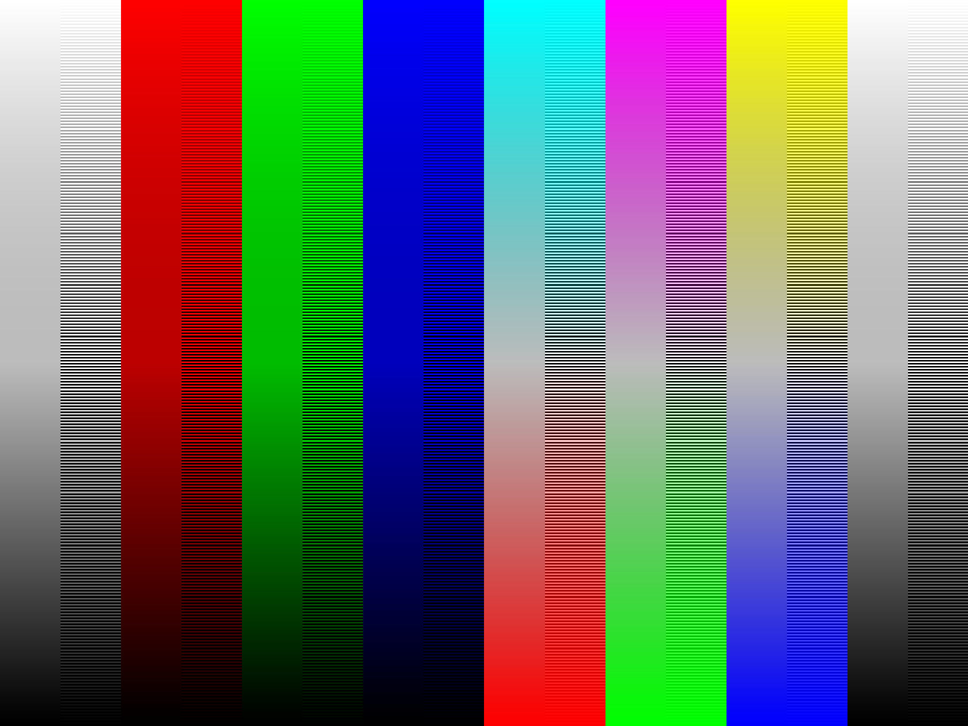


Induced matchings and the algebraic stability of persistence barcodes

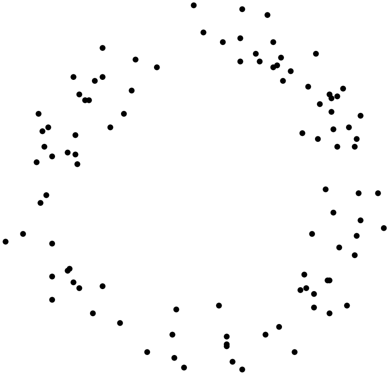
Ulrich Bauer

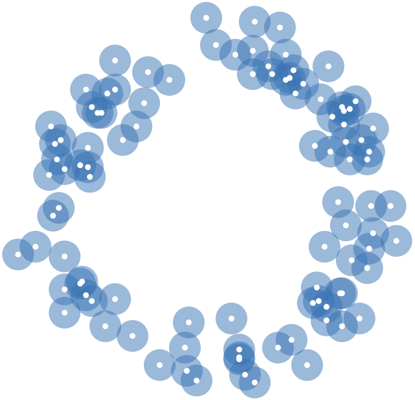
TUM

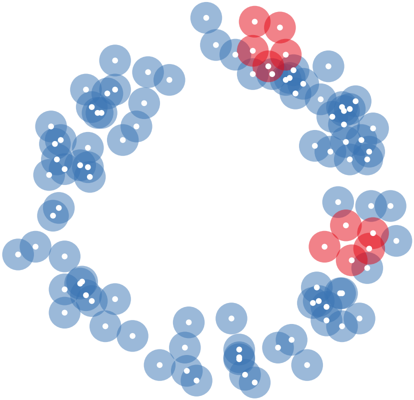
Apr 7, 2015

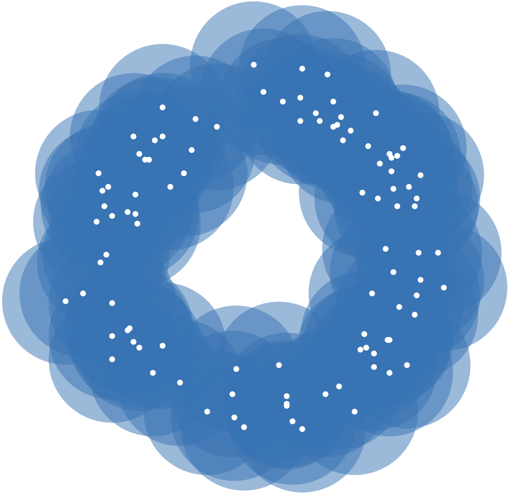
GETCO 2015, Aalborg

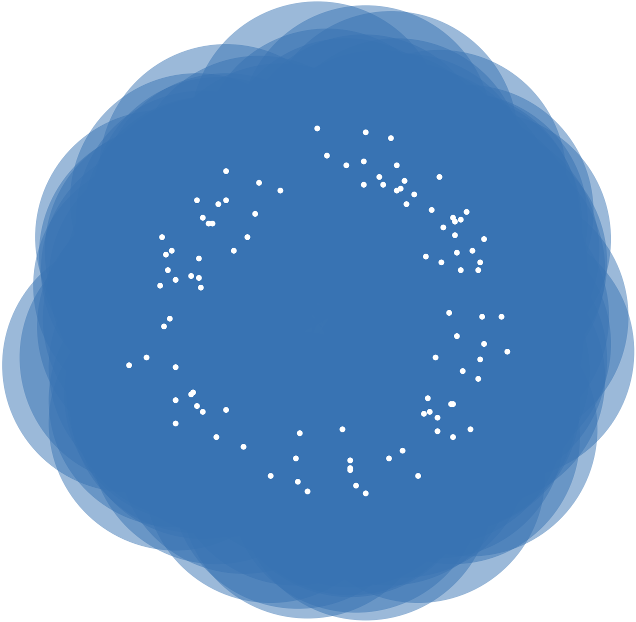
Joint work with Michael Lesnick (IMA)

