ON CHRISTOFFEL SYMBOLS AND TEOREMA EGREGIUM

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1. CHRISTOFFEL SYMBOLS

This is a section on a technical device which is indispensable both in the proof of Gauss' Theorema egregium and when handling geodesics and geodesic curvature. To compare with C. Bär: Elermentary Differential geometry, notice that a chart is denoted $x(u,v)$, $x : U \to \mathbb{R}^3$. The entries g_{ij} in the Gram matrix of the first fundamental form are denoted $E = g_{11}$. $F = g_{12} = g_{21}$ and $G = g_{22}$

1.1. **Definition.** In the following, we fix a parametrization $\mathbf{x} : U \to V \cap S$ for the surface S. At every point $p = \mathbf{x}(q)$, the three vectors $\mathbf{x}_u(q), \mathbf{x}_v(q)$ and $\mathbf{N}(p)$ form a (moving) basis for $\mathbf{R}^3$. Let us express the double derivatives $x_{uu}(q), x_{uv}(q)$ and $x_{vv}(q)$ as linear combinations of these basis vectors (we omit the ps and qs and regard the derivatives etc. as vector fields):

$$\begin{aligned} \mathbf{x_{uu}} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + \Gamma^3_{11}\mathbf{N} \\ \mathbf{x_{uv}} &= \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + \Gamma^3_{12}\mathbf{N} \\ \mathbf{x_{vv}} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + \Gamma^3_{22}\mathbf{N}. \end{aligned} \tag{1}$$

with unknown real coefficient functions $\Gamma^k_{ij} : U \to \mathbb{R}$. Since the vector fields $\mathbf{x}_u, \mathbf{x}_v$ and N are linearly independet, the system (1) has a *unique* solution at every point $p \in S$.

1.2. **The metric determines the Christoffel symbols.** We know already the coefficients Γ^3_{ij} of $\mathbf{N}$:

Lemma 1.1. *The coefficients Γ^3_{ij} coincide with the coefficients e, f and g of the second fundamental form:*

$$\Gamma^3_{11} = e, \qquad \Gamma^3_{12} = f, \qquad \Gamma^3_{22} = g.$$

Bevis. Perform the dot-product with the normal vector $\mathbf{N}$ to the three equations in (1). The result is:

$$\begin{aligned} e = \mathbf{x}_{uu} \cdot \mathbf{N} &= \Gamma^3_{11} \\ f = \mathbf{x}_{uv} \cdot \mathbf{N} &= \Gamma^3_{12} \\ g = \mathbf{x}_{vv} \cdot \mathbf{N} &= \Gamma^3_{22}. \end{aligned}$$

$\square$

Hence, we can reformulate (1) in matrix language as follows: Let $\mathbf{\Gamma}$ denote the $3\times$ 3-matrix

$$\mathbf{\Gamma} = \begin{bmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \\ e & f & g \end{bmatrix}.$$

Then, the following matrix equation holds:

$$[\mathbf{x}_{uu}\ \mathbf{x}_{uv}\ \mathbf{x}_{vv}] = [\mathbf{x}_u\ \mathbf{x}_v\ \mathbf{N}]\mathbf{\Gamma}. \tag{2}$$

Multiplying this equation (from the left) with the 3 by 3-matrix $[\mathbf{x}_u, \mathbf{x}_v, \mathbf{N}]^T$, we obtain:

$$\begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_{uu} & \mathbf{x}_u \cdot \mathbf{x}_{uv} & \mathbf{x}_u \cdot \mathbf{x}_{vv} \\ \mathbf{x}_v \cdot \mathbf{x}_{uu} & \mathbf{x}_v \cdot \mathbf{x}_{uv} & \mathbf{x}_v \cdot \mathbf{x}_{vv} \\ \mathbf{N} \cdot \mathbf{x}_{uu} & \mathbf{N} \cdot \mathbf{x}_{uv} & \mathbf{N} \cdot \mathbf{x}_{vv} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_v \\ \mathbf{N} \end{bmatrix} [\mathbf{x}_{uu}\ \mathbf{x}_{uv}\ \mathbf{x}_{vv}] = \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_v \\ \mathbf{N} \end{bmatrix} [\mathbf{x}_u\ \mathbf{x}_v\ \mathbf{N}]\mathbf{\Gamma} =$$

$$\begin{bmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{\Gamma},$$

and hence

$$\mathbf{\Gamma} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F & 0 \\ -F & E & 0 \\ 0 & 0 & EG - F^2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_{uu} & \mathbf{x}_u \cdot \mathbf{x}_{uv} & \mathbf{x}_u \cdot \mathbf{x}_{vv} \\ \mathbf{x}_v \cdot \mathbf{x}_{uu} & \mathbf{x}_v \cdot \mathbf{x}_{uv} & \mathbf{x}_v \cdot \mathbf{x}_{vv} \\ \mathbf{N} \cdot \mathbf{x}_{uu} & \mathbf{N} \cdot \mathbf{x}_{uv} & \mathbf{N} \cdot \mathbf{x}_{vv} \end{bmatrix}. \tag{3}$$

