Workshop at Aalborg University: *Random Number Representations, Stochastic Processes, and Quantum Transport*

May 19-22, 2024

This workshop has an exploratory nature and concerns various topics in metric number theory, probability theory, and mathematical physics. The presenters of the talks are assumed to account for the fact that the participants have diverse backgrounds in mathematics.

The participants from abroad are planned to arrive on Sunday, May 19, or sooner, and to leave on Wednesday, May 22, or later.

The scientific program will take place on Monday and Tuesday, with five one-hour talks (including discussion and breaks) each day. Please see below.

The workshop is financed by the Independent Research Fund Denmark through a project entitled Peculiar Distribution Functions and Interesting Stochastic Processes.

Program Monday-Tuesday

The lectures take place in Room 3.565 at AAU CREATE (Rendsburggade 14, DK-9000 Aalborg).

Monday:

9:00–10:00 *Manon Stipulanti*: Parry's 1960 theorem and some applications in combinatorics on words.
10:00–11:00 *Tom Kempton*: Counting representations of integers as sums of Fibonacci numbers.
11:00–12:00 *Anne Marie Svane*: Asymptotic distribution of the scaled remainder in base-q digit expansions.
12:00–13:30 *Lunch* at Hos Henius (Slotsgade 33, DK-9000 Aalborg).
13:30–14:30 *Jesper Møller*: The sufficient digits of continuous random variables.
14:30–15:30 *Horia Cornean*: On a certain discrete dynamical system, analyzed with tools from operator theory.
18:30–21:30 *Dinner* at Det Glade Vanvid (Budolfi Plads 26, DK-9000 Aalborg).

Tuesday:

9:00-10:00 *Karma Dajani*: Equilibrium states for the random β-transformation through g-measures
10:00–11:00 *Slade Sanderson*: Matching for a family of symmetric golden maps.
11:00–12:00 *Ira Herbst*: Mild L^p regularity of Leray-Hopf solutions of the Navier-Stokes equations, $p \in [1, 2)$.
12:00–13:00 *Lunch* at the canteen at Create.
13:00–14:00 *Giovanna Marcelli*: On the self-consistent Landauer–Büttiker formalism.
14:00–15:00 *Abdelmalek Abdesselam*: On correlation inequalities for non-Abelian spin systems and lattice gauge.
18:30–20:30 *Dinner* ("on your own" – remember to keep the receipt for reimbursement) at Restaurant Sanya (Restaurant Sanya, Rendsburggade 18, 9000 Aalborg).

Monday or Tuesday (depending on the weather conditions):
Walk along Limfjorden (the inlet which many visitors think is a river through Aalborg).

Abstracts for the talks on Monday

Manon Stipulanti (University of Liege)

Parry's 1960 theorem and some applications in combinatorics on words

Abstract: To manipulate numbers, we need to represent them. A *numeration system* is nothing but a set of rules that enables us to write each number as a sequence of symbols belonging to an alphabet of digits. In combinatorics on words, such a sequence is called a *word*. A result of Parry, dating back to 1960 and now classical, allows us to describe the *numeration language*, that is, the set of all admissible representations, in a numeration system based on a real number. In particular, in such a numeration system, the representation of 1 plays an crucial role. For so-called *Parry numbers*, the latter representation is particular: it is either finite or eventually periodic. This property gives rise to particularly rich numeration systems having nice exploitable properties. In this talk, I will present the framework of these numeration systems and show some applications that come up in my work in combinatorics on words.

Tom Kempton (University of Manchester)

Counting representations of integers as sums of Fibonacci numbers

Abstract: The Fibonacci partition function R(n) counts the number of ways of representing a natural number n as the sum of distinct Fibonacci numbers. For example, R(6)=2

since 6=5+1 and 6=3+2+1. An explicit formula for R(n) was recently given by Chow and Slattery. In this talk we express R(n) in terms of ergodic sums over an irrational rotation, which allows us to prove lots of statements about the local structure of R(n). This talk should be accessible to all, no knowledge of dynamics or number theory will be assumed.

