WAVELET TRANSFORMS FOR HOMOGENEOUS MIXED-NORM TRIEBEL–LIZORKIN SPACES

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Abstract. Homogeneous mixed-norm Triebel–Lizorkin spaces are introduced and studied with the use of a discrete wavelet transformation, the so-called φ-transform. This extends the classical φ-transform approach introduced by Frazier and Jawerth to the setting of mixed-norm spaces. Moreover, the theory of the φ-transform is enhanced through a precise definition of the synthesis operator, in terms of a Pettis integral, and a number of rigorous results for this operator.

1. Introduction

One of the central problems in harmonic analysis is to estimate the norm of a distribution in a smoothness space by the norm of a related sequence, through a discrete representation. Perhaps the most well-known example being Parseval's identity connecting the L_2-norm of a function with the ℓ^2-norm of the sequence of its Fourier coefficients.

The φ-transform has been systematically exploited to obtain such discrete representations since the celebrated papers of Frazier and Jawerth [10, 11]. Using the φ-transform, Frazier and Jawerth explored the deeper properties of the homogeneous Triebel–Lizorkin spaces $\dot{F}^s_{pq}$, and this led to a wealth of subsequent work by many authors. For homogeneous Triebel–Lizorkin spaces on $\mathbb{R}^n$ we may refer the reader to works of Bownik and Ho and of Kyriazis and Petrushev [6, 20–22], and for anisotropic decompositions to papers of Borup and Nielsen and of Bownik and Ho [3, 4, 6]; Bownik treated non-diagonal dilations and doubling measures [5]. For spaces on other domains such as the sphere, see for example [14, 19, 20, 23, 24].

In the past decade there has been an interest in analysing regularity of functions by means of inhomogeneous *mixed norm* Triebel–Lizorkin spaces $F^s_{\vec{p},q}$. This is a way to measure the degree of smoothness s as well as integrability $\vec{p} = (p_1, \dots, p_n)$ with different integral exponents in different directions, and a certain microscopic parameter q, in an efficacious environment of harmonic analysis.

For contributions on the embedding properties and traces on hyperplanes, the reader is referred to works of Johnsen and Sickel [17, 18], who partly in collaboration with Munch Hansen analysed embeddings and equivalent characterisations of such mixed-norm spaces, their invariance under coordinate transformations and traces on hyperplanes and domains; cf. [15, 16]. Continuity of pseudo-differential operators in this set-up has been treated by Georgiadis and Nielsen [12].

In this paper we take up the construction of wavelet bases for the mixed norm Triebel–Lizorkin spaces. This has seemingly not been done in the context before, perhaps because of certain difficulties in handling the mixed norms. But for the basic mixed-norm Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^n)$, there is a recent construction for $n = 2$ in Section 6 of the work of Torres and Ward [29]. Anyhow, the wavelets seem useful for implementation of the $F^s_{\vec{p},q}$-spaces in most applied branches of mathematics.

In our treatment of wavelets, we adopt the approach of Frazier and Jawerth by modifying their φ-transform; cf. [10, 11]. To do so, we introduce the homogeneous mixed-norm Triebel–Lizorkin spaces $\dot{F}^s_{\vec{p},q}$ with $\vec{p} = (p_1, \dots, p_n)$ for $0 < p_j < \infty$, $s \in \mathbb{R}$, $0 < q \le \infty$, and we also develop their basic theory along the way.

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Following Frazier and Jawerth, we define corresponding sequence spaces $\dot{f}^s_{\vec{p},q}$ and introduce two operators, depending on some admissible functions φ, ψ:

$$S_\varphi \colon \dot{F}^s_{\vec{p},q} \to \dot{f}^s_{\vec{p},q} \quad (\varphi\text{-transform})$$
$$T_\psi \colon \dot{f}^s_{\vec{p},q} \to \dot{F}^s_{\vec{p},q} \quad (\text{“inverse” } \varphi\text{-transform})$$

Hereby S_φ maps a function f to its wavelet coefficients (sends signals to sequences), whereas T_ψ transforms sequences to functions.

The main result of this article is the following variant of the classical φ-transform result, which states inter alia that every function f in $\dot{F}^s_{\vec{p},q}$ will be reconstructed by using T_ψ on its wavelet coefficients $S_\varphi f$:

Theorem 1.1. *The linear transformations S_φ and T_ψ are bounded operators, and $T_\psi(S_\varphi f) = f$ holds for every $f \in \dot{F}^s_{\vec{p},q}$.*

A convenient consequence of the induced formula $T_\psi \circ S_\varphi = I$ is that completeness of $\dot{F}^s_{\vec{p},q}$ follows at once from that of the simpler sequence space $\dot{f}^s_{\vec{p},q}$.