We conclude:

Proposition 1.2. *The Christoffel-symbols Γ^k_{ij}, $1 \le i, j, k \le \mathbf{2}$ are intrinsic entities, i.e., they depend only on the metric (first fundamental form) along the surface S.*

Bevis. Since the Christoffel symbols are the first two rows of $\mathbf{\Gamma}$, and since the entries (13) and (23) in the inverse matrix are 0, it is enough to show that the first two rows of the matrix $\begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_{uu} & \mathbf{x}_u \cdot \mathbf{x}_{uv} & \mathbf{x}_u \cdot \mathbf{x}_{vv} \\ \mathbf{x}_v \cdot \mathbf{x}_{uu} & \mathbf{x}_v \cdot \mathbf{x}_{uv} & \mathbf{x}_v \cdot \mathbf{x}_{vv} \\ \mathbf{N} \cdot \mathbf{x}_{uu} & \mathbf{N} \cdot \mathbf{x}_{uv} & \mathbf{N} \cdot \mathbf{x}_{vv} \end{bmatrix}$

are expressible in terms of the coefficients E, F, G of the first fundamental form. To this end, we form the partial derivatives of the functions E, F, G with respect to the variables u and v and obtain the following six equations:

$$E_u = 2\mathbf{x}_u \cdot \mathbf{x}_{uu},\ E_v = 2\mathbf{x}_u \cdot \mathbf{x}_{uv},\ F_u = \mathbf{x}_v \cdot \mathbf{x}_{uu} + \mathbf{x}_u \cdot \mathbf{x}_{uv},\ F_v = \mathbf{x}_v \cdot \mathbf{x}_{uv} + \mathbf{x}_u \cdot \mathbf{x}_{vv},$$

$$G_u = 2\mathbf{x}_v \cdot \mathbf{x}_{uv},\ G_v = 2\mathbf{x}_v \cdot \mathbf{x}_{vv}.$$

This system of equations has the following solutions:

$$\mathbf{x}_u \cdot \mathbf{x}_{uu} = \frac{E_u}{2},\ \mathbf{x}_u \cdot \mathbf{x}_{uv} = \frac{E_v}{2},\ \mathbf{x}_u \cdot \mathbf{x}_{vv} = F_v - \frac{G_u}{2},$$

$$\mathbf{x}_v \cdot \mathbf{x}_{uu} = F_u - \frac{E_v}{2},\ \mathbf{x}_v \cdot \mathbf{x}_{uv} = \frac{G_u}{2},\ \mathbf{x}_v \cdot \mathbf{x}_{vv} = \frac{G_v}{2}.$$

□

2. Teorema Egregium

The Gauss curvature is defined in terms of the second fundamental form, and it is certainly not true, that the coefficients h_{ij} can be expressed by g_{ij}. Nor is it the case that the principal curvatures are intrinsic. However, the remarkable theorem, Teorema Egregium says, that the Gauss curvature is in fact intrinsic. So the product $\kappa_1\kappa_2$ is intrinsic, even if κ_1 and κ_2 are not.

Theorem 2.1. *Let S_1 and S_2 be regular surfaces and let $\phi : S_1 \to S_2$ be a local isometry. Then the Gauss curvature is preserved; $K(p) = K(\phi(p))$ for all $p \in S_1$.*

Bevis. Let $p \in S_1$. By Bär exercise 4.3, it suffices to see for a chart (U, F, V), that $K(p)$ may be expressed via the koefficients of the first fundamental form.
From Corollary 2.3, we get

$$d_pN(x_u) \times d_pN(x_v) = K(p)x_u \times x_v.$$

On both sides of this equation, take the inner product with $x_u \times x_v$ to get

$$K(g_{11}g_{22} - g_{12}^2) = (d_pN(x_u) \cdot x_u)(d_pN(x_v) \cdot x_v) - (d_pN(x_u) \cdot x_v)(d_pN(x_v) \cdot x_u)$$