Anne Marie Svane (Aalborg University)

Asymptotic distribution of the scaled remainder in base-q digit expansions

Abstract: Let $X_1, X_2, ...$ be the digits in the base-q expansion of a random variable X defined on $[0, 1)$ where $q \geqslant 2$ is an integer. For $n = 1, 2, ...$, we consider the probability distribution P_n of the (scaled) remainder $T^n(X) = \sum_{k=n+1}^{\infty} X_k q^{n-k}$. If X has a lower semi-continuous probability density, then we establish a coupling between X and a non-negative integer valued random variable N so that $T^N(X)$ follows the uniform distribution μ on the unit interval and is independent of $(X_1, ..., X_N)$. This implies that P_n converges in the total variation metric to μ. Under weak smoothness conditions, the convergence rate can be shown to be exponential. The latter convergence result can be generalized to base-β expansions, where β is the golden mean, but the convergence rate is slower in this case. This is joint work with I. W. Herbst and J. Møller.

Jesper Møller (Aalborg University)

The sufficient digits of continuous random variables

Abstract: A general setting for nested subdivisions of a bounded real set into intervals defining the digits $X_1, X_2, ...$ of a random variable X with a probability density function f is considered. Under the weak condition that f is almost everywhere lower semi-continuous, a coupling between X and a non-negative integer-valued random variable N is established so that $X_1, ..., X_N$ have an interpretation as the "sufficient digits", since the distribution of $(X_{N+1}, X_{N+2}, ...)$ conditioned on $(X_1, ..., X_N)$ does not depend on f. The importance of this coupling result and some suggestions and open problems for future research are discussed. Related papers are available on arXiv: `arxiv.org/abs/2404.09525`, `arxiv.org/abs/2307.06685`.

Horia Cornean (Aalborg University)

On a certain discrete dynamical system, analyzed with tools from operator theory

Abstract: Let $q \geqslant 1$ and $n \geqslant 2$ be integers, and let $q < \beta < q+1$ be the unique positive root of the polynomial $\beta^n - q(\beta^{n-1} + \cdots + \beta + 1)$. Let $t_\beta : [0,1] \mapsto [0,1]$ be given by $t_\beta(x) = \beta x - \lfloor \beta x \rfloor$. For $p \geqslant 1$ we introduce the Koopman operator $K : L^p([0,1]) \mapsto L^p([0,1])$ given by $\big(K(f)\big)(x) = f(t_\beta(x))$, and we also consider the Perron-Frobenius operator P given by the (dual) adjoint of K. We first show that for every $f \in L^1([0,1])$ with $\int_0^1 f(x)dx = 1$, the sequence $P^m(f)$ converges to a positive u_1 which has n piecewise constant components, such that $P(u_1) = u_1$ and it integrates to 1. If f is smooth, we can show that the convergence is exponentially fast and we estimate its speed of convergence. It turns out that t_β preserves the measure $u_1(x)dx$. Let f be smooth and consider the random variables $X_m(x) = \big(K^m(f)\big)(x)$, $m \geqslant 1$, with $\Omega = [0,1]$ and probability measure $u_1 dx$. We prove that the random variables $\{X_m\}_{m \geqslant 1}$ have exponential decay of correlations and obey the law of large numbers.

All these things can be related to greedy expansions of the type $x = \sum_{j \geqslant 1} x_j \beta^{-j}$, which go back at least to Parry in the '60s, and with the ergodicity of the discrete dynamical system induced by t_β.

This is joint work with G. Marcelli (Rome 3) and I. Herbst (Virginia).

