Reduction of higher-dimensional automata

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Higher-dimensional automata [Pratt, van Glabbeek]

A *higher-dimensional automaton* over a monoid M (M-HDA) is a tuple $\mathcal{A} = (P, I, F, \lambda)$ where

- P is a precubical set, i.e., a graded set with *boundary operators*

  $$d_i^k : P_n \to P_{n-1} \quad (n > 0,\ k = 0, 1,\ i = 1, \ldots, n)$$

  satisfying the relations $d_i^k \circ d_j^l = d_{j-1}^l \circ d_i^k \quad (k, l = 0, 1,\ i < j)$.
- $I \subseteq P_0$ is a set of *initial states*,
- $F \subseteq P_0$ is a set of *final states*,
- $\lambda \colon P_1 \to M$ is a map, called the *labeling function*, such that

  $$\lambda(d_i^0 x) = \lambda(d_i^1 x)$$

  for all $x \in P_2$ and $i \in \{1, 2\}$.

Higher-dimensional automata [Pratt, van Glabbeek]

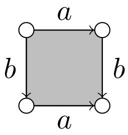


Figure: Cubes represent independence of actions

A simple concurrent system

Consider the concurrent system where two identical processes P_0 and P_1 modify two shared boolean variables x and y, initially zero, by executing the program given by the following *program graph*:

$$\to 0 \xrightarrow{x=0\,:\,x:=1} 1 \xrightarrow{x:=0} 2 \xrightarrow{y=0\,:\,y:=1} 3 \xrightarrow{y:=0} 4$$

In the first and third instructions, the assignment action is only executable when the guard condition, indicated before the colon, holds.

A simple concurrent system

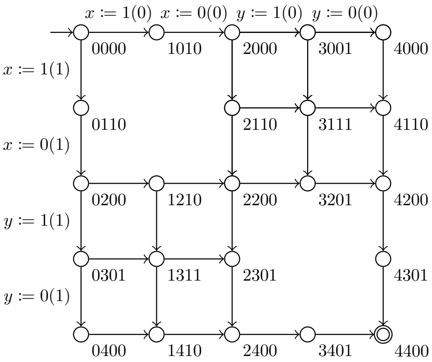


Figure: Transition system representing the reachable part of the system

A simple concurrent system

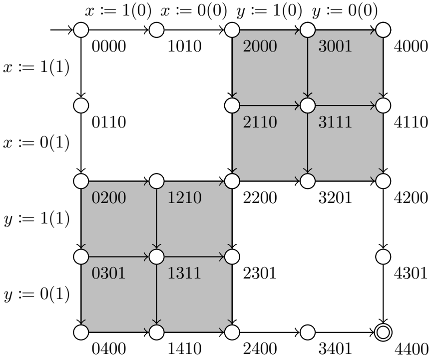


Figure: HDA model of the reachable part of the system

Paths

Let k and l be integers such that $k \leq l$. The *precubical interval* $[\![k, l]\!]$ is the precubical set

$$\overset{k}{\bullet} \to \overset{k+1}{\bullet} \to \cdots \to \overset{l-1}{\bullet} \to \overset{l}{\bullet} .$$

A *path of length* k in a precubical set P is a morphism of precubical sets $\omega \colon [\![0, k]\!] \to P$.

The set of paths in P is denoted by $P^{\mathbb{I}}$.

Remark

A path of length $k \geq 1$ can be identified with a sequence $(x_1, \ldots, x_k)$ of elements of P_1 such that $d_1^0 x_{j+1} = d_1^1 x_j$ $(1 \leq j < k)$.