We used Lagranges formula 2.4 on the right hand side. The normal vector is

$$N \circ x(u, v) = \frac{x_u \times x_v}{|x_u \times x_v|}(u, v)$$

and $|x_u \times x_v|^2(u,v) = (g_{11}g_{22} - g_{12}^2)(u,v)$ so

$$d_pN(x_u) = \frac{\partial N \circ x}{\partial u} = \frac{x_{uu} \times x_v + x_u \times x_{vu}}{\sqrt{g_{11}g_{22} - g_{12}^2}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{g_{11}g_{22} - g_{12}^2}}\right)(x_u \times x_v)$$

and symmetrically:

$$d_pN(x_v) = \frac{\partial N \circ x}{\partial v} = \frac{x_{uv} \times x_v + x_u \times x_{vv}}{\sqrt{g_{11}g_{22} - g_{12}^2}} + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{g_{11}g_{22} - g_{12}^2}}\right)(x_u \times x_v)$$

Substitute these in the equation for Gauss curvature to get:

$$K(g_{11}g_{22} - g_{12}^2)^2 = ((x_{uu} \times x_v) \cdot x_u)((x_u \times x_{vv}) \cdot x_v) - ((x_u \times x_{uv}) \cdot x_v)((x_{uv} \times x_v) \cdot x_u)$$

The determinant of a 3x3 matrix is expressed via the cross product and inner product of rows (or columns) and here this gives

$$= \det(x_{uu}x_vx_u)\det(x_ux_{vv}x_v) - \det(x_ux_{uv}x_v)\det(x_{uv}x_vx_u)$$

where the vectors x_{ij} are columns. Interchange columns and transpose to get

$$= \det((x_{uu}x_vx_u)\begin{pmatrix} x_u \\ x_{vv} \\ x_v \end{pmatrix}) - \det((x_ux_{uv}x_v)\begin{pmatrix} x_{uv} \\ x_v \\ x_u \end{pmatrix})$$

$$= \det((x_{uu}x_ux_v)\begin{pmatrix} x_{vv} \\ x_u \\ x_v \end{pmatrix}) - \det((x_{uv}x_ux_v)\begin{pmatrix} x_{uv} \\ x_u \\ x_v \end{pmatrix})$$

$$= \det\begin{pmatrix} x_{uu} \cdot x_{vv} & x_{uu} \cdot x_u & x_{uu} \cdot x_v \\ x_u \cdot x_{vv} & g_{11} & g_{12} \\ x_v \cdot x_{vv} & g_{12} & g_{22} \end{pmatrix} -$$

$$\det\begin{pmatrix} x_{uv} \cdot x_{uv} & x_{uv} \cdot x_u & x_{uv} \cdot x_v \\ x_u \cdot x_{uv} & g_{11} & g_{12} \\ x_v \cdot x_{uv} & g_{12} & g_{22} \end{pmatrix}.$$

We have already expresses the entrances of these matrices in terms of the first fundamental form (via the Christoffel symbols), except the upper left entrances. Since we take the determinant, it suffices to express the difference $x_{uu} \cdot x_{vv} - x_{uv} \cdot x_{uv}$ (Why?).
Since $x_u \cdot x_{vv}$ and $x_u \cdot x_{uv}$ have already been expressed in terms of the first fundamental form, there are expressions of their derivatives.
Now $\frac{\partial}{\partial u}(x_u \cdot x_{vv}) = x_{uu} \cdot x_{vv} + x_u \cdot x_{vvu}$ and $\frac{\partial}{\partial v}(x_u \cdot x_{uv}) = x_{uv} \cdot x_{uv} + x_u \cdot x_{uvv}$. The chart x is C^∞, so $x_{vvu} = x_{uvv}$. Subtract the two expressions and conclude that $x_{uu} \cdot x_{vv} - x_{uv} \cdot x_{uv}$ is expressible by g_{ij} and their derivatives. □

Lemma 2.2. *Let* $[\mathbf{v_1 v}_2]$ *be a 3x2 matrix with columns* $\mathbf{v_i}$. *Let* A *be a 2x2 matrix and define* $[\mathbf{w_1 w}_2] = [\mathbf{v_1 v}_2]A$. *Then the vector product satisfies* $\mathbf{w}_1 \times \mathbf{w}_2 = \mathbf{v}_1 \times \mathbf{v}_2 det(A)$

Bevis. Left to the reader. □

Corollary 2.3. $d_pN(x_u) \times d_pN(x_v) = K(p)x_u \times x_v$

Bevis. Let W be the matrix of the Weingarten map wrt. x_u, x_v. Then $-d_pN(x_u) = w_{11}x_u + w_{21}x_v$ and $-d_pN(x_v) = w_{12}x_u + w_{22}x_v$. Hence the lemma applies with $A = W$, and $K(p) = det(W)$ □

Lemma 2.4. *Lagrange's formula: For cross product of vectors in* $\mathbb{R}^3$, *we have*

$$(v_1 \times v_2) \cdot (w_1 \times w_2) = (v_1 \cdot w_1)(v_2 \cdot w_2) - (v_1 \cdot w_2)(v_2 \cdot w_1)$$