# Oriented Syzygies for Monoids

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**Joint works with**

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- In low dimensions : **coherent presentations**
  - generators, **oriented** relations, **oriented syzygies**.
  - Applications:
    - Explicit description of actions of a monoid on categories (representation theory),
    - Coherence theorems for monoids.

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▶ A **Coxeter system** $(\mathbf{W}, S)$ is a data made of a group $\mathbf{W}$ with a presentation by a (finite) set $S$ of involutions, $s^2 = 1$, satisfying **braid relations**

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▶ Forgetting the involutive character of generators, one gets the **Artin's presentation**

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**Objective.**

▷ Push further Artin's presentation and study the **relations amongst the braid relations**. (Brieskorn-Saito, 1972, Deligne, 1972, Deligne, 1997, Tits, 1981, Michel, 1999).

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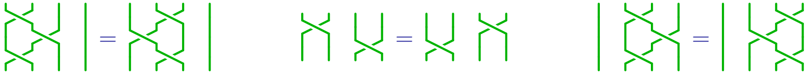

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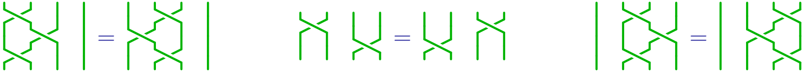

▶ The relations amongst the braid relations on 4 strands are generated by the following **Zamolodchikov relation** (Deligne, 1997).

$$\begin{array}{ccccccccc} & stsrst & = & strsrt & = & srtstr & = & srstsr & = rsrtsr & \\ tstrst & & & & \Big\| \; Z_{r,s,t} & & & & & rstrsr \\ & tsrtst & = & tsrsts & = & trsrts & = & rtstrs & = rstsrs & \end{array}$$

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$$\mathbf{P}_n = \left\langle\, 1, \ldots, n \;\middle|\; \begin{array}{ll} zxy = xzy & \text{for all } 1 \leqslant x \leqslant y < z \leqslant n \\ yzx = yxz & \text{for all } 1 \leqslant x < y \leqslant z \leqslant n \end{array} \right\rangle$$

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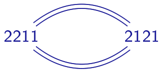

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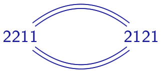


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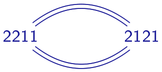


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**Objective.**

  - Compute finite coherent presentation for $\mathbf{P}_n$.

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- The **Knuth-Bendix** procedure does not terminate for
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  and a way to homotopically reduce them.

# Plan

## I. Coherent presentations of categories

- Polygraphs as higher-dimensional rewriting systems
- Coherent presentations as cofibrant approximations

## II. Homotopical completion-reduction procedure

- Tietze transformations
- Rewriting properties of polygraphs
- Completion-reduction procedure

## III. Applications to Artin and plactic monoids

**References**


- Hage-M., *Coherent presentations of plactic monoids*, 2015.
- Gaussent-Guiraud-M., *Coherent presentations of Artin monoids*, 2015.
- Guiraud-M.-Mimram, *A homotopical completion procedure with applications to coherence of monoids*, 2013.

# Part I. Coherent presentations of categories

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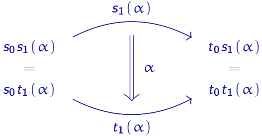

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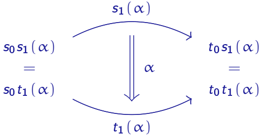


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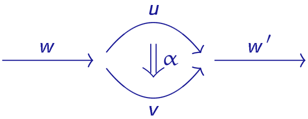


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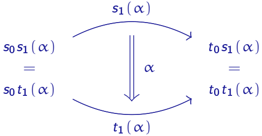


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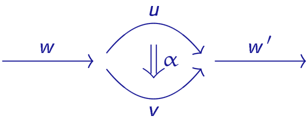