Labels of paths

The *extended labeling function* of an an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ is the map

$$\overline{\lambda} \colon P^{\mathbb{I}} \to M$$

defined by

$$\overline{\lambda}(x_1, \ldots, x_k) = \lambda(x_1) \cdots \lambda(x_k).$$

If ω is a path of length 0, then we set

$$\overline{\lambda}(\omega) = 1.$$

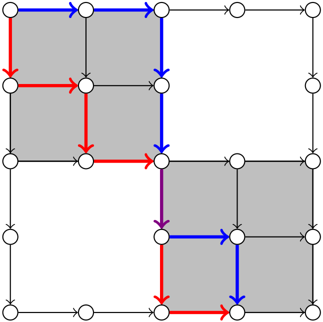
Dihomotopy [Goubault]

Two paths ω and ν in a precubical set P are said to be *elementarily dihomotopic* if there exist paths $\alpha, \beta \in P^{\mathbb{I}}$ and an element $z \in P_2$ such that

- $d_1^0 d_1^0 z = \alpha(\mathrm{length}(\alpha)), d_1^1 d_1^1 z = \beta(0)$,
- $\{\omega, \nu\} = \{\alpha \cdot (d_1^0 z, d_2^1 z) \cdot \beta, \alpha \cdot (d_2^0 z, d_1^1 z) \cdot \beta\}$.

Dihomotopy is the equivalence relation generated by elementary dihomotopy.

Dihomotopy [Goubault]



Dihomotopic paths

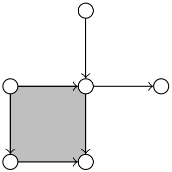
The trace category of an HDA [Bubenik]

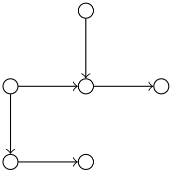
The *fundamental category* (or *path category* [Jardine]) of a precubical set P is the category $\vec{\pi}_1(P)$ whose objects are the vertices of P and whose morphisms are the dihomotopy classes of paths in P.

A vertex v of a precubical set P is said to be *maximal (minimal)* if there is no element $x \in P_1$ such that $d_1^0 x = v$ ($d_1^1 x = v$). The sets of maximal and minimal elements of P are denoted by $M(P)$ and $m(P)$ respectively.

The *trace category* of an M-HDA $\mathcal{A} = (P, I, F, \lambda)$, $TC(\mathcal{A})$, is the full subcategory of $\vec{\pi}_1(P)$ generated by $I \cup F \cup m(P) \cup M(P)$.

Bad collapse





Invariance of the trace category under elementary collapses

Let $\mathcal{A} = (P, I, F, \lambda)$ be an M-HDA, and let x be an n-cube with free face $d_i^1 x$. Suppose that x is *regular* (or *non-self-linked* [Fajstrup, Raussen, Goubault]), i.e., that the characteristic map

$$x_\sharp \colon [\![0,1]\!]^{\otimes n} \to P$$

is injective. Consider the precubical subset $Q = P \setminus \{x, d_i^1 x\}$ of P and the M-HDA $\mathcal{B} = (Q, I, F, \lambda|_{Q_1})$.

Proposition

If $n \geq 4$, then the inclusion induces an isomorphism $TC(\mathcal{B}) \cong TC(\mathcal{A})$.

Invariance of the trace category under elementary collapses

Theorem

Suppose that $n = 3$ and that every path from $I \cup F \cup M(P) \cup m(P)$ to $d_1^0 d_1^0 d_i^1 x$ factors up to dihomotopy through the edge leading from $d_1^0 d_1^0 d_1^0 x$ to $d_1^0 d_1^0 d_i^1 x$. Then the inclusion induces an isomorphism $TC(\mathcal{B}) \cong TC(\mathcal{A})$.

Theorem

Suppose that $n = 2$ and that

1. *for at least two edges $y \in P_1$, $d_1^0 y = d_1^0 d_i^1 x$;*
2. *every path $\omega \in Q^{\mathbb{I}}$ from $I \cup F \cup M(P) \cup m(P) \cup \{d_1^1 d_1^1 x\}$ to $d_1^0 d_i^1 x$ factors uniquely up to dihomotopy through $d_{3-i}^0 x$.*

Then the inclusion induces an isomorphism $TC(\mathcal{B}) \cong TC(\mathcal{A})$.