Our treatment is based on standard formulas for S_φ and T_ψ, namely

$$(S_\varphi f)_Q = \langle f, \varphi_Q \rangle \qquad \text{for each dyadic cube } Q, \tag{1.1}$$

$$T_\psi a(x) = \sum_{Q \in \mathcal{Q}} a_Q \psi_Q(x) \qquad \text{for each sequence } a = (a_Q)_{Q \in \mathcal{Q}}. \tag{1.2}$$

However, while $S_\varphi f$ is well defined for $f \in \mathcal{S}'/\mathcal{P}$, the operator T_ψ is a more delicate object since the sum in (1.2), and hence T_ψ itself, only makes sense a priori on sequences (a_Q) of finite support.

The synthesis operator T_ψ has been further studied in the homogeneous set-up in by e.g. Kyriazis [20], later by Bownik and Ho [6] and Bownik [5], who partially resolved the question of interpretation of (1.2). Here we would like to present a new perspective on T_ψ and put the study of it in a rigorous framework.

So let us recall that previously boundedness of T_ψ on sequences of finite support has been followed up by extension by continuity—that for $q = \infty$ is inadequate due to a lack of density. And often this extension $\widetilde{T}_\psi$ has entered composition formulas, like $T_\psi \circ S_\varphi = I$, in a heuristic way with a tacit assumption that also $\widetilde{T}_\psi$ acts as in (1.2)—although the notation $\sum_{Q \in \mathcal{Q}}$ was never explicitly assigned any specific meaning. Indeed, the set of dyadic cubes, $\mathcal{Q}$, can be numbered in many ways, so some condition of *integrability* must be imposed on (a_Q) to get a consistent theory. To our knowledge, neither the foundational papers by Frazier and Jawerth [10, 11] nor the subsequent literature have explicitly addressed this integrability.

On these ground it seems well motivated that we here revise the whole foundation of the synthesis operator T_ψ by suggesting a concise definition of it. Specifically we

- define $T_\psi a$ in terms of a Pettis integral (or weak Bochner integral) over $\mathcal{Q}$ with respect to the counting measure τ_{1+n},

$$\langle T_\psi a, \phi \rangle = \int_{\mathcal{Q}} \langle a_Q \psi_Q, \phi \rangle \, d\tau_{1+n}; \tag{1.3}$$

- obtain that $R(S_\varphi) \subset D(T_\psi)$, i.e. the above T_ψ is defined on the entire range of the wavelet transform S_φ;
- rigorously prove that $T_\psi(S_\varphi f) = f$ for every $f \in \mathcal{S}'/\mathcal{P}$, when φ, ψ are admissible test functions satisfying the well-known *reconstruction* identity,

$$\sum_{\nu=-\infty}^{\infty} \overline{\hat{\varphi}(2^{-\nu}\xi)} \hat{\psi}(2^{-\nu}\xi) = 1 \quad \text{for } \xi \neq 0; \tag{1.4}$$

- deduce from (1.4) that $P = S_\varphi \circ T_\psi$ is a projection for which $P = I$ holds if and only if φ, ψ fulfil the *biorthogonality* condition

$$\int_{\mathbb{R}^n} \psi_Q \overline{\varphi_J} \, dx = \delta_{Q,J} \qquad \text{(Kronecker delta)}; \tag{1.5}$$

- show explicitly for our sequence spaces that $\dot{f}^{s}_{\vec{p},q} \subset D(T_\psi)$.

By virtue of the Pettis integral, when given (1.4), then the biorthogonality (1.5) implies that any numbering of the wavelets $(\psi_Q)_{Q\in\mathcal{Q}}$ always constitutes an *unconditional* basis for $\mathcal{S}'/\mathcal{P}$; and vice versa. Indeed, the a priori existence of $T_\psi(S_\varphi f)$ vastly simplifies the discussion of bases.