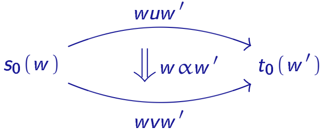


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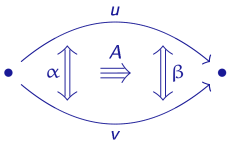

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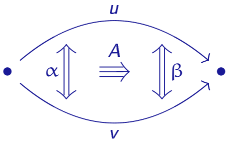


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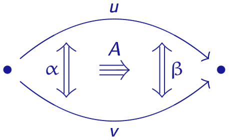


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- An **extended presentation** of $\mathbf{C}$ is a $(3,1)$-polygraph $\Sigma$ such that

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**Theorem.** [Gaussent-Guiraud-M., 2015]

Let $\Sigma$ be an extended presentation of a category $\mathbf{C}$. For the Lack's model structure on $2$-categories, the following assertions are equivalent:

**i)** The $(3,1)$-polygraph $\Sigma$ is a coherent presentation of $\mathbf{C}$.

**ii)** The $(2,1)$-category $\Sigma_2^\top/\Sigma_3$ is a **cofibrant approximation** of $\mathbf{C}$, that is, a cofibrant $2$-category weakly equivalent to $\mathbf{C}$.

# Examples

- **Free monoid** : no relation, an empty homotopy basis:

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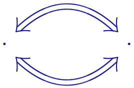
- **Free commutative monoid** of rank 3:
  - ▷ the full coherent presentation:

$$\langle\, r, s, t \;\mid\; sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\; ts \overset{\gamma_{st}}{\Longrightarrow} st,\; tr \overset{\gamma_{rt}}{\Longrightarrow} rt \;\mid\; \text{all the 2-spheres}$$



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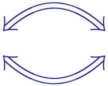

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  - a homotopy basis can be made with only one $3$-cell

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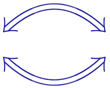
# Examples

- **Free monoid** : no relation, an empty homotopy basis:

$$\langle\, x_1, \ldots, x_n \;\mid\; \emptyset \;\mid\; \emptyset \,\rangle$$

- **Free commutative monoid** of rank 3:
  - ▷ the full coherent presentation:

$$\langle\, r, s, t \;\mid\; sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\ ts \overset{\gamma_{st}}{\Longrightarrow} st,\ tr \overset{\gamma_{rt}}{\Longrightarrow} rt \;\mid\; \text{all the 2-spheres}\ \cdot$$  $$\cdot \,\rangle$$

  - ▷ a homotopy basis can be made with only one 3-cell

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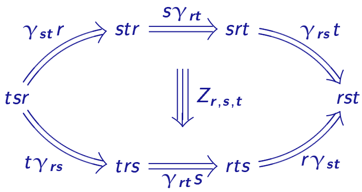
where the 3-cell $Z_{r,s,t}$ is the **permutohedron**



# Examples

- **Artin monoid $\mathbf{B}^+(\mathbf{S}_3)$**

$s =$  $t =$ 

 $\Rightarrow$ 

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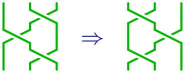

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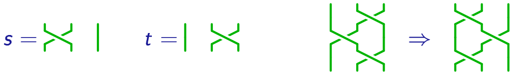


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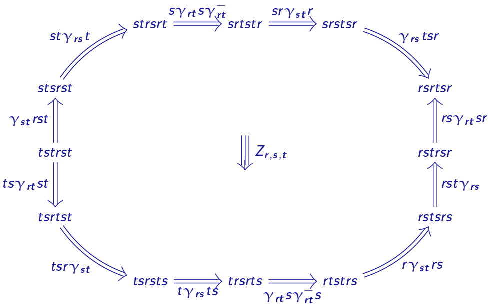

# Coherent presentations

**Problems.**

1. How to compute a coherent presentation ?
2. How to transform a coherent presentation ?

# Part II. Homotopical completion-reduction procedure

# Tietze transformations

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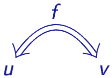

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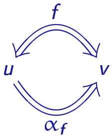

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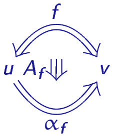