Two HDAs

(a) $\mathcal{A}$ (b) $\mathcal{B}$

Figure: Two HDAs $\mathcal{A}$ and $\mathcal{B}$ over the free monoid on $\{a, b, c\}$

Tensor product

Given two precubical sets P and Q, the *tensor product* $P \otimes Q$ is the precubical set defined by

$$(P \otimes Q)_n = \coprod_{p+q=n} P_p \times Q_q.$$

and

$$d_i^k(x,y) = \begin{cases} (d_i^k x, y), & 1 \leq i \leq \deg(x), \\ (x, d_{i-\deg(x)}^k y), & \deg(x)+1 \leq i \leq \deg(x)+\deg(y). \end{cases}$$

Remark

$$|[\![0,k_1]\!] \otimes \cdots \otimes [\![0,k_n]\!]| = [0,k_1] \times \cdots \times [0,k_n].$$

Weak morphisms

A *weak morphism* from a precubical set Q to a precubical set P is a continuous map $f\colon |Q| \to |P|$ such that the following two conditions hold:

1. f sends vertices to vertices;
2. for all integers $n, k_1, \dots, k_n \geq 1$ and every morphism of precubical sets $\xi\colon [\![0, k_1]\!] \otimes \cdots \otimes [\![0, k_n]\!] \to Q$, there exist integers $l_1, \dots, l_n \geq 1$, a morphism of precubical sets

$$\chi\colon [\![0, l_1]\!] \otimes \cdots \otimes [\![0, l_n]\!] \to P,$$

and a homeomorphism

$$\begin{aligned} \phi\colon & |[\![0, k_1]\!] \otimes \cdots \otimes [\![0, k_n]\!]| = [0, k_1] \times \cdots \times [0, k_n] \\ \to \quad & |[\![0, l_1]\!] \otimes \cdots \otimes [\![0, l_n]\!]| = [0, l_1] \times \cdots \times [0, l_n] \end{aligned}$$

such that $f \circ |\xi| = |\chi| \circ \phi$ and ϕ is a dihomeomorphism, i.e., ϕ and ϕ^{-1} preserve the natural partial order of $\mathbb{R}^n$.

Weak morphisms

Let $f\colon |Q| \to |P|$ be a weak morphism of precubical sets, and let $\omega\colon [\![0,k]\!] \to Q$ $(k \geq 0)$ be a path. We denote by $f^{\mathbb{I}}(\omega)$ the unique path $\nu\colon [\![0,l]\!] \to P$ for which there exists a dihomeomorphism $\phi\colon |[\![0,k]\!]| = [0,k] \to |[\![0,l]\!]| = [0,l]$ such that $f \circ |\omega| = |\nu| \circ \phi$.

A *weak morphism* from an M-HDA $\mathcal{B} = (Q, J, G, \mu)$ to an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ is a weak morphism $f\colon |Q| \to |P|$ such that $f(J) \subseteq I$, $f(G) \subseteq F$ and $\overline{\lambda} \circ f^{\mathbb{I}} = \overline{\mu}$.

Proposition

Weak morphisms preserve dihomotopy. Consequently, if f is a weak morphism from an M-HDA $\mathcal{B} = (Q, J, G, \mu)$ to an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ such that $f(m(Q)) \subseteq m(P)$ and $f(M(Q)) \subseteq M(P)$, then f induces a functor $f_*\colon TC(\mathcal{B}) \to TC(\mathcal{A})$.