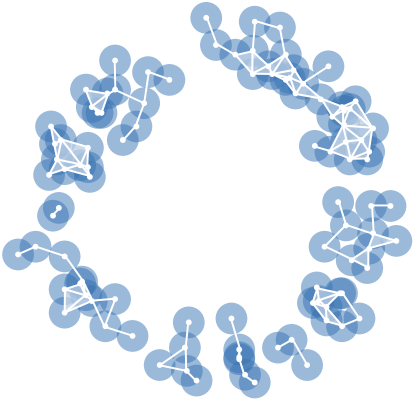


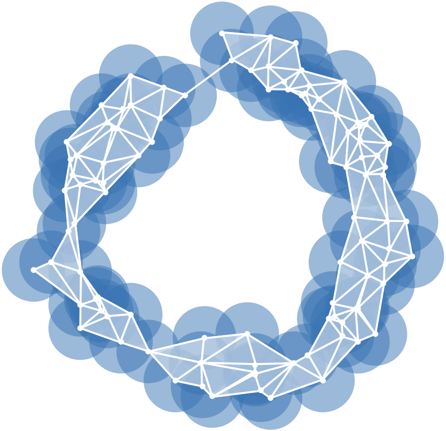


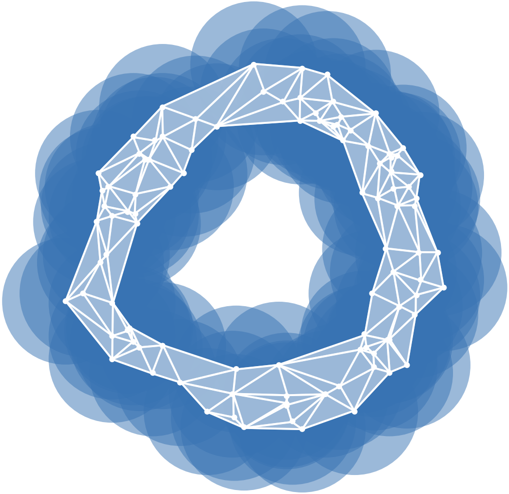


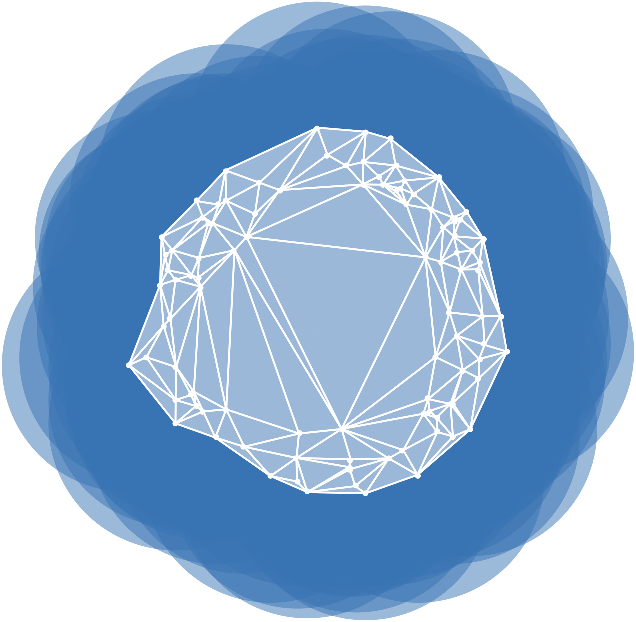




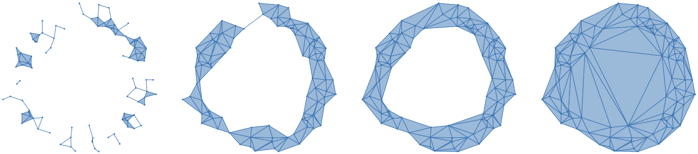


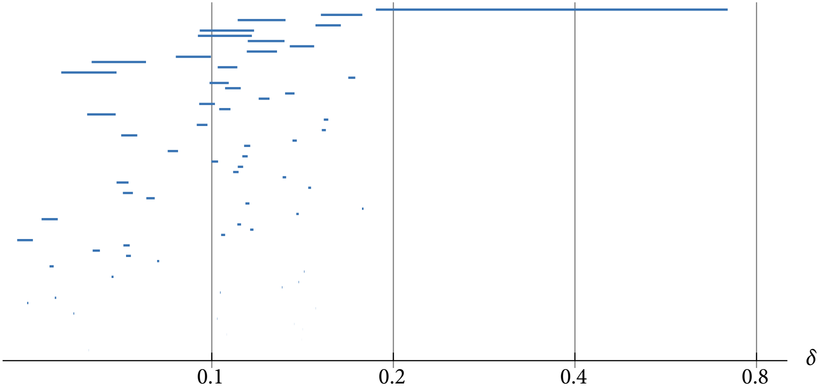






What is persistent homology?





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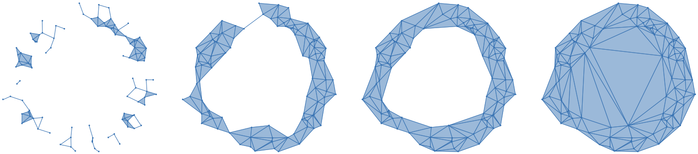

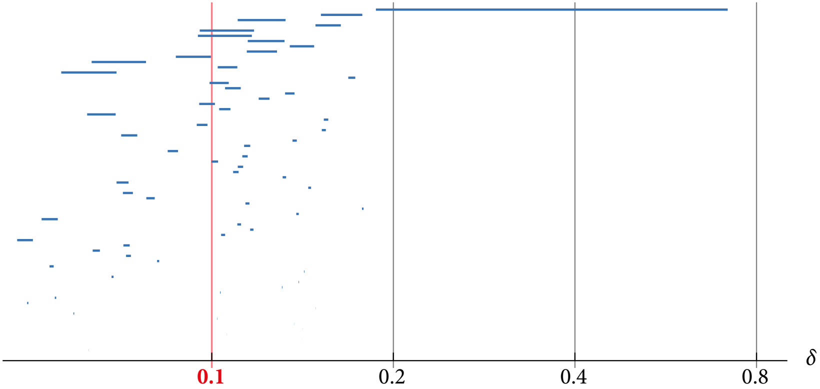

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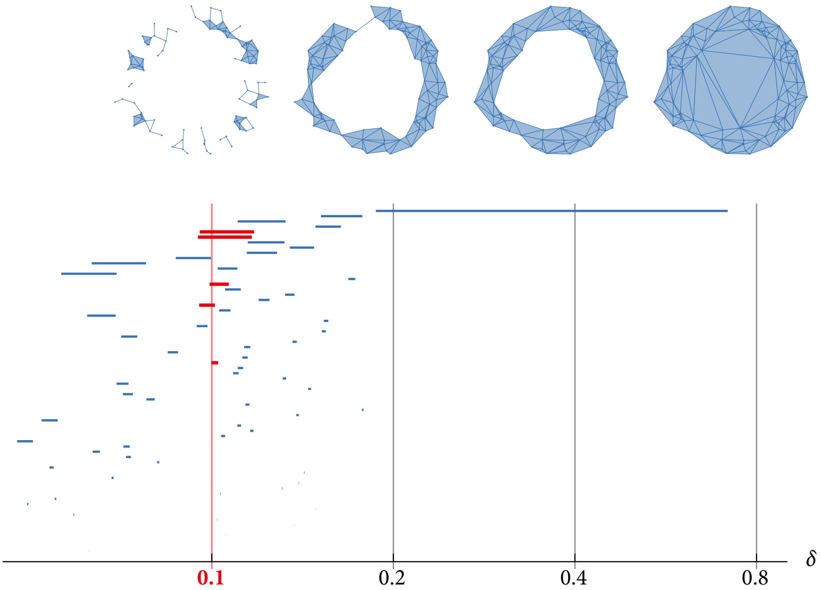

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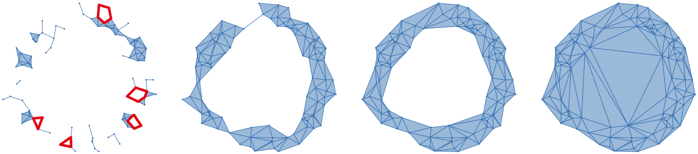