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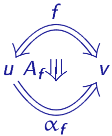


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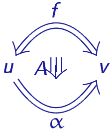

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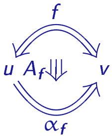


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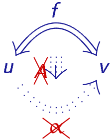

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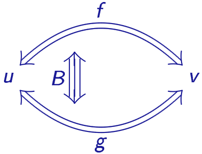

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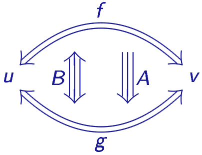

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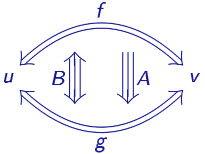


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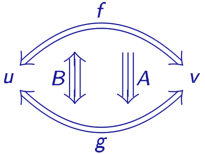

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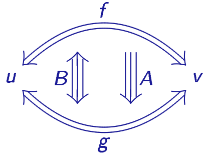


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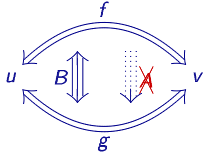

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**Theorem.** [Gaussent-Guiraud-M., 2015]

Two (finite) $(3,1)$-polygraphs $\Sigma$ and $\Upsilon$ are Tietze equivalent if, and only if, there exists a (finite) Tietze transformation

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**Consequence.**

If $\Sigma$ is a coherent presentation of a category $\mathbf{C}$ and if there exists a Tietze transformation

$$\mathcal{T} : \Sigma^{\top} \longrightarrow \Upsilon^{\top}$$

then $\Upsilon$ is a coherent presentation of $\mathbf{C}$.

# Rewriting properties of $2$-polygraphs

Let $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ be a $2$-polygraph.

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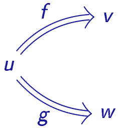
▶ A **branching** of $\Sigma$ is a pair $(f, g)$ of 2-cells of $\Sigma_2^*$ with a common source



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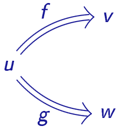


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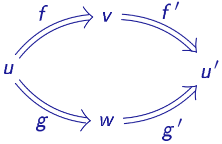

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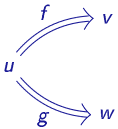


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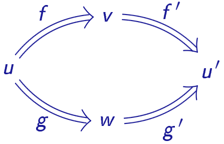
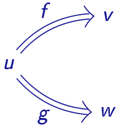



- $\Sigma$ is **convergent** if it terminates and it is confluent.

# Rewriting properties of 2-polygraphs

▶ A branching

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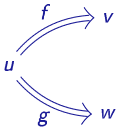


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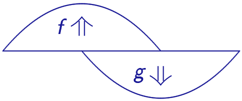
- A **critical branching** is a local branching of the form



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- The $2$-polygraph

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$$\mathbf{tst}\,st \overset{\gamma_{\mathbf{st}}\,st}{\Longrightarrow} \mathbf{sts}\,st$$

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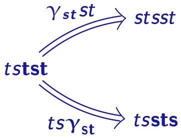

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$$tsr$$

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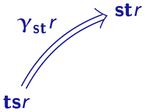

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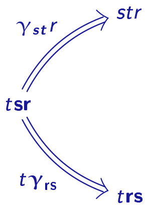

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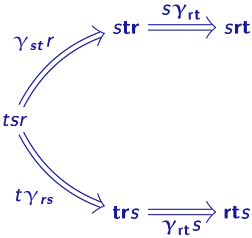

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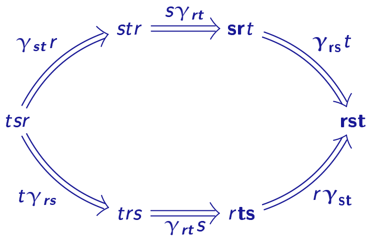

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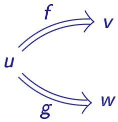

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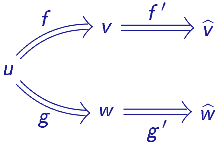


