Solutions to exercises Kreyszig 15.1 16-19

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Please notice that $|z_1 z_2| = |z_1||z_2|$ and $|z_1/z_2| = |z_1|/|z_2|$. If you do not feel 100% confident dealing with complex numbers, I advice you to read sections 13.1 and 13.2 of Kreyszig.

Is the series convergent or divergent? Give a reason. Show details.

16) $\sum_{n=0}^{\infty} z_n$, with $z_n = \frac{(20+30i)^n}{n!}$. By ratio test:
$|\frac{z_{n+1}}{z_n}| = |\frac{(20+30i)^{(n+1)}}{(n+1)!} / \frac{(20+30i)^n}{n!}| = |\frac{(20+30i)}{(n+1)}|$. As $\lim_{n\to\infty} |\frac{z_{n+1}}{z_n}| = L = 0 < 1$, the serie is convergent. See page 676 and 677 (theorem 7 and 8, ratio-test).

17) $\sum_{n=2}^{\infty} z_n$, with $z_n = \frac{(-i)^n}{ln\, n}$. This serie can be split into two alternating series that fulfill convergense criterium as given on *Oversigt nr. 3*:
$\sum_{n=2}^{\infty} z_n = \frac{(-i)^2}{ln2} + \frac{(-i)^3}{ln3} + \frac{(-i)^4}{ln4} + \frac{(-i)^5}{ln5} + \frac{(-i)^6}{ln6} + \cdots = \frac{-1}{ln2} + \frac{i}{ln3} + \frac{1}{ln4} + \frac{-i}{ln5} + \frac{-1}{ln6} + \cdots = \sum_{n=1}^{\infty} a_n + i\sum_{n=1}^{\infty} b_n$, with $a_n = \frac{(-1)^n}{ln(2n)}$ and $b_n = \frac{(-1)^{(n+1)}}{ln(2n+1)}$

18) $\sum_{n=1}^{\infty} z_n$, with $z_n = n^2(\frac{i}{4})^n$. By ratio test:
$|\frac{z_{n+1}}{z_n}| = |(n+1)^2(\frac{i}{4})^{(n+1)}/n^2(\frac{i}{4})^n| = |\frac{(n+1)^2}{n^2}\frac{i}{4})| = \frac{(n+1)^2}{n^2}\frac{|i|}{4})$. As $\lim_{n\to\infty} |\frac{z_{n+1}}{z_n}| = L = \frac{1}{4} < 1$, the serie is convergent. See page 676 and 677 (theorem 7 and 8, ratio-test).

19) $\sum_{n=0}^{\infty} z_n$, with $z_n = \frac{i^n}{n^2-i}$. By comparision test:
$|z_n| = |\frac{i^n}{n^2-i}| = \frac{|i^n|}{|n^2-i|} = \frac{1}{[(n^2-i)(n^2+i)]^{1/2}} = \frac{1}{(n^4+1)^{1/2}} \leq \frac{1}{n^2}$.
As $\sum_{n=0}^{\infty} \frac{1}{n^2}$ is convergent, so is the serie under consideration.

20) $\sum_{n=0}^{\infty} z_n$, with $z_n = \frac{n+i}{3n^2+2i}$. By comparision test:
$|z_n| = |\frac{n+i}{3n^2+2i}| = \frac{|n+i|}{|3n^2+2i|} = \frac{(n^2+1)^{1/2}}{(9n^4+4)^{1/2}} = (\frac{1+1/n^2}{9n^2+4/n^2})^{1/2} > \frac{1}{3n}$. As $\sum_{n=0}^{1\infty} \frac{1}{n}$ is *divergent*, so is the serie under consideration. NB: Is indeed $\frac{1+1/n^2}{9n^2+4/n^2} > \frac{1}{9n^2}$? Yes, as:
$\frac{1+1/n^2}{9n^2+4/n^2} > \frac{1}{9n^2} \to (1+1/n^2) > (1+4/9n^4) \to 1/n^2 > 4/9n^4 \to n^2 > 4/9$ and this is actually true for all $n > 0$.

21) $\sum_{n=0}^{\infty} z_n$, with $z_n = \frac{(c+ci)^{2n+1}}{(2n+1)!}$. By ratio-test:
$|\frac{z_{n+1}}{z_n}| = \frac{(c+ci)^{2n+3}}{(2n+3)!} / \frac{(c+ci)^{2n+1}}{(2n+1)!} = \frac{(c+ci)^2}{(2n+3)(2n+2)}$. As $\lim_{n\to\infty} |\frac{z_{n+1}}{z_n}| = L = 0 < 1$, the serie is convergent. See page 676 and 677 (theorem 7 and 8, ratio-test).