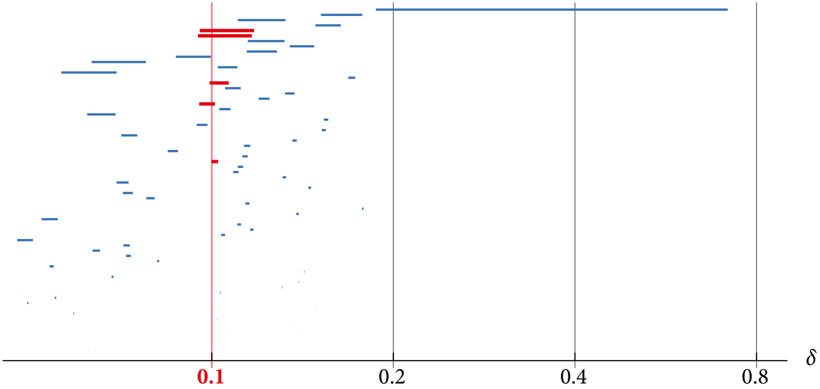

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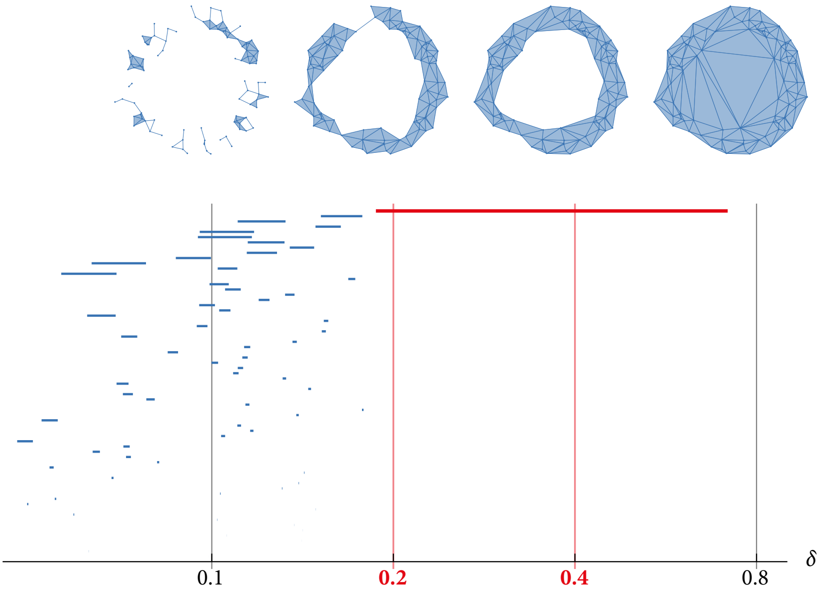

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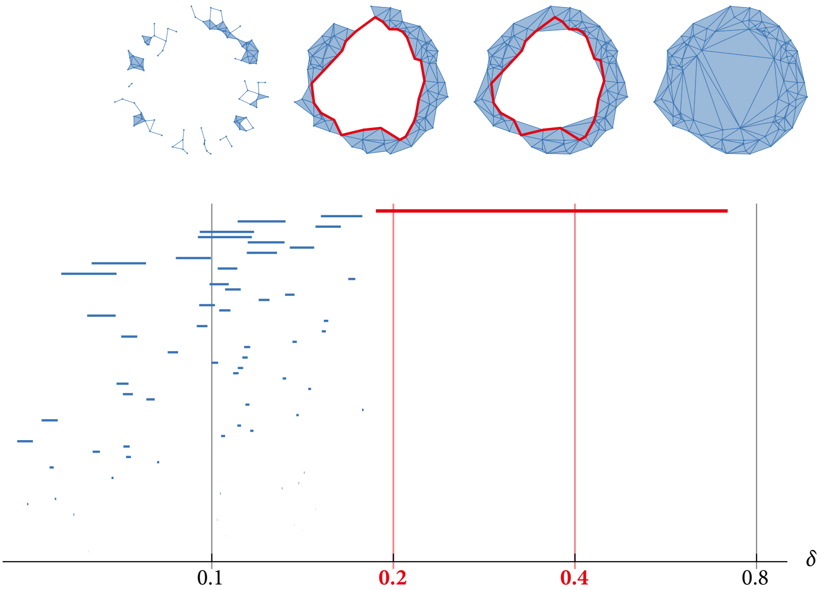

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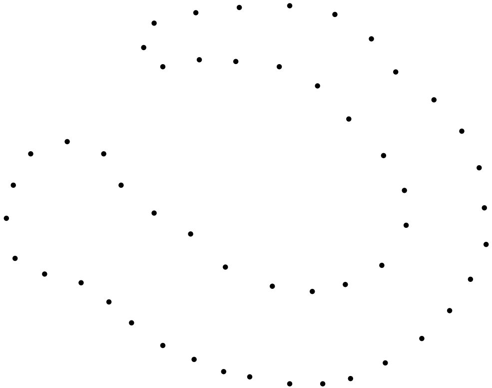