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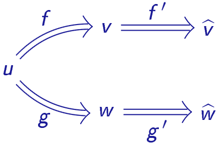


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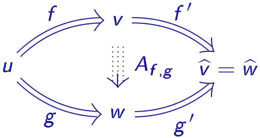

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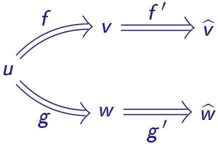


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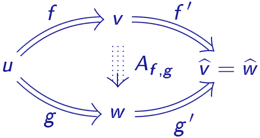
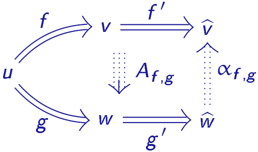



- ▷ if $\widehat{v} < \widehat{w}$, add the $2$-cell $\alpha_{f,g}$ and the $3$-cell $A_{f,g}$



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- Potential adjunction of additional 2-cells $\alpha_{f,g}$ can create new critical branchings,
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**Proof.**

- $\mathcal{S}(\Sigma)$ obtained from $\Sigma$ by successive application of Knuth-Bendix's procedure
- Squier's coherence theorem.

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**Example.** The **Kapur-Narendran's presentation** of $\mathbf{B}^+(\mathbf{S}_3)$, obtained from Artin's presentation by coherent adjunction of the Coxeter element $st$

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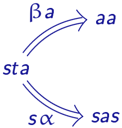

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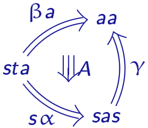

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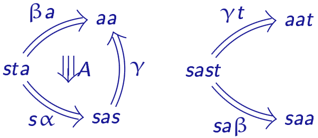

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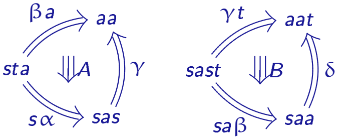

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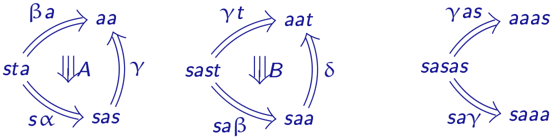

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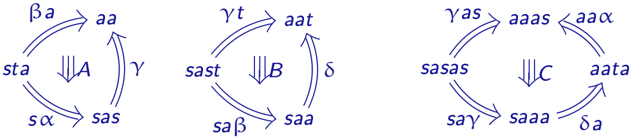

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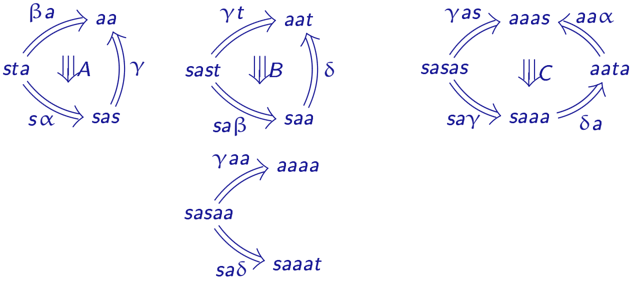

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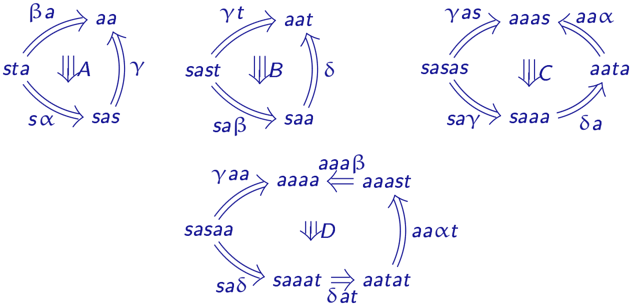