Alongside this rigorous definition of T_ψ, we have also worked out a precise version of Peetre's homogeneous Littlewood–Paley decomposition; cf. Appendix C. As this corrects the previous literature in two ways, and enters our proof of the formula $T_\psi \circ S_\varphi = I$, we review it here:

When $\hat{\phi} \in \mathcal{S}$ is supported in an annulus $0 < C_0 \le |\xi| \le C^0$ and $1 = \sum_{\nu=-\infty}^{\infty} \hat{\phi}(2^{-\nu}\xi)$ for $\xi \ne 0$, then every $f \in \mathcal{S}'$ has a homogeneous Littlewood–Paley decomposition with an *explicit* asymptotic behaviour for $\nu \to -\infty$. Namely, by working with polynomial corrections $P_{m,N}$ with $N \in \mathbb{N}$ and a fixed degree $m \ge -1$, one has

$$f(x) = \sum_{\nu=-N}^{\infty} \phi(2^{-\nu}\cdot) * f(x) + P_{m,N}(x) + R_m(x). \tag{1.6}$$

Here the remainder term fulfils $R_m = \mathcal{O}(2^{-N(n+m+1-d)})$ in $\mathcal{S}'$-seminorm, d being the $\mathcal{S}'$-order of f. So for degrees $m \ge d-n$, clearly $R_m \to 0$ exponentially fast for $N \to \infty$, whence f is well represented in $\mathcal{S}'$ by the band-limited series $\sum_{\nu \ge -N} \phi(2^{-\nu}\cdot) * f$ corrected by $P_{m,N}$.

As a novelty the $P_{m,N}$ are *uniquely* determined (asymptotically for $N \to \infty$) as the degree m Taylor polynomials at $x = 0$ of a convolution $2^{-nN}\Phi(2^{-N}\cdot) * f$; cf. (C.4). The correcting $P_{m,N}$ can moreover be omitted ($m = -1$) for distributions having $d < n$, in view of the estimate of R_m.

Previously the literature has indicated, through several contributions, that in general one would meet arbitrary polynomials P and P_N on the left- and right-hand sides of (1.6), whilst a few authors have claimed some restrictions for the degrees of P, P_N; cf. Remark C.8 below.

But this picture is misleading in two ways: our analysis shows that such P, P_N must be interrelated, as $P - P_N$ asymptotically equals the Taylor polynomial $P_{m,N}$ due to the uniqueness—and on the contrary the degree $m \ge -1$ can be arbitrary. Our remainder estimate, which seems to be new in itself, shows that even by omitting polynomials, f will have a specific asymptotic representation by $\sum_{\nu\in\mathbb{Z}} \phi(2^{-\nu}\cdot) * f$, which e.g. is exact for $d < n$ and, because of formula (1.6), has an error with $P_{d-n,N}(x)$ as the leading term for $d \ge n$.

This improved insight results at once from a general analysis of the "wrong" limit $t \to 0^+$ of convolutions of the form

$$t^n\Phi(t\cdot) * f(x), \qquad \Phi \in \mathcal{S}, f \in \mathcal{S}'. \tag{1.7}$$

For details on how such convolutions behave asymptotically in $\mathcal{S}'$ as their Taylor polynomial P_m of degree m, the reader is referred to our analysis in Proposition C.2.

For the wavelet reconstruction formula $f = T_\psi(S_\varphi f)$ the consequences of the above are immediate, because the right-hand side $T_\psi(S_\varphi f)$ identifies with the series in (1.6) for a special choice of ϕ. Thus one always has that $f = T_\psi(S_\varphi f)$ in the quotient space $\mathcal{S}' \setminus \mathcal{P}$ (if (1.4) holds), but it even holds in $\mathcal{S}'$ itself for every f having $d < n$. In general the above shows which polynomials to add and how fast the wavelet reconstruction converges.

Contents. Preliminaries and notation are summed up in Section 2. General results for the synthesis operator T_ψ defined by the Pettis integral are developed in Section 3. Triebel–Lizorkin spaces with mixed norms are introduced and studied in Section 4 together with the corresponding sequence spaces. Section 5 is devoted to our results on S_φ and T_ψ in the scales of mixed-norm Triebel–Lizorkin spaces. Some technical proofs are given in Appendix A–C.

2. Preliminaries

2.1. **Notions and notation.** Generally we follow the notation of Hörmander [13] for the Fourier transform $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int e^{-\mathrm{i}\, x\cdot\xi} f(x)\,dx$ and the distribution spaces $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$ that are dual to the spaces of Schwartz functions $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and smooth functions of compact support $C_0^\infty(\mathbb{R}^n)$, respectively. In particular $D^\alpha = (-\mathrm{i})^{|\alpha|}\partial^\alpha$ for each multiindex α.

However, we use for convenience bracket notation and take functionals to be anti-linear (unless it is stated otherwise), so if e.g. f is locally integrable $\langle f,\varphi\rangle = \int f(x)\overline{\varphi(x)}\,dx$ for $\varphi \in C_0^\infty$.

