Zorn's Lemma

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A set S is partially ordered if there exists an order relation $\leq$ which is reflexive ($x \leq x$ for all x), antisymmetric (if $x \leq y$ and $y \leq x$ then $x = y$) and transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$). If $x \leq y$ and $x \neq y$, then we write $x < y$ or $y > x$.

A chain in S is a subset C in which any two elements are comparable, that is for every $x, y \in C$ then either $x \leq y$ or $y \leq x$.

An element $m \in S$ is called maximal if there is no other $x \in S$ such that $m < x$. This does not mean that m is the largest element, which would be an element $M \in S$ such that $x \leq M$ for every $x \in S$.

Theorem 0.1. *(Zorn's lemma). Let S be a partially ordered set in which every chain has an upper bound. Then S has at least one maximal element.*

Proof. We first prove a weaker version, in which we assume that every chain has a least upper bound (supremum). More precisely, for every chain C there exists an element called $\sup(C)$ which *(i)* is an upper bound for C, i.e. for every $x \in C$ we have $x \leq \sup(C)$, and *(ii)* is the smallest upper bound, i.e. for every $x \in C$ such that $x < \sup(C)$ there exists $z \in C$ such that $x < z \leq \sup(C)$.

Proposition 0.2. *Let S be a partially ordered set in which every chain has a supremum. Then S has at least one maximal element.*

Proof. Define a "successor" operation on S as follows: if x is non-maximal, choose some $y > x$ and set $\phi(x) = y$. If x is maximal, put $\phi(x) = x$. Note that the existence of ϕ is insured by the axiom of choice.

Now we say that a subset $N \subseteq S$ is a *tower* if we have the following two properties:

$$P1. \text{ If } x \in N, \text{ then } \phi(x) \in N;$$
$$P2. \text{ For any chain } C \subseteq N, \text{ then } \sup(C) \in N.$$

Let us note that S itself is a tower, and the intersection of any family of towers is a tower (exercise). In particular, the intersection of all possible towers is a tower. Denote the smallest (non-empty) tower of S with M.

Definition 0.3. *($P3$) We say that $x \in M$ has the property $P3$ if for any $y \in M$, we either have $y \leq x$ or $y \geq \phi(x)$.*

Lemma 0.4. *Assume that $P3$ holds for all $x \in M$. Then M is a chain, and M has a largest element which also is a maximal element of S.*

Proof. Let us prove that M is chain. For, take $x, y \in M$. Then $P3$ applied for x says that if we do not have $y \leq x$, then we must have $\phi(x) \leq y$. But $\phi(x) \in M$ due to $P1$, and $x \leq \phi(x)$. The transitivity then gives $x \leq y$, hence M is a chain. Now because M is a chain, then due to $P2$ it must contain its supremum $\sup(M)$. But then $\sup(M)$ is a maximal element, because on one hand $\phi(\sup(M)) \geq$

sup(M), and on the other hand due to $P1$ we have that $\phi(\sup(M)) \in M$ hence $\phi(\sup(M)) \leq \sup(M)$. The antisymmetry gives $\phi(\sup(M)) = \sup(M)$ and we get our maximal element, thus proving the lemma. □

From the above lemma we see that the proposition is proved if we can show that $P3$ holds for all point of M. In order to do that, we first need another definition:

Definition 0.5. *($P4$) We say that $x \in M$ has property $P4$ if for any $y \in M$ with $y < x$ we have $\phi(y) \leq x$.*

Lemma 0.6. *If $x \in M$ obeys $P4$, then it also obeys $P3$.*

Proof. Let

$$M' := \{y \in M : \; y \leq x \quad \text{or} \quad y \geq \phi(x)\}.$$

If we can prove that M' is a tower, then $M' = M$ because M is the smallest tower of S. So we need to verify $P1$ and $P2$ for M'. We start with $P1$, that is we need to show that for any $y \in M'$ we have $\phi(y) \in M'$. Indeed, if $y \in M'$ then we either have a) $y < x$, b) $y = x$ or c) $y \geq \phi(x)$. If a) holds, then $P4(x)$ says that $\phi(y) \leq x$ hence $\phi(y) \in M'$, thus $P1$ holds. If either b) or c) holds, then we trivially have $\phi(y) \geq \phi(x)$, thus $\phi(y) \in M'$, hence $P1$ holds.

In order to verify $P2$, we need to show that if $C \subseteq M'$ is a chain, then $\sup(C) \in M'$. Clearly, because C is also a chain in M, we have that $\sup(C) \in M$.