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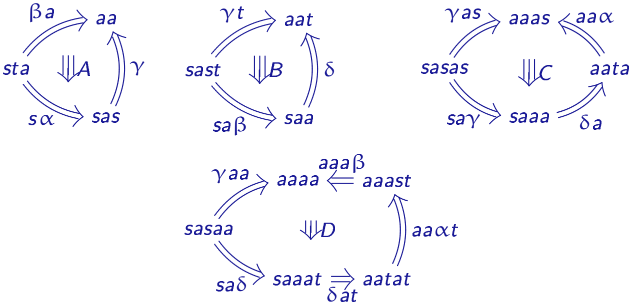


**However.** The extended presentation $\mathcal{S}(\Sigma_2^{\mathrm{KN}})$ obtained is bigger than necessary.

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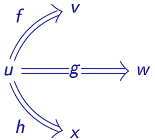

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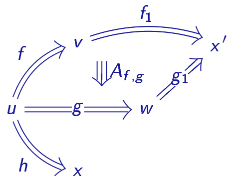

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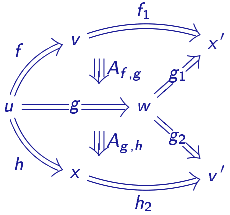

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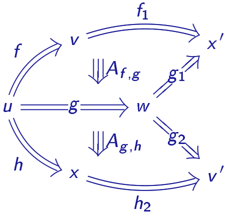


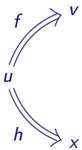

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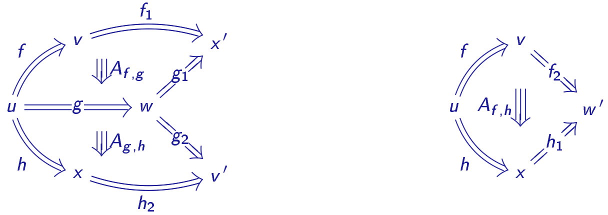

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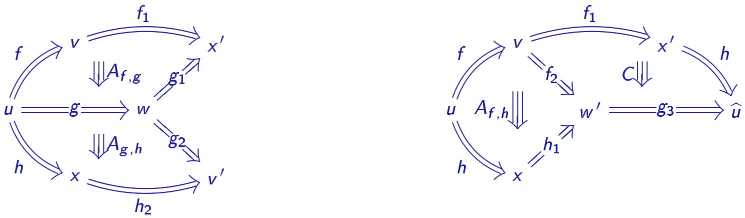

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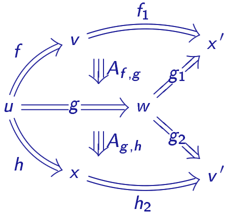


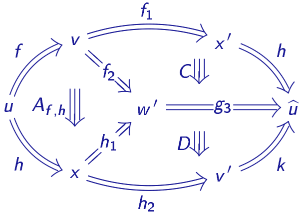

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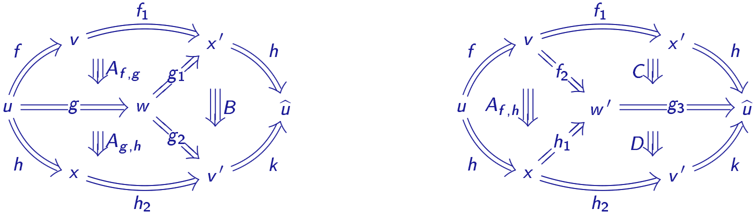

# Homotopical completion-reduction procedure

**INPUT:** A terminating $2$-polygraph $\Sigma$.

**Step 1.** Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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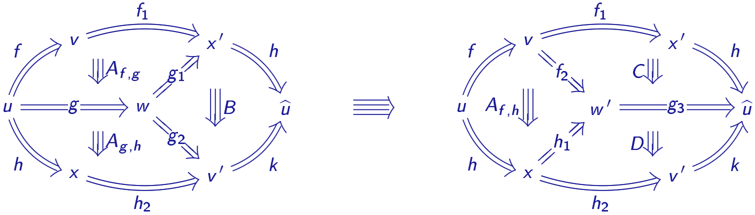