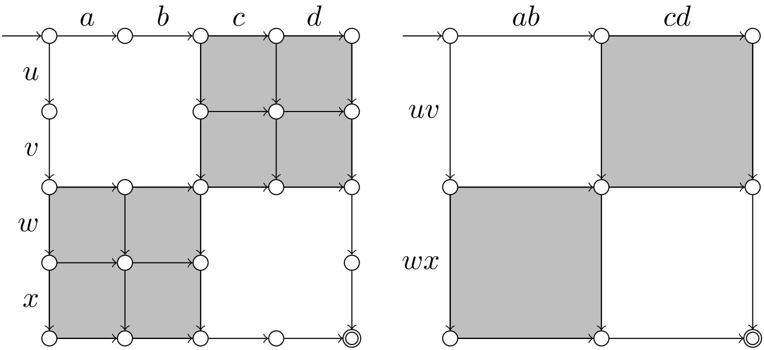
Homeomorphic abstraction

Consider two M-HDAs $\mathcal{A} = (P, I, F, \lambda)$ and $\mathcal{B} = (Q, J, G, \mu)$. We say that $\mathcal{B}$ is a *homeomorphic abstraction* of $\mathcal{A}$, or that $\mathcal{A}$ is a *homeomorphic refinement* of $\mathcal{B}$, if there exists a weak morphism f from $\mathcal{B}$ to $\mathcal{A}$ that is a homeomorphism and satisfies $f(J) = I$ and $f(G) = F$. We use the notation $\mathcal{B} \overset{\approx}{\to} \mathcal{A}$ to indicate that $\mathcal{B}$ is a homeomorphic abstraction of $\mathcal{A}$.

Remarks

- The relation $\overset{\approx}{\to}$ is a preorder on the class of M-HDAs.
- Homeomorphic abstraction is a labeled version of *T-homotopy equivalence* [Gaucher, Goubault].

Homeomorphic abstraction



(a) An HDA

(b) Homeomorphic abstraction

Invariance of the trace category

Definition

An M-HDA is said to be *weakly regular* if for every element x of degree 2, $d_1^0 x \neq d_2^0 x$ and $d_1^1 x \neq d_2^1 x$.

Theorem

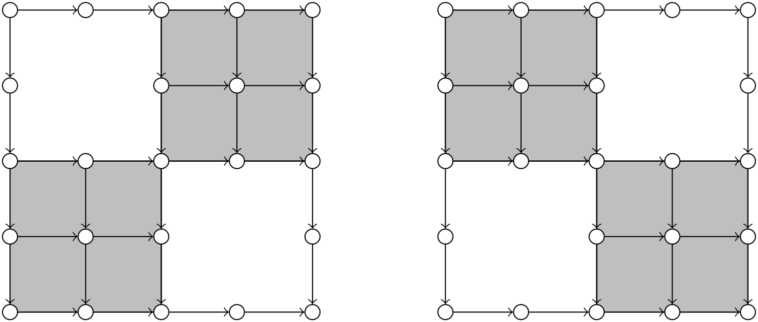
Suppose that $\mathcal{B} \overset{\approx}{\to} \mathcal{A}$*. If* $\mathcal{A}$ *is weakly regular, then* $TC(\mathcal{B}) \cong TC(\mathcal{A})$.

The homology graph

Let P be a precubical set. We say that a homology class $\alpha \in H_*(|P|)$ *points* to a homology class $\beta \in H_*(|P|)$ and write $\alpha \nearrow \beta$ if there exist precubical subsets $X, Y \subseteq P$ such that $\alpha \in \operatorname{im} H_*(|X| \hookrightarrow |P|)$, $\beta \in \operatorname{im} H_*(|Y| \hookrightarrow |P|)$, and for all $x \in X_0$ and $y \in Y_0$ there exists a path in P from x to y.

The *homology graph* of P is the directed graph whose vertices are the homology classes of $|P|$ and whose edges are given by the relation $\nearrow$.

Ordered vs. unordered holes



(a) The homology class representing the upper hole points to the homology class representing the lower hole

(b) The homology graph has no edges between non-zero classes of H_1

Invariance of the homology graph

Theorem

Let $f\colon |Q| \to |P|$ be a weak morphism of precubical sets that is a homeomorphism. Then $f_*\colon H_*(|Q|) \to H_*(|P|)$ is a graph isomorphism.

Definition

Let C and C' be precubical subsets of P. We say that C is *deformable into* C' if there exists a precubical subset $\hat{C} \subseteq P$ such that $C \subseteq \hat{C} \supseteq C'$ and the inclusion $|C'| \hookrightarrow |\hat{C}|$ is a homotopy equivalence.