If $\vec{p} = (p_1,\dots,p_n)$ with $0 < p_1,\dots,p_n < \infty$ a function $f\colon \mathbb{R}^n \to \mathbb{C}$ belongs to $L_{\vec{p}} = L_{\vec{p}}(\mathbb{R}^n)$ if

$$\|f\|_{\vec{p}} := \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |f(x_1,\dots,x_n)|^{p_1}dx_1\Big)^{\frac{p_2}{p_1}}dx_2\Big)^{\frac{p_3}{p_2}} \cdots dx_n\Big)^{\frac{1}{p_n}} < \infty. \tag{2.1}$$

Here $\|\cdot\|_{\vec{p}}$ is a quasi-norm, but $(L_{\vec{p}}, \|\cdot\|_{\vec{p}})$ is a Banach space if $\min(p_1,\dots,p_n) \ge 1$.

Throughout we use the involution $\tilde{\phi}(x) = \overline{\phi(-x)}$, for which $\mathcal{F}\tilde{\varphi} = \overline{\mathcal{F}\varphi}$. When a vector space X has two equivalent quasi-norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$, i.e. some numbers c, C fulfil $c\|x\| \le |\!|\!|x|\!|\!| \le C\|x\|$ for all $x \in X$, we indicate this by writing $\|\cdot\| \approx |\!|\!|\cdot|\!|\!|$.

A topological vector space E is said to have a sequence (x_n) as basis when each $x \in E$ can be written $x = \sum_{n=1}^\infty \lambda_n x_n$, with convergence in E, for a unique sequence of scalars λ_n. Moreover, (x_n) is called a Schauder basis if the linear forms $x \mapsto \lambda_n(x)$ are continuous (this extension beyond the category of Banach spaces goes back at least to Arsove and Edwards [1]). It is an unconditional basis of E when, moreover, $x = \sum_{n=1}^\infty \lambda_{p(n)} x_{p(n)}$ for any bijection $p\colon \mathbb{N} \to \mathbb{N}$.

Unimportant positive constants are denoted by c, although the value may depend on the place of occurrence. As usual $t_+ = \max(t,0)$, and $\mathbb{1}_S$ stands for the characteristic function of the set S.

2.2. **The wavelet set-up.** Our basic building block is a function $\varphi \in \mathcal{S}$ satisfying

$$\operatorname{supp}\hat{\varphi} \subset \{\, \xi \in \mathbb{R}^n \mid K_0 \le |\xi| \le K^0 \,\}, \tag{2.2}$$

$$|\hat{\varphi}(\xi)| \ge c > 0 \quad \text{ for } K_1 \le |\xi| \le K^1 \tag{2.3}$$

for some fixed constants $K_0 < K_1 < 1 < K^1 < K^0$ (where superscripts refer to the upper bounds in (2.2)–(2.3)) chosen so that

$$2K_1 < K^1, \qquad K^0 < \pi. \tag{2.4}$$

The choice $K_0 = 1/2$, $K^0 = 2$ and $K_1 = 3/5$, $K^1 = 5/3$ was used in [11], but we extend the framework as described.

Definition 2.1. *Functions $\varphi \in \mathcal{S}$ satisfying (2.2)–(2.4) will be called admissible.*

Admissible functions obviously exist, but they are needed in pairs $(\varphi,\,\psi)$ that fulfil the reconstruction identity in the following classical lemma:

Lemma 2.2. *To each admissible φ there exist a function $\psi \in \mathcal{S}$ which is admissible (for the same constants) and satisfies*

$$\sum_{\nu\in\mathbb{Z}} \overline{\hat{\varphi}(2^{-\nu}\xi)}\hat{\psi}(2^{-\nu}\xi) = 1 \qquad \textit{for} \quad \xi \ne 0. \tag{2.5}$$

We recall that $\hat{\psi}(\xi) = (h(\xi) - h(2\xi))/\overline{\hat{\varphi}(\xi)}$ reduces the claim in Lemma 2.2 to a telescopic sum, if the auxiliary function $h \in C_0^\infty(\mathbb{R}^n)$ is chosen thus: when $|\hat{\varphi}(\xi)| > c/2$ for $|\xi| \in [\tilde{K}_1, \tilde{K}^1] \subset]K_0, K^0[$ with $\tilde{K}_1 < K_1$ and $K^1 < \tilde{K}^1$, then $h(\xi) = 0$ should hold for $|\xi| \ge \tilde{K}^1$ and $h(\xi) = 1$ for $|\xi| \le 2\tilde{K}_1$ (where $2\tilde{K}_1 < \tilde{K}^1$ holds by (2.4)), whilst $0 < h < 1$ elsewhere, for then $h(\xi) - h(2\xi) > 0$ if and only if $|\xi| \in]\tilde{K}_1, \tilde{K}^1[$, so that (2.2) and (2.3) hold for $\hat{\psi}$.