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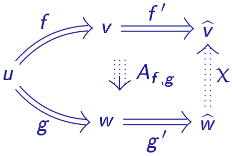

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The **homotopical completion-reduction** of terminating $2$-polygraph $\Sigma$ is the $(3,1)$-polygraph

$$\mathcal{R}(\Sigma) = \pi_\Gamma(\mathcal{S}(\Sigma))$$

**Theorem.** [Gaussent-Guiraud-M., 2015]
For every terminating presentation $\Sigma$ of a category $\mathbf{C}$, the homotopical completion-reduction $\mathcal{R}(\Sigma)$ of $\Sigma$ is a coherent presentation of $\mathbf{C}$.

# The homotopical completion-reduction procedure

**Example.**

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \quad \big| \quad ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

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- There are four critical triple branchings, overlapping on

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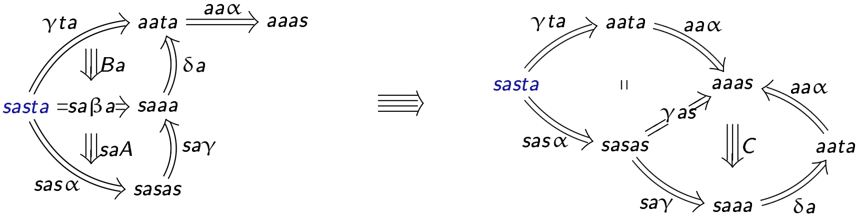
  - Critical triple branching on $sasta$ proves that $C$ is redundant:




$$C = sas\alpha^{-1} \star_1 (Ba \star_1 aa\alpha) \star_2 (saA \star_1 \delta a \star_1 aa\alpha)$$

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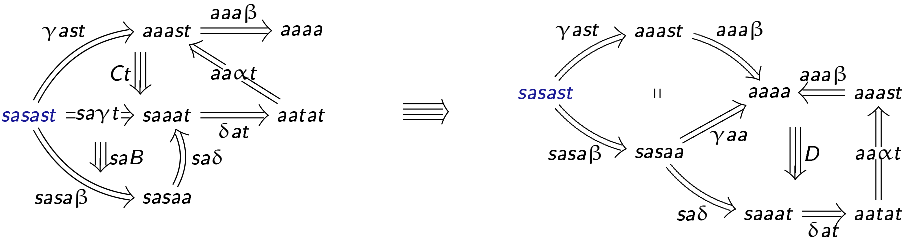


$$D = sasa\beta^{-1} \star_1 \big( (Ct \star_1 aaa\beta) \star_2 (saB \star_1 \delta at \star_1 aa\alpha t \star_1 aaa\beta) \big)$$

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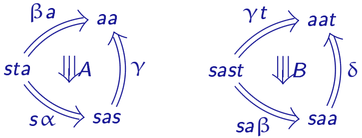
▷ The $3$-cells $A$ and $B$ are collapsible and the rules $\gamma$ and $\delta$ are redundant.



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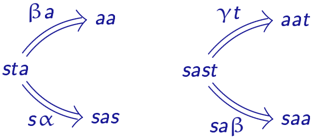

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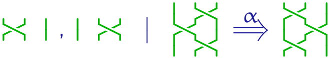
$$\mathcal{R}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t \;\mid\; tst \overset{\alpha}{\Longrightarrow} sts \;\mid\; \emptyset \,\rangle$$

$$= \mathrm{Art}_3(\mathbf{S}_3)$$

$= \langle$  $\mid \emptyset \rangle$

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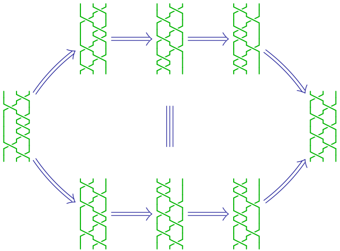

With presentation $\mathrm{Art}_2(\mathbf{S}_3)$ two proofs of the same equality in $\mathbf{B}_3^+$ are equal.