Theorem

Let P be a precubical set, and let $x \in P_{\geq 1}$ be regular with free face $d_i^1 x$. Consider the precubical set $Q = P \setminus \{x, d_i^1 x\}$, and suppose that every precubical subset C of P is deformable into a precubical subset C' of Q such that from every vertex v in C', there exists a path in Q to a vertex in C from which $d_1^0 \cdots d_1^0 d_i^1 x$ is only reachable if $d_1^0 \cdots d_1^0 x$ is reachable in Q from v. Then $H_*(|Q| \hookrightarrow |P|)$ is a graph isomorphism.

Topological abstraction

Consider two M-HDAs $\mathcal{A} = (P, I, F, \lambda)$ and $\mathcal{B} = (Q, J, G, \mu)$. We write $\mathcal{B} \overset{\sim}{\to} \mathcal{A}$ and say that $\mathcal{B}$ is a *topological abstraction* of $\mathcal{A}$, or that $\mathcal{A}$ is a *topological refinement* of $\mathcal{B}$, if there exists a weak morphism f from $\mathcal{B}$ to $\mathcal{A}$ such that

1. $f(J) = I$, $f(G) = F$, $f(M(Q)) = M(P)$, $f(m(Q)) = m(P)$,
2. f is a homotopy equivalence,
3. the functor $f_* \colon TC(\mathcal{B}) \to TC(\mathcal{A})$ is an isomorphism,
4. the map $f_* \colon H_*(|Q|) \to H_*(|P|)$ is a graph isomorphism.

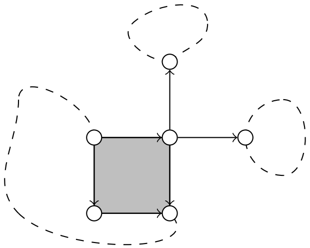
Theorem

Suppose that $\mathcal{B} \overset{\approx}{\to} \mathcal{A}$. If $\mathcal{A}$ is weakly regular, then $\mathcal{B} \overset{\sim}{\to} \mathcal{A}$.

Elementary collapses

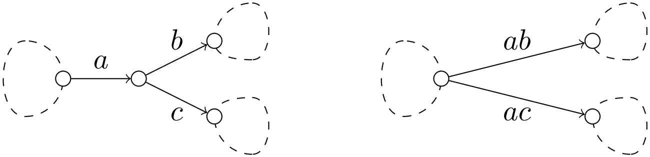
Theorem

Let $\mathcal{A} = (P, I, F, \lambda)$ be a weakly regular M-HDA, and let $x \in P_{\geq 2}$ be a regular cube with free face $d_i^1 x$. Suppose that there is precisely one edge ending in $d_1^0 \cdots d_1^0 d_i^1 x$. If $n \leq 3$, it is also required that $d_1^0 \cdots d_1^0 d_i^1 x \notin I \cup F$. If $n = 2$, it is finally required that at least two edges begin in $d_1^0 d_i^1 x$. Then $Q = P \setminus \{x, d_i^1 x\}$ is a precubical subset of P such that $I \cup F \subseteq Q$, $m(Q) = m(P)$, $M(Q) = M(P)$, and $(Q, I, F, \lambda|_{Q_1}) \xrightarrow{\sim} \mathcal{A}$.



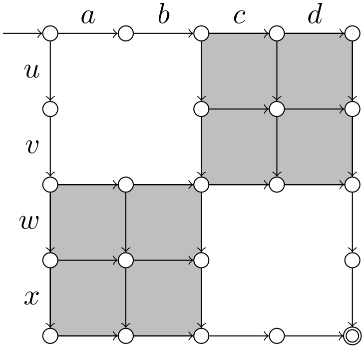
Further collapsing operations

(a) Collapsing the star of a vertex

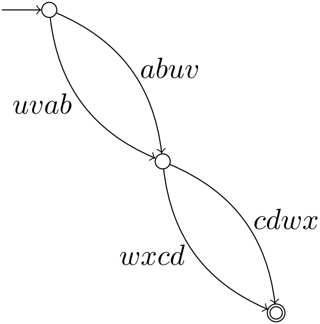


(b) Collapsing an edge

Topological abstraction: example



(a) Ordered holes



(b) Topological abstraction