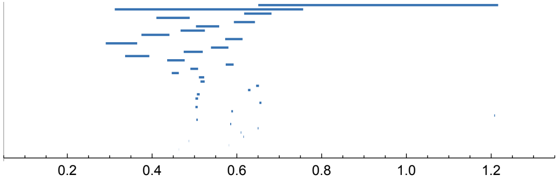
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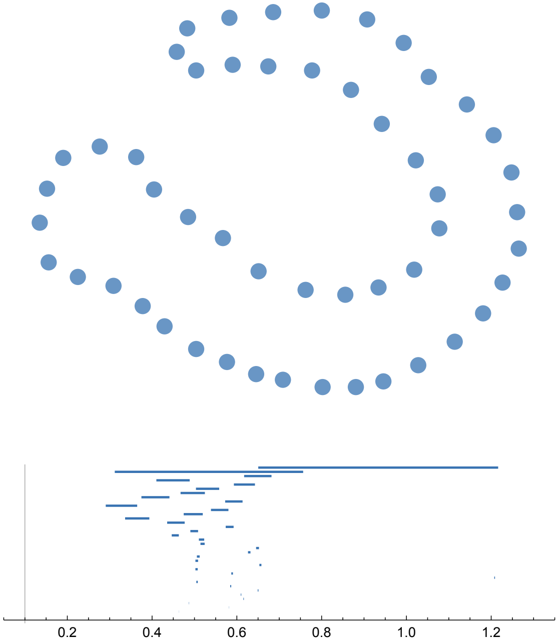
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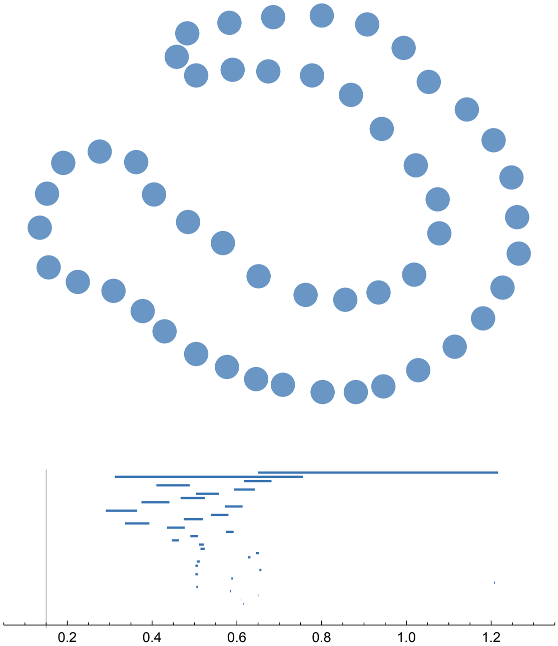
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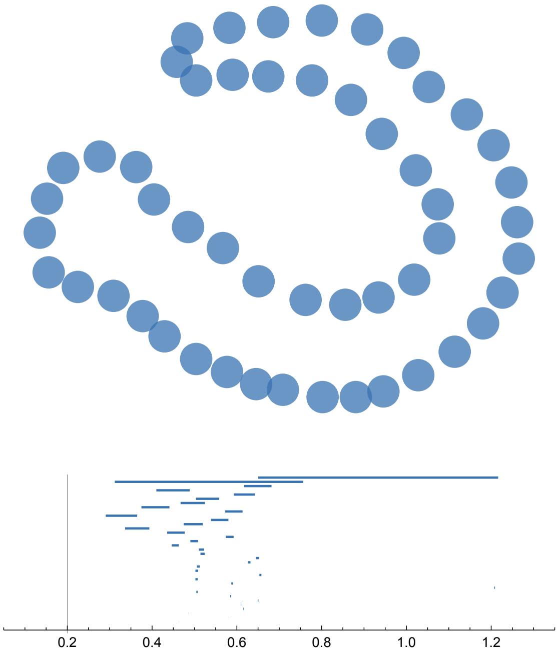
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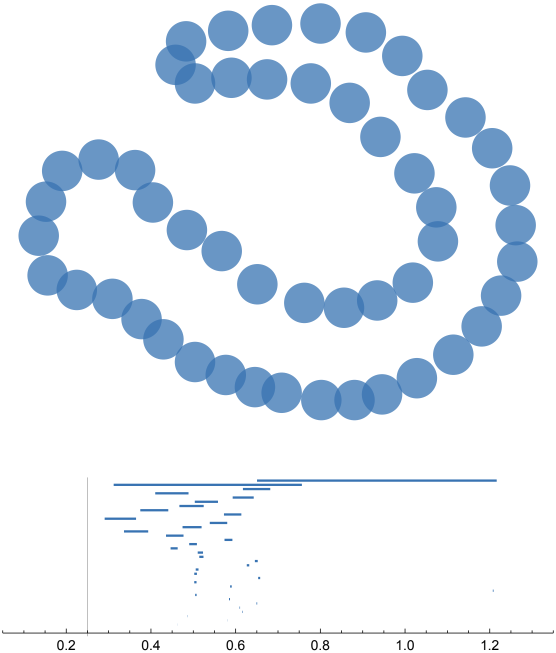
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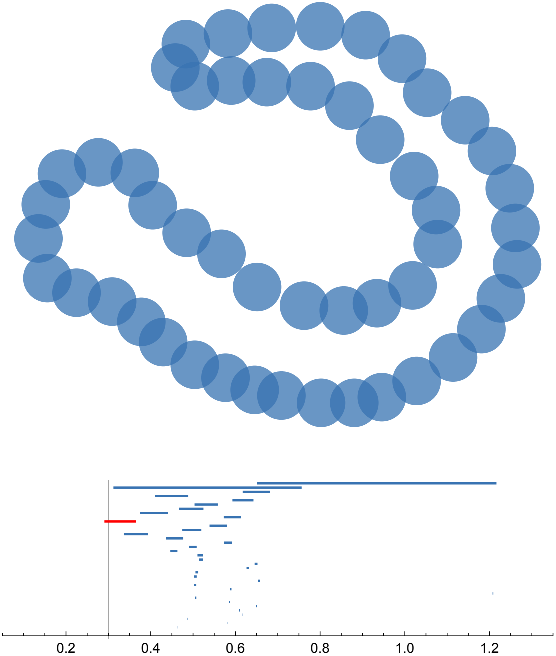
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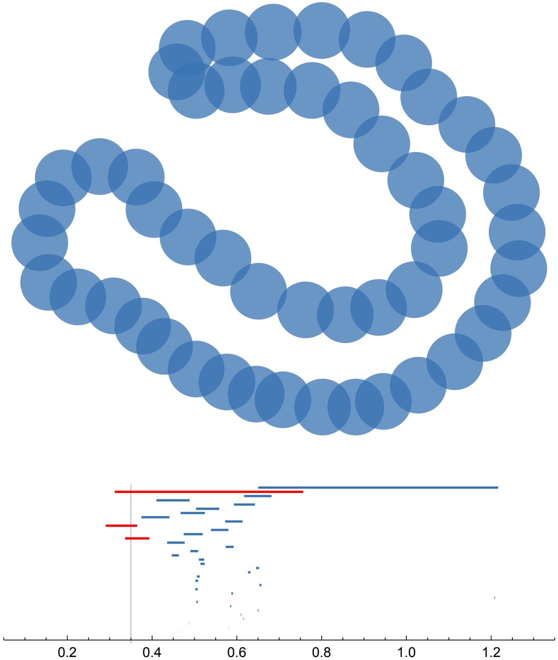
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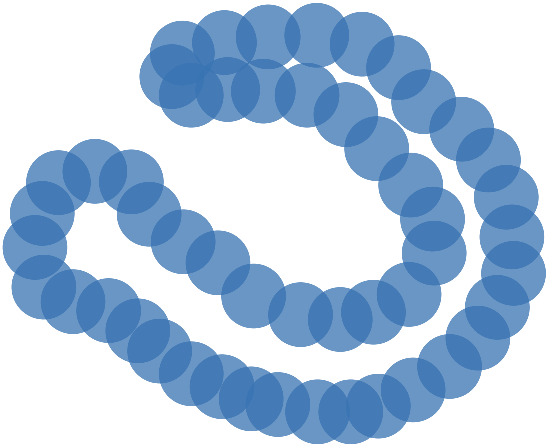