Abstracts for the talks on Tuesday

Karma Dajani (Utrecht University)

Equilibrium states for the random β-transformation through g-measures

Abstract: We consider the random β-transformation K_β, defined on $\{0,1\}^{\mathbb{N}} \times [0, \frac{\lfloor \beta \rfloor}{\beta - 1}]$, that generates all possible expansions of the form $x = \sum_{i=0}^{\infty} \frac{a_i}{\beta^i}$, where $a_i \in \{0, 1, \cdots, \lfloor \beta \rfloor\}$. We exhibit an uncountable family of K_β-invariant exact g-measures for a certain collection of algebraic β's. The construction of these g-measures is explicit and the corresponding potentials are not locally constant. This is a joint work with Kieran Power.

Slade Sanderson (Utrecht University)

Matching for a family of symmetric golden maps

Abstract: In 2020, Dajani and Kalle introduced a one-parameter family of piecewise-linear interval maps of slope two and studied the asymptotic frequencies of digits of the signed binary expansions they produce. Central to their analysis is a property called 'matching,' which occurs when the orbits of the left and right limits at discontinuity points eventually coincide, and which allows for the construction of absolutely continuous invariant measures whose densities are step functions. We introduce and obtain analogous results for 'signed golden' expansions arising from a one-parameter family of 'symmetric golden maps' of constant slope equal to the golden mean. We characterize the (more delicate)

matching phenomenon in our setting, present explicit absolutely continuous invariant measures and—via Birkhoff's ergodic theorem—determine asymptotic frequencies of digits of the corresponding expansions. Time permitting, we show further implications for another family of number expansions arising from 'skewed symmetric golden maps' of non-constant slope. (Joint work with Karma Dajani.)

Ira Herbst (University of Virginia)

Mild L^p regularity of Leray-Hopf solutions of the Navier-Stokes equations, $p \in [1, 2)$

Abstract: We study Leray-Hopf weak solutions of the Navier-Stokes equations in $\mathbb{R}^m, m \geqslant 3$. These solutions are global in space and time, $t \geqslant 0$, given an initial fluid velocity, $u_0 \in L^2$ but uniqueness is not known. Nor is it known that if one imposes smoothness and decay conditions on u_0, that this smoothness is not lost at some later time. One of the Millennium Prize problems is to show that if u_0 is given in the Schwartz space of smooth rapidly decaying divergence free vector fields, then there is a smooth solution, $u(t, x)$, of the equations with $\int |u(t, x)|^2 dx <$ const for all $t \geqslant 0$ and $u(0, x) = u_0(x)$. In this talk which I will try to make understandable to someone who knows nothing about Navier-Stokes, I will explain that without imposing any additional smoothness on u_0, the weak Leray-Hopf solutions have the property that $u(t) - e^{t\Delta}u_0$ has some mild regularity in $L^p, p \in [1, 2)$. Here Δ is the Laplacian and $e^{t\Delta}u_0$ satisfies the heat equation, $\partial_t v(t, x) = \Delta_x v(t, x)$.

Giovanna Marcelli (University of Roma Tre)

On the self-consistent Landauer–Büttiker formalism

Abstract: We provide sufficient conditions such that the time evolution of a mesoscopic tight-binding open system with a local Hartree–Fock non-linearity converges to a self-consistent non-equilibrium steady state, which is independent of the initial condition from the "small sample". We also show that the steady charge current intensities are given by Landauer–Büttiker-like formulas, and make the connection with the case of weakly self-interacting many-body systems. This is a joint work with Horia D. Cornean `arxiv.org/abs/2309.01564`.

Abdelmalek Abdesselam (University of Virginia)

On correlation inequalities for non-Abelian spin systems and lattice gauge theories

Abstract: I will present a conjecture about inequalities for correlations of invariant observables in non-Abelian spin systems, and lattice gauge theories. These inequalities are expressed in terms of the non-interacting probability measure, but they would imply the analogue of the GKS inequality at non-zero coupling. A test of the conjectured inequalities is when integrands are taken to a large power. In that regime, one can prove the conjectured inequalities. This talk is partly about joint work with Gennady Uraltsev and Joe Webster.