{d\xi_i}{d\mu}$$

$$= \frac{y_i - \mu}{a(\varphi)\, V(\mu_i)}$$

$$\left(\text{dermed} \quad \frac{d}{d\mu} d_i = -2\, \frac{y_i - \mu}{a(\varphi)\, V(\mu)} \right)$$

Sæt $q(\alpha) = \frac{1}{2}\sum_{i=1}^{n} d_i + \frac{1}{2}\alpha^T G^{-1}\alpha$

Dermed (Laplace-approximation) $(q'(\tilde{\alpha}) = 0)$

$$L(\beta) \propto |G|^{-\frac{1}{2}} \, |q''(\tilde{\alpha})|^{-\frac{1}{2}} \exp(-q(\tilde{\alpha}))$$

$$q'(\alpha) = \frac{1}{2}\sum_{i=1}^{n} \frac{d}{d\mu} d_i \frac{d\mu}{d\xi_i}\frac{d\xi_i}{d\alpha} + G^{-1}\alpha$$

(kanonisk link $\Rightarrow \xi = \eta$) (antager kanonisk link)

$$= -\sum_{i=1}^{n} \frac{y_i - \mu_i}{a(\phi)\cancel{V(\mu_i)}} \cancel{V(\mu_i)} Z_i + G^{-1}\alpha$$

$$q''(\alpha) = \sum_{i=1}^{n} \frac{V(\mu_i)}{a(\phi)} Z_i Z_i^T + G^{-1}$$

$$= Z^T W Z + G^{-1}$$

$$Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \qquad W = \operatorname{diag}\left(\frac{V(\mu_i)}{a(\phi)}\right)$$

$$L(\beta) \propto |G|^{-\frac{1}{2}} \, |Z^T W Z G + I|^{-\frac{1}{2}} \, |G|^{\frac{1}{2}} \exp(-q(\tilde{\alpha}))$$

$$= |Z^T W Z G + I|^{-\frac{1}{2}} \exp(-q(\tilde{\alpha}))$$

(dvs. Laplace-approximationen)

Antag yderligere $\frac{d}{d\beta} w \approx 0$

$$l(\beta) = \log L(\beta) \approx -\frac{1}{2}\sum_{i=1}^{n} \tilde{d}_i - \frac{1}{2}\tilde{\alpha}^T G^{-1}\tilde{\alpha}$$

Definer

$$l_{PQ}(\beta,\alpha) = -\frac{1}{2}\sum_{i=1}^{n} d_i - \frac{1}{2}\alpha^T G^{-1}\alpha \quad \left(= \log f(y|\alpha;\beta) f(\alpha)\right)$$

Bemærk: $\boxed{l_{PQ}(\beta,\alpha) = q(\alpha)!}$

Dvs. for givet β maksimerer $\tilde{\alpha} = \tilde{\alpha}(\beta)$ $l_{PQ}(\beta,\alpha)$ mht α.

- og for givet $\tilde{\alpha}$ maksimerer $\hat{\beta}(\tilde{\alpha})$ $l_{PQ}(\beta,\tilde{\alpha})$.

Det svarer til en iterativ proces som munder

ud i $(\hat{\beta},\hat{\alpha}) = \arg\max\, l_{PQ}(\alpha,\beta)$

<u>Equations</u> (løse $\frac{d}{d(\alpha,\beta)} l_{PQ}(\alpha,\beta) = 0$)

$$\frac{d}{d\beta} l_{PQ}(\beta) = \sum_{i=1}^{n} \frac{y_i - \mu_i}{a(\varphi)} X_i = X^T W\left[(y_i-\mu_i)g'(\mu_i)\right] \quad \left(g'(\mu) = \frac{1}{V(\mu)}\right)$$

$$\frac{d}{d\alpha} l_{PQ}(\alpha) = \sum_{i=1}^{n} \frac{y_i-\mu_i}{a(\varphi)} \underset{m\times 1}{Z_i} - G^{-1}\alpha = Z^T W\left[(y_i-\mu_i)g'(\mu_i)\right] - G^{-1}\alpha$$

$$\frac{d^2}{d\beta^T d\beta} l_{PQ}(\beta) = -X^T W X \qquad \frac{d^2}{d\alpha^T d\beta} l_{PQ}(\beta) = -Z^T W X$$

$$\frac{d^2}{d\alpha^T d\alpha} l_{PQ}(\beta) = -Z^T W Z - G^{-1} \qquad W = \mathrm{diag}\left(\frac{V(\mu_i)}{a(\varphi)}\right)$$

Newton-raphson:

$$\begin{pmatrix}\tilde{\beta}_{l+1}\\ \tilde{\alpha}_{l+1}\end{pmatrix} = \begin{pmatrix}\tilde{\beta}_l\\ \tilde{\alpha}_l\end{pmatrix} + \begin{bmatrix} X^T W X & X^T W Z \\ Z^T W X & Z^T W Z + G \end{bmatrix}^{-1} \left[\begin{bmatrix} \overset{p\times n}{X^T}\,\overset{n\times n}{W} \\ \overset{m\times n}{Z^T} W \end{bmatrix} \overset{n\times 1}{\left[(y_i-\mu_i)g'(\mu_i)\right]} - \begin{bmatrix} 0 \\ G^{-1}\alpha \end{bmatrix}\right]$$

$$\begin{bmatrix} X^T W X & X^T W Z \\ Z^T W X & Z^T W Z + G^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\beta}_{l+1} - \tilde{\beta}_l \\ \tilde{\alpha}_{l+1} - \tilde{\alpha}_l \end{bmatrix} = \underbrace{\begin{bmatrix} X^T W \\ Z^T W \end{bmatrix} \left[(y_i - \mu_i)\, g'(\mu_i) \right] - \begin{bmatrix} 0 \\ G^{-1}\alpha \end{bmatrix}}$$

$$\begin{bmatrix} X^T W X & X^T W Z \\ Z^T W X & Z^T W Z + G^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\beta}_{l+1} \\ \tilde{\alpha}_{l+1} \end{bmatrix} - \begin{bmatrix} X^T W \\ Z^T W \end{bmatrix} \underbrace{\begin{bmatrix} X\tilde{\beta}_l & Z\tilde{\alpha}_l \end{bmatrix}}_{\tilde{\eta}_l} - \begin{bmatrix} 0 \\ G\tilde{\alpha}_l \end{bmatrix} = \begin{bmatrix} X^T W \\ Z^T W \end{bmatrix} \left[(y_i - \mu_i)\, g'(\mu_i) \right] - \begin{bmatrix} 0 \\ G^{-1}\tilde{\alpha}_l \end{bmatrix}$$

$$\eta = X\beta + Z\alpha =$$

$$\begin{bmatrix} X^T W X & X^T W Z \\ Z^T W X & Z^T W Z + G^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\beta}_{l+1} \\ \tilde{\alpha}_{l+1} \end{bmatrix} = \begin{bmatrix} X^T W \\ Z^T W \end{bmatrix} \left[\tilde{\eta}_l + (y_i - \mu_i)\, g'(\mu_i) \right]$$

Svarer til mixed models equations (2.37) side 76 i Jiang