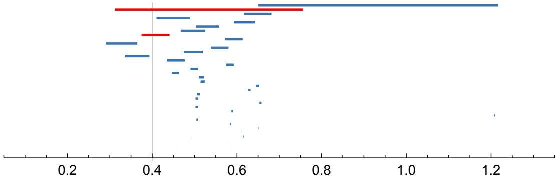
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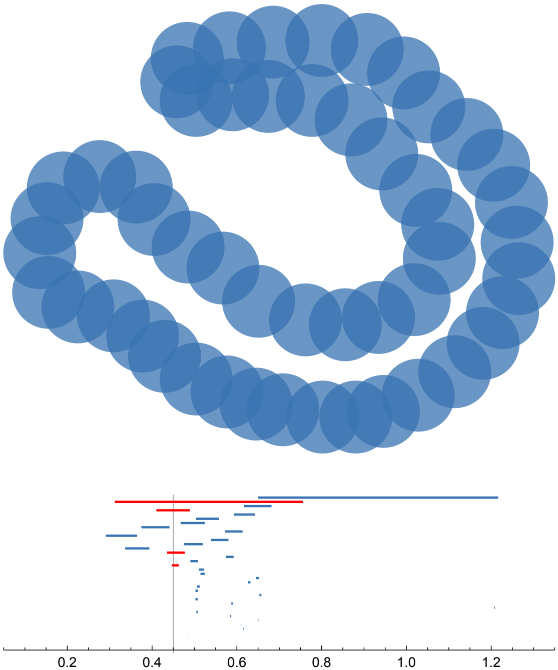
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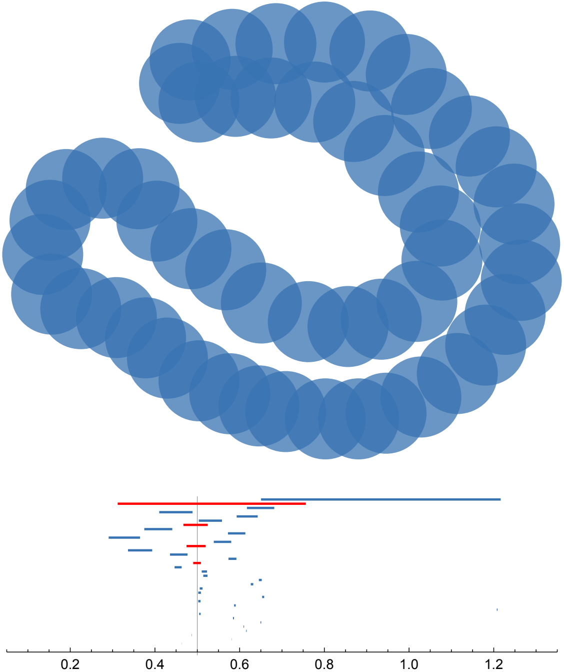
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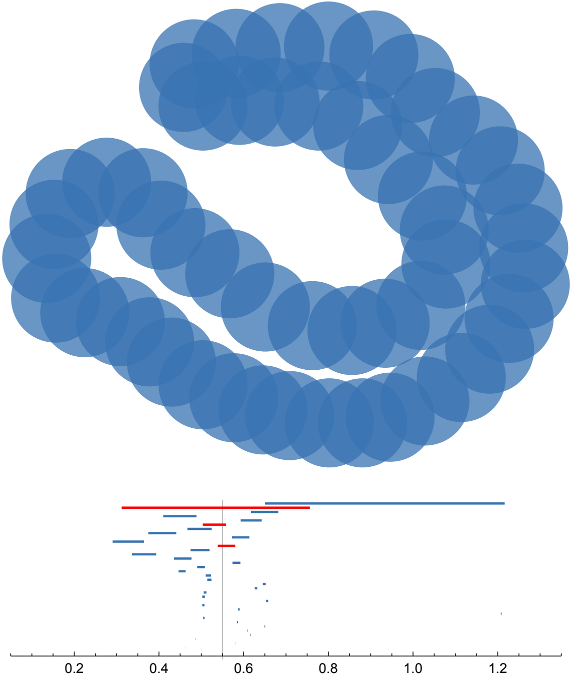
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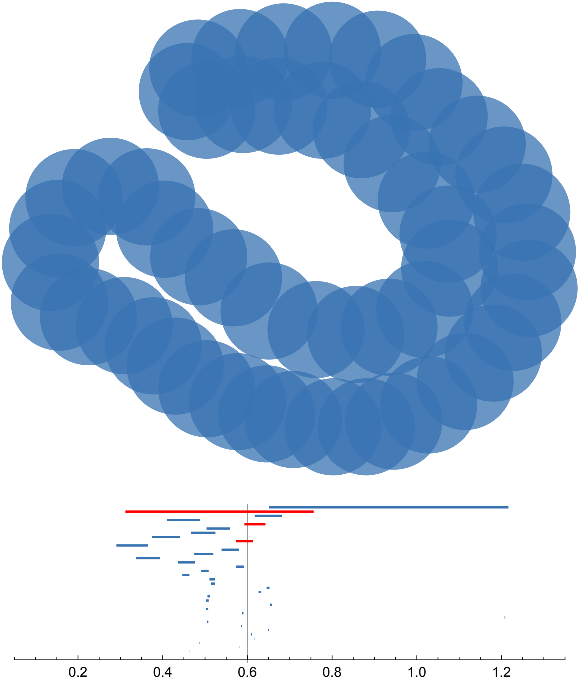
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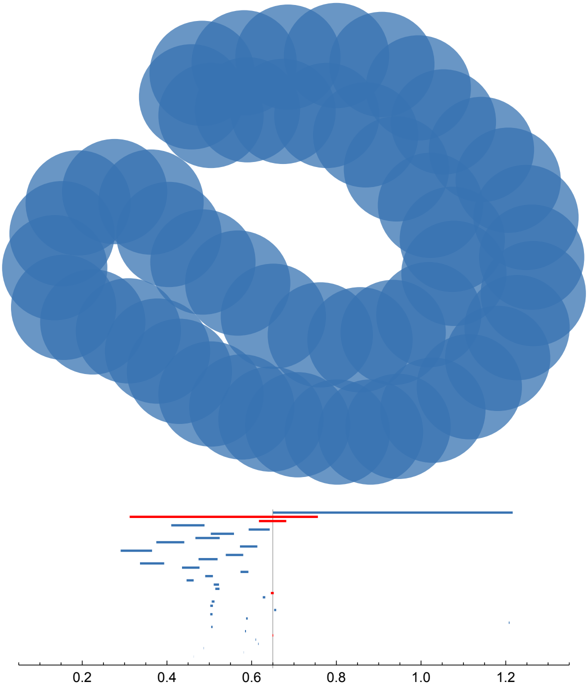
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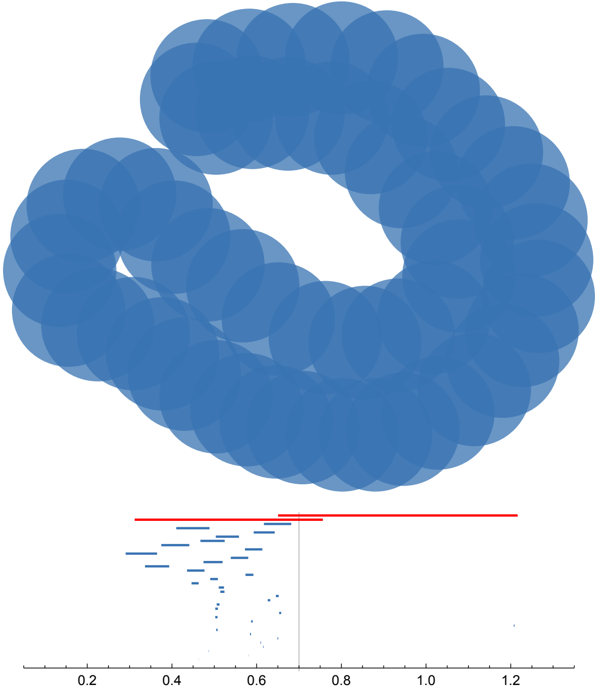
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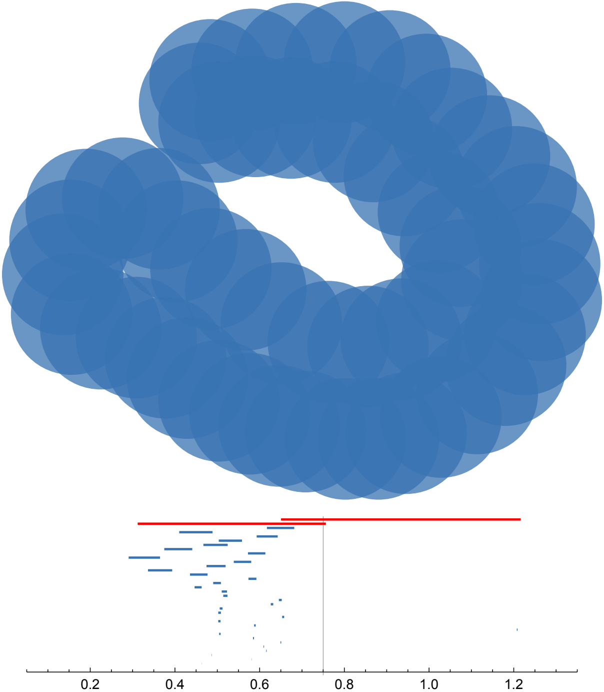
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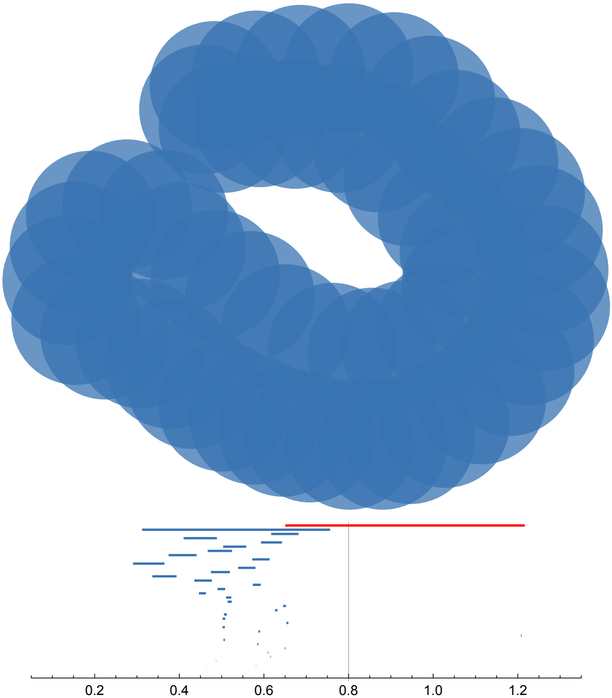
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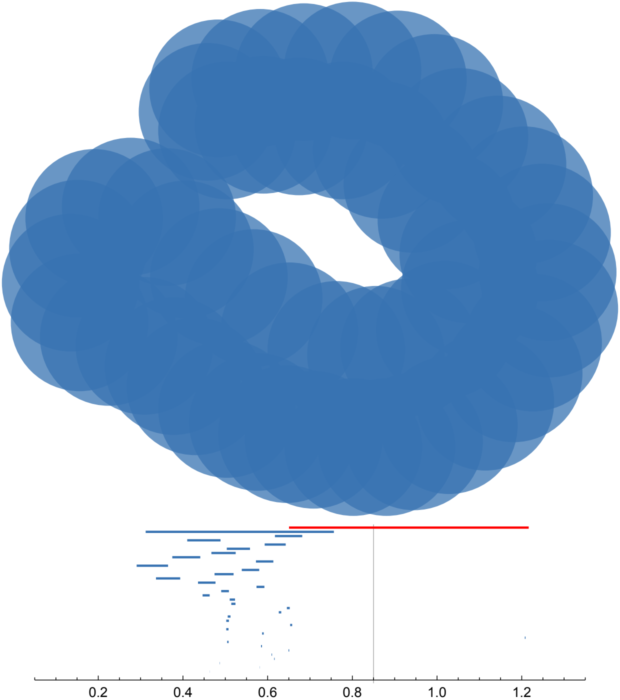
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Homology inference using persistent homology