Now we have two possibilities: a) $z \leq x$ for all $z \in C$, and b) there exists some $z \in C$, $z > x$. If a) holds, then x is an upper bound hence $\sup(C) \leq x$, thus $\sup(C) \in M'$. If b) holds, then because $z > x$ and $z \in M'$ implies that $z \geq \phi(x)$, hence $\sup(C) \geq \phi(x)$, thus $\sup(C) \in M'$. Since $P2$ is also verified, then M' is a tower and $M = M'$. □

The last step in the proof of the proposition, is showing that $P4$ holds true for every $x \in M$. For, denote by N the set of points $x \in M$ which obey $P4$. As above, it suffices to show that N is a tower.

We start with proving $P1$ for N. Take $x \in N$, and we want to show that $\phi(x) \in N$. For that, look at all $y \in M$ with $y < \phi(x)$ and try to show that $\phi(y) \leq \phi(x)$.

Because we know from Lemma 0.6 that x obeys $P3$, the only possibility is to have $y \leq x$. Then a) $y < x$ or b) $y = x$. If a) holds, then because x was supposed to obey $P4$ we get $\phi(y) \leq x$, hence $\phi(y) \leq \phi(x)$. If b) holds, then trivially $\phi(y) \leq \phi(x)$. In both cases we proved that $\phi(x) \in N$ hence $P1$ is fulfilled.

We now prove that $P2$ holds. Consider a chain $C \subseteq N$; we want to show that $\sup(C) \in N$, i.e. $\sup(C)$ has the property $P4$. In other words, for every $y \in M$ with $y < \sup(C)$ we need to show that $\phi(y) \leq \sup(C)$. Now from $y < \sup(C)$ it means that y is not an upper bound for C, so it exists $z \in C$ such that $z \not\leq y$. This means that either a) y and z are not comparable, or b) $y < z$. But $z \in N$ has property $P4$ and hence $P3$ (from Lemma 0.6), thus z and $y \in M$ are comparable, hence b) holds. Now apply $P4(z)$: it gives $\phi(y) \leq z$, hence $\phi(y) \leq \sup(C)$. Therefore $\sup(C) \in N$, and $P2$ is verified. We conclude that N is a tower, therefore $N = M$.

Finishing the proof of Proposition 0.2. We have just shown that all points of M have the property $P4$. Lemma 0.6 showed that $P4$ implies $P3$. Then Lemma 0.4 says that M must have a largest element, which was shown to be a maximal element of S. Thus Proposition 0.2 is proved. $\square$

We now use Proposition 0.2 for proving the *Hausdorff maximal principle*:

Lemma 0.7. *(The Hausdorff maximal principle). Let Q be a partially ordered set. Then Q contains a maximal chain (i.e. a chain which is not contained in a bigger chain).*

Proof. Define S to be the set of all chains of Q, partially ordered with respect to the set inclusion. More precisely, if C_1 and C_2 are chains in Q (and elements of S), then we say that $C_1 \leq_S C_2$ in S if $C_1 \subseteq C_2$ in Q. It is easy to prove that $\leq_S$ is a partial order (exercise).

Another important property is that the intersection in Q of two chains is a chain, and in fact an arbitrary intersection of chains from Q is a chain (exercise).

Now let us denote an arbitrary chain in S by K. Note that the elements of K in S consist of chains in Q. Denote by $\#K$ the set K seen as a set formed of elements of Q; clearly, $\#K$ is a chain in Q. If Q has no maximal chain, then there should exist at least one element $k \in Q$ such that $x <_Q k$ for all $x \in \#K$. Then $\#\tilde{K} := \#K \cup \{k\}$ is a chain in Q, and $\tilde{K} := K \cup \{k\}$ is an upper bound for K in S.

Now if A_1 and A_2 in S are upper bounds for K, then $A_1 \cap A_2$ is also an upper bound for K (exercise). Define $\sup(K)$ as the intersection of all possible upper bounds of K; then $\sup(K)$ is a chain in Q, and by construction it is a least upper bound for K.

Therefore S is a partially ordered set where all chains K have a supremum. Proposition 0.2 now states that there exists a maximal element $K \in S$. But this is the same with saying that $\#K$ is a chain in Q which is not included in a longer chain, and the proof of this lemma is over.

Finishing the proof of Zorn's Lemma. We can now lift the extra-condition in Proposition 0.2. Assume that S is a partially ordered set, where every chain has an upper bound. According to the Hausdorff maximum principle, there exists a maximal chain $C \subseteq S$. Being a chain, C must have an upper bound $x \in S$, and this means that $C \cup \{x\}$ is another chain in S. But C is maximal, therefore $x \in C$. Moreover, x must be the largest element of C. Finally, x is a maximal element in S and $\phi(x) = x$, because if $\phi(x) > x$ we can consider $C \cup \{\phi(x)\}$ which would contradict the maximality of C. The proof of the theorem is over. $\square$