# Part III. Applications : Artin and plactic monoids

# Artin monoids: Garside's presentation

- Let $\mathbf{W}$ be a **Coxeter group**

$$\mathbf{W} = \langle\ S \quad | \quad s^2 = 1,\ \ \langle ts \rangle^{m_{st}}\ = \ \langle st \rangle^{m_{st}}\ \rangle$$

where $\langle ts \rangle^{m_{st}}$ stands for the word $tsts\dots$ with $m_{st}$ letters.

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- **Artin's presentation** of the Artin monoid $\mathbf{B}^+(\mathbf{W})$

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  - $1$-cells:

$$\mathrm{Gar}_1(\mathbf{W}) = \mathbf{W} \setminus \{1\}$$

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$$\mathrm{Gar}_2(\mathbf{W}) = \left\{ \; u|v \overset{\alpha_{u,v}}{\Longrightarrow} uv \;\text{ whenever } l(uv) = l(u) + l(v) \; \right\}$$

where $uv$ is the product in $\mathbf{W}$ and $u|v$ is the product in the free monoid over $\mathbf{W}$.

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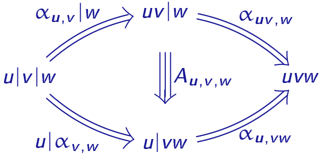
▷ $\mathrm{Gar}_3(\mathbf{W})$ made of one 3-cell




for every $u$, $v$, $w$ in $\mathbf{W} \setminus \{1\}$ such that the lengths can be added.

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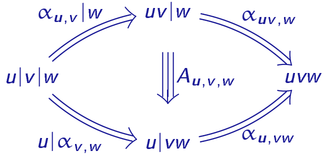


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**Theorem.** [Gaussent-Guiraud-M., 2015]
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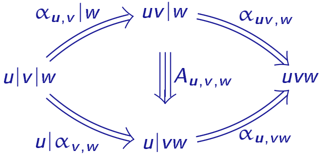


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**Theorem.** [Gaussent-Guiraud-M., 2015]

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**Proof.**

By homotopical completion-reduction of the 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$.

# Artin monoids: Artin's coherent presentation

**Theorem.** [Gaussent-Guiraud-M., 2015]

The Artin monoid $\mathbf{B}^+(\mathbf{W})$ admits the coherent presentation $\mathrm{Art}_3(\mathbf{W})$ made of

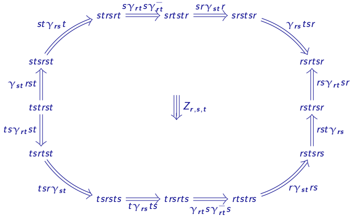
▷ Artin's presentation

$$\mathrm{Art}_2(\mathbf{W}) = \langle\ S \quad \big| \quad \langle ts\rangle^{m_{st}} = \langle st\rangle^{m_{st}}\ \rangle$$

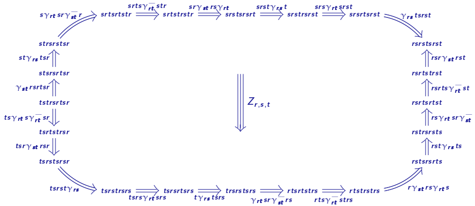
▷ one 3-cell $Z_{r,s,t}$ for every $t > s > r$ in $S$ such that the subgroup $\mathbf{W}_{\{r,s,t\}}$ is finite.