$P_\delta = B_\delta(P)$: δ-neighborhood (union of balls) around P

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. Let $P \subset \Omega$ be such that

- $\Omega \subseteq P_\delta$ *for some* $\delta > 0$ *and*
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Then

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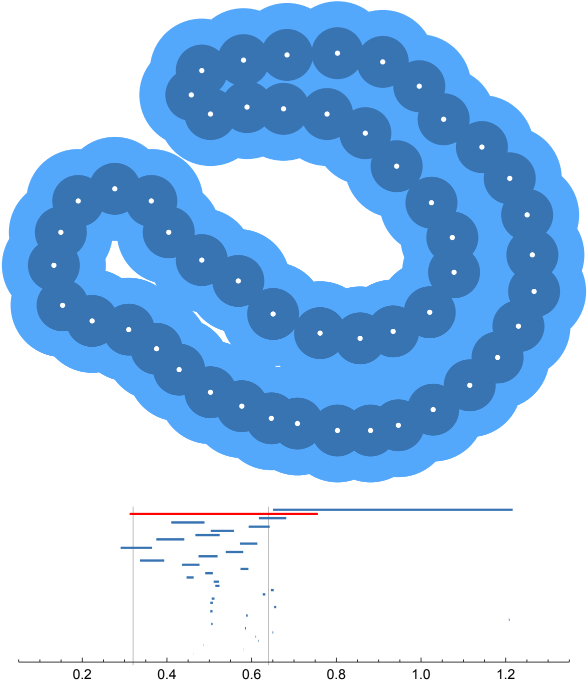
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The pipeline of topological data analysis

point cloud

↓ distance

function

↓ sublevel sets

topological spaces

↓ homology

vector spaces

↓ barcode

intervals

The pipeline of topological data analysis

point cloud $\quad P \subset \mathbb{R}^d$

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The pipeline of topological data analysis

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topological spaces	$K : \mathbf{R} \to \mathbf{Top}$
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vector spaces	$M : \mathbf{R} \to \mathbf{Vect}$
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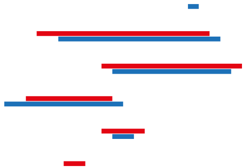
Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

If two functions $f, g : K \to \mathbb{R}$ have distance $\|f - g\|_\infty \le \delta$ then there exists a δ-matching of their barcodes.



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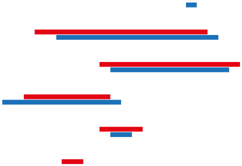

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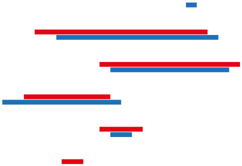

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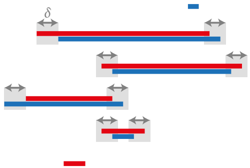

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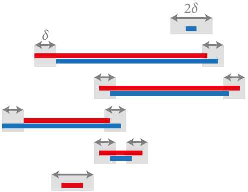

- *matching* $A \nrightarrow B$: bijection of subsets $A' \subseteq A$, $B' \subseteq B$
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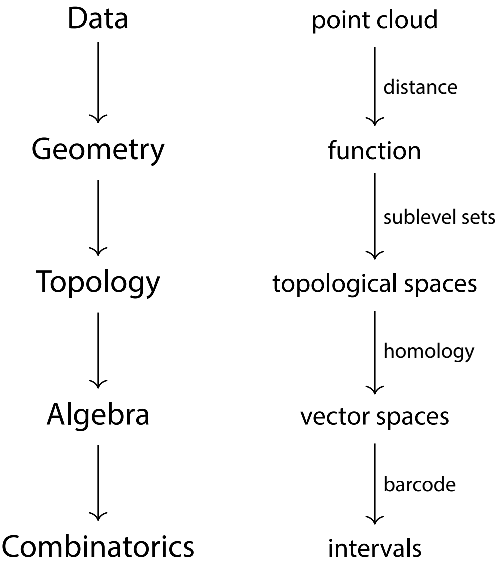
  - unmatched intervals have length $\leq 2\delta$

Stability for functions in the big picture



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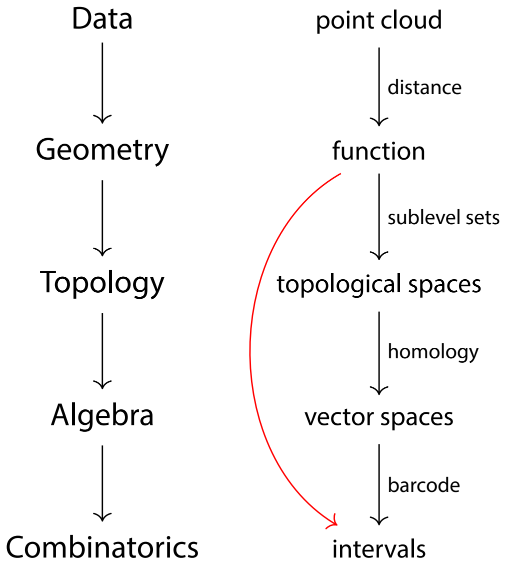

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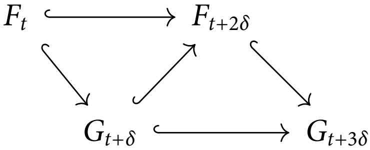

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Homology is a *functor:* homology groups are interleaved too.

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Interval Persistence Modules

Let $\mathbb{K}$ be a field. For an arbitrary interval $I \subseteq \mathbb{R}$, define the *interval persistence module* $C(I)$ by

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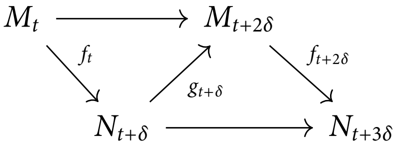

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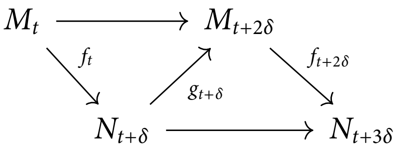

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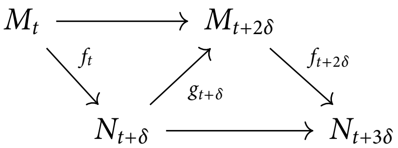
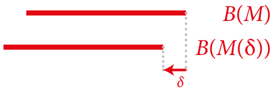


- define $M(\delta)$ by $M(\delta)_t = M_{t+\delta}$ (shift barcode to the left by δ)



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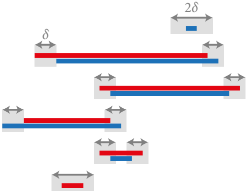

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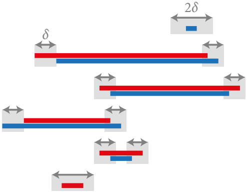

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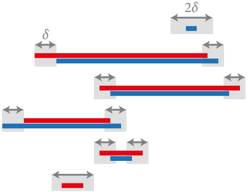

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- indirect proof, 80 page paper (Chazal et al. 2012)

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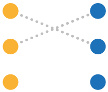
- relies on *partial functoriality* of the induced matching

The matching category

A *matching* $\sigma : S \nrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$.



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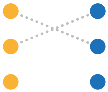

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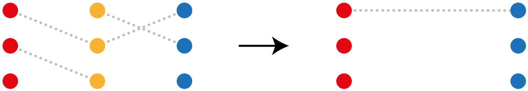

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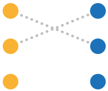

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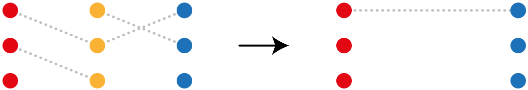
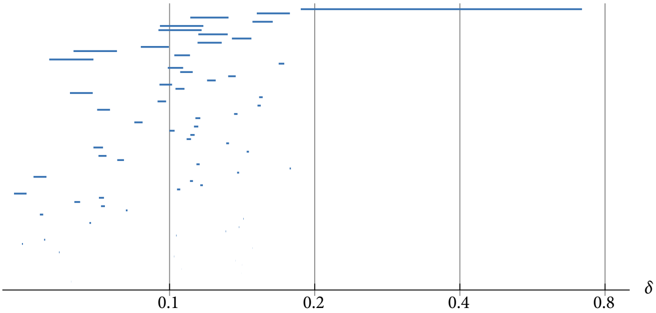


Matchings form a category $\mathbf{Mch}$

- objects: sets
- morphisms: matchings

Barcodes as matching diagrams

We can regard a barcode B as a functor $\mathbf{R} \to \mathbf{Mch}$:



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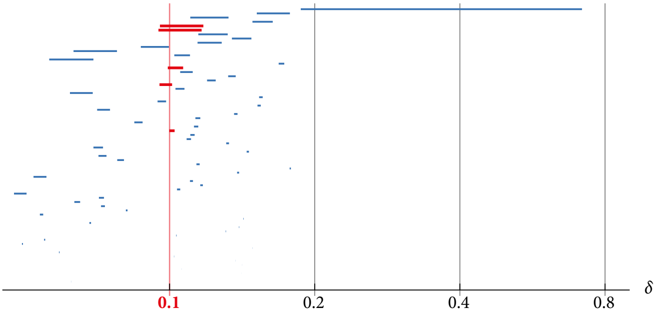

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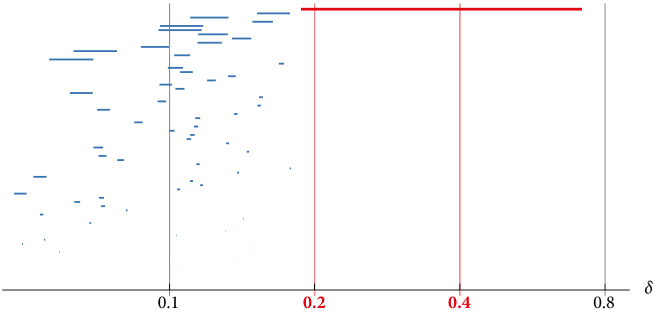

Barcode matchings as natural transformations

We can regard certain matchings of barcodes $\sigma : A \nrightarrow B$
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- consider restrictions $\sigma_t : A_t \nrightarrow B_t$ of σ to $A_t \times B_t$:

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- In particular, there is no natural choice of basis for vector spaces

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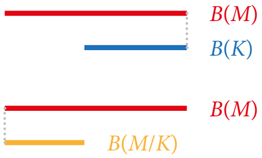

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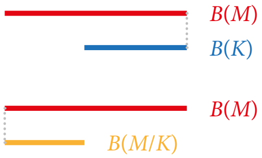

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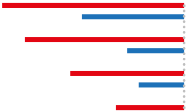

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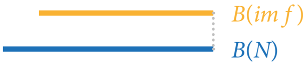

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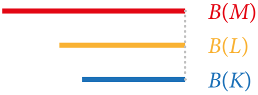
This matching is functorial *for injections*:

$$B(K \hookrightarrow M) = B(L \hookrightarrow M) \circ B(K \hookrightarrow L)$$



Induced matchings

For any morphism $f : M \to N$ between persistence modules:

- decompose into $M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$
- $\operatorname{im} f \cong M / \ker f$ is a quotient of M
- $\operatorname{im} f$ is a submodule of N
- Composing the canonical matchings yields a matching $B(f) : B(M) \nrightarrow B(N)$ induced by f

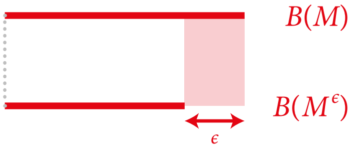
This matching is functorial *for injections*:

$$B(K \hookrightarrow M) = B(L \hookrightarrow M) \circ B(K \hookrightarrow L)$$

Similar for surjections.

The induced matching theorem

Define M^ϵ by shrinking bars of $B(M)$ from the right by ϵ.

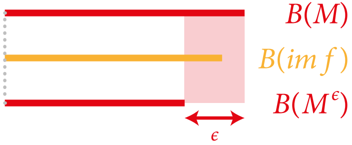


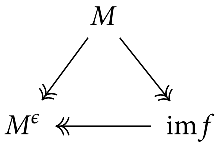
The induced matching theorem

Define M^ϵ by shrinking bars of $B(M)$ from the right by ϵ.

Lemma

Let $f : M \to N$ be a morphism such that $\ker f$ is ϵ-trivial (all bars of $B(\ker f)$ are shorter than ϵ). Then M^ϵ is a quotient module of $\operatorname{im} f$.