# Artin monoids: Zamolodchikov $Z_{r,s,t}$ according to Coxeter type

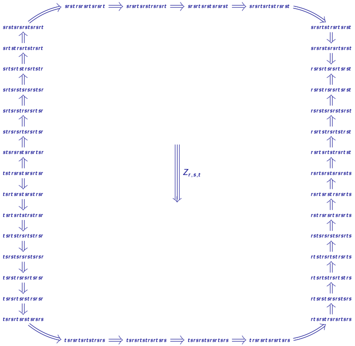
Type $A_3$



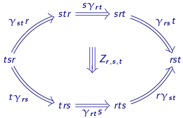

Type $B_3$



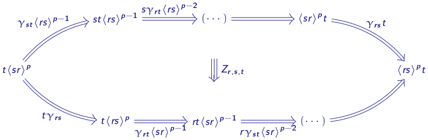

Type $H_3$




Type $A_1 \times A_1 \times A_1$




Type $I_2(p) \times A_1$, $p \geqslant 3$,



# Plactic monoids

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$$\mathrm{Knuth}_2(n) = \left\{ \begin{array}{ll} zxy = xzy & \text{for all } 1 \leqslant x \leqslant y < z \leqslant n \\ yzx = yxz & \text{for all } 1 \leqslant x < y \leqslant z \leqslant n \end{array} \right\}$$

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| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 | 3 | 4 |
| 2 | 2 | 3 | 3 | 4 | 6 | |
| 4 | 5 | 6 | 6 | | | |
| 6 | 7 | | | | | |

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  - ▷ 2-cells are **Knuth relations**:

$$\mathrm{Knuth}_2(n) = \left\{ \begin{array}{ll} zxy = xzy & \text{for all } 1 \leqslant x \leqslant y < z \leqslant n \\ yzx = yxz & \text{for all } 1 \leqslant x < y \leqslant z \leqslant n \end{array} \right\}$$

- For $n \geqslant 4$, there is no finite completion of $\mathrm{Knuth}_2(n)$ on $\mathrm{Knuth}_1(n)$.

- Any 1-cell $w$ in $\mathrm{Knuth}_1^*(n)$ is equals to its **Schensted's tableau** $P(w)$:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 | 3 | 4 |
| 2 | 2 | 3 | 3 | 4 | 6 | |
| 4 | 5 | 6 | 6 | | | |
| 6 | 7 | | | | | |

- **Column presentation** (Cain-Gray-Malheiro, 2015)
  - ▷ add **columns** as generators:

$$c_u = x_p \ldots x_2 x_1 \in \mathrm{Knuth}_1^*(n) \quad \text{such that} \quad x_p > \ldots > x_2 > x_1.$$

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$$c_u c_v \overset{\alpha_{u,v}}{\Longrightarrow} c_w c_{w'}$$

such that $u$ and $v$ are columns, the planar representation of the Schensted tableau $P(uv)$ is not the juxtaposition of columns $u$ and $v$ and where $w$ and $w'$ are respectively the left and right columns of $P(uv)$.

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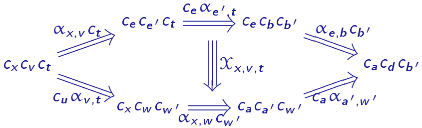


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**Theorem.** [Hage-M., 2015]

For $n \geqslant 2$, $\mathrm{Col}_3(n)$ is a finite coherent presentation of the plactic monoid $\mathbf{P}_n$.

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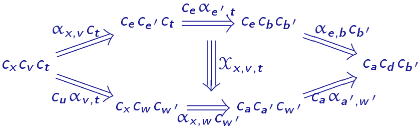


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**Theorem.** [Hage-M., 2015]

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**Proof.**

By homotopical completion-reduction of the 2-polygraph $\mathrm{Col}_2(n)$.

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  - ▷ Cubical polygraphic resolutions could help to explicit formulas for higher syzygies of $\mathbf{B}_n^+$ and $\mathbf{P}_n$.

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  - ▷ **Objective:** explicit resolutions for $\mathbf{B}_n^+$ and $\mathbf{P}_n$.
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  - ▷ Cubical polygraphic resolutions could help to explicit formulas for higher syzygies of $\mathbf{B}_n^+$ and $\mathbf{P}_n$.
- Prototype implementation of homotopical completion-reduction procedure, (Mimram, 2013)
  - ▷ `http://www.pps.univ-paris-diderot.fr/~smimram/rewr`
  - ▷ **Objective:** computations for higher ranks and higher syzygies.