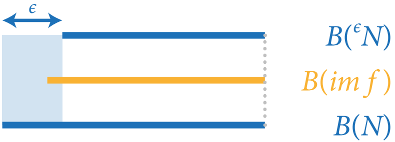


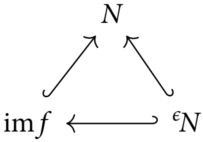
The induced matching theorem

Define ${}^{\epsilon}N$ by shrinking bars of $B(N)$ from the left by ϵ.

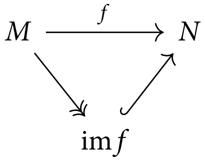
Lemma

Let $f : M \to N$ be a morphism such that $\operatorname{coker} f$ is ϵ-trivial (all bars of $B(\operatorname{coker} f)$ are shorter than ϵ). Then ${}^{\epsilon}N$ is a submodule of $\operatorname{im} f$.

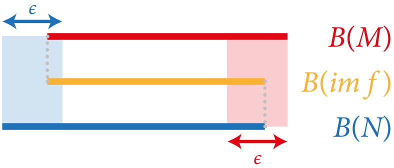


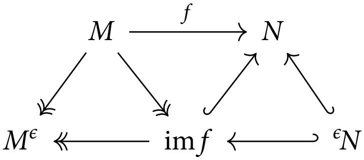


The induced matching theorem

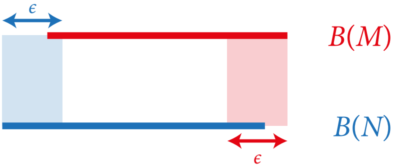


The induced matching theorem





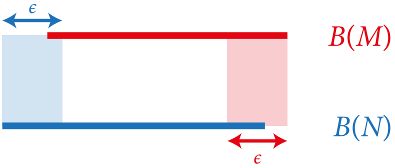
The induced matching theorem



The induced matching theorem

Theorem (B, Lesnick 2013)

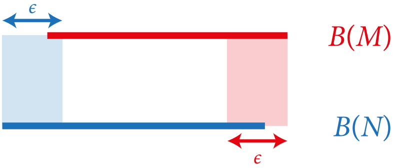
Let $f : M \to N$ *be a morphism with* $\ker f$ *and* $\operatorname{coker} f$ ϵ*-trivial.*



The induced matching theorem

Theorem (B, Lesnick 2013)

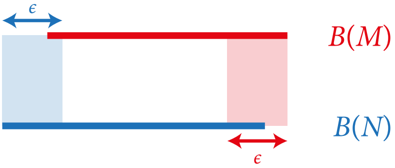
Let $f : M \to N$ *be a morphism with* $\ker f$ *and* $\operatorname{coker} f$ ϵ*-trivial.*
Then each interval of length $\geq \epsilon$ *is matched by* $B(f)$*.*



The induced matching theorem

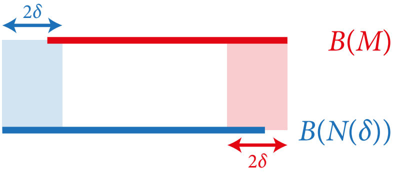
Theorem (B, Lesnick 2013)

Let $f : M \to N$ be a morphism with $\ker f$ and $\operatorname{coker} f$ ϵ-trivial.
Then each interval of length $\geq \epsilon$ is matched by $B(f)$.
If $B(f)$ matches $[b, d) \in B(M)$ to $[b', d') \in B(N)$, then
$b' \leq b \leq b' + \epsilon$ *and* $d - \epsilon \leq d' \leq d$.



The induced matching theorem

Let $f : M \to N(\delta)$ be an interleaving morphism.
Then $\ker f$ and $\operatorname{coker} f$ are 2δ-trivial.



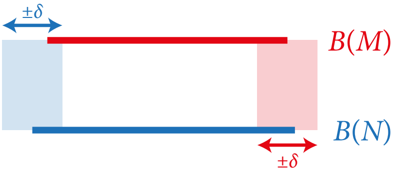
The induced matching theorem

Let $f : M \to N(\delta)$ be an interleaving morphism.

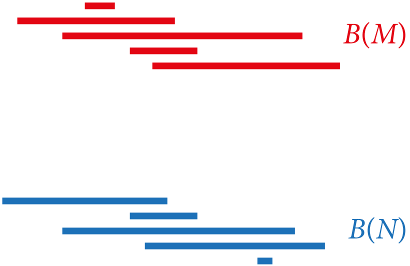
Then $\ker f$ and $\operatorname{coker} f$ are 2δ-trivial.

Corollary (Algebraic stability via induced matchings)

A δ-interleaving between persistence modules induces a δ-matching of their persistence barcodes.



Stability via induced matchings



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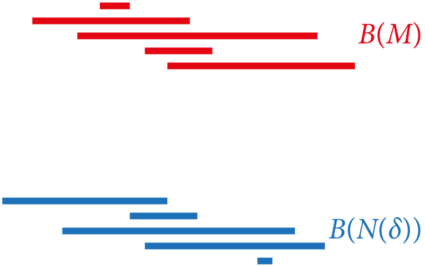

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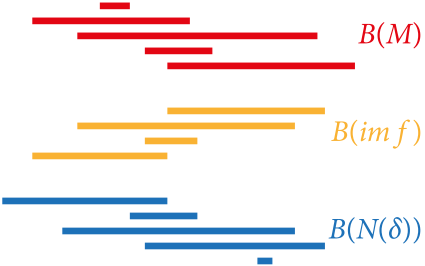
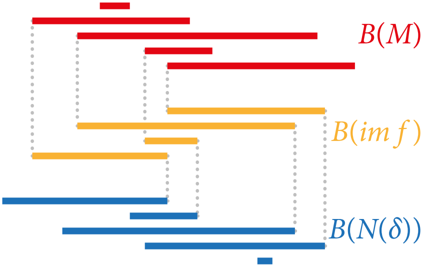
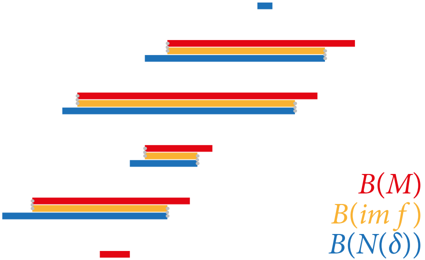


Stability via induced matchings



Stability via induced matchings



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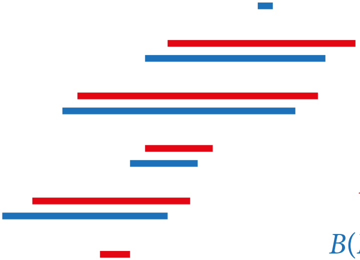
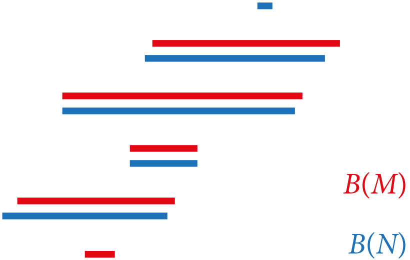


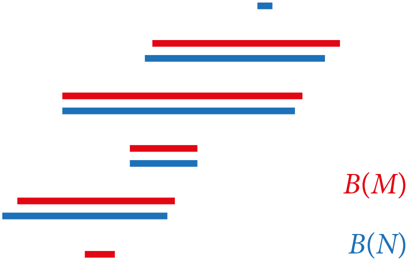
$B(M)$

$B(N(\delta))$

Stability via induced matchings



Stability via induced matchings



Thanks for your attention!