We generally set $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$ for $\nu \in \mathbb{Z}$, and any $\varphi \in \mathcal{S}$. For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we denote by $Q_{\nu k}$ the dyadic cube

$$Q_{\nu k} = \{(x_1,\dots,x_n) \in \mathbb{R}^n : k_i \le 2^\nu x_i < k_i + 1, i = 1,\dots,n\}$$

and let $\mathcal{Q}$ be the set of dyadic cubes; Q will designate an arbitrary cube in $\mathcal{Q}$. Moreover, $x_Q = 2^{-\nu}k$ stands for the "lower left-corner" of Q. By $\ell(Q) = 2^{-\nu}$ we indicate the side length of Q, and by $|Q| = 2^{-\nu n}$ its Lebesgue measure.

We recall that a frame of wavelets consists of an admissible function φ subjected to translation and dilation, associated with an arbitrary dyadic cube $Q = Q_{\nu k}$,

$$\varphi_Q(x) := 2^{n\nu/2}\varphi(2^\nu x - k) = |Q|^{1/2}\varphi_\nu(x - x_Q). \tag{2.6}$$

For all dyadic Q of length $\ell(Q)=2^{-\nu}$

$$\operatorname{supp}\hat{\varphi}_Q \subset \{\, \xi \mid K_0 2^{\nu} \le |\xi| \le K^0 2^{\nu} \,\}, \tag{2.7}$$

and since $(1+2^{\mu}|x-k|)^L D^{\gamma}\varphi(2^{\nu}x-k)$ is a bounded function, there are estimates

$$|D^{\gamma}\varphi_Q(x)| \le C_{\gamma,L}|Q|^{-1/2-|\gamma|/n}(1+\ell(Q)^{-1}|x-x_Q|)^{-L} \tag{2.8}$$

for each $L\in\mathbb{N}$ and multi-index γ of length $|\gamma|\ge 0$.

Moreover, we need pointwise estimates of convolutions with two parameters of dilation and a translation; and it will often be crucial to have improved estimates in case one factor has vanishing moments. So we recall that $\psi\in\mathcal{S}$ is said to fulfill a moment condition of order $M\ge -1$ if it annihilates all polynomials of degree M; that is, if

$$\int_{\mathbb{R}^n} x^{\alpha}\psi(x)\,dx = 0 \quad \text{for } |\alpha|\le M. \tag{2.9}$$

Lemma 2.3. *If $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ and J is a dyadic cube of length $2^{-\mu}$ then there is for each $N>0$ a uniform estimate for $x\in\mathbb{R}^n$ and $\nu\in\mathbb{Z}$,*

$$(1+2^{\mu}|x-x_J|)^N|\psi(2^{\mu}(\cdot-x_J))*\varphi_{\nu}(x)| \le C_N 2^{(N-n)(\mu-\nu)_+}. \tag{2.10}$$

When ψ moreover fulfils a moment condition of order $M\in\mathbb{N}_0$, then the above improves to

$$(1+2^{\mu}|x-x_J|)^N|\psi(2^{\mu}(\cdot-x_J))*\varphi_{\nu}(x)| \le C'_{N,M} 2^{(N-n-(M+1))(\mu-\nu)_+}. \tag{2.11}$$

Similarly, when φ satisfies a moment condition of order $M\in\mathbb{N}_0$,

$$(1+2^{\mu}|x-x_J|)^N|\psi(2^{\mu}(\cdot-x_J))*\varphi_{\nu}(x)| \le C''_{N,M} 2^{(N-n)(\mu-\nu)_+-(M+1)(\nu-\mu)_+}. \tag{2.12}$$

In particular (2.11) *or* (2.12) *holds for all M if ψ or φ, respectively, is admissible.*

Details on these estimates can be found in Appendix A.

To elucidate the limitations of the present set-up, we first give the following account:

Example 2.4. *In a fundamental contribution Meyer proved that an orthonormal basis of Schwartz function wavelets exists in $L_2(\mathbb{R})$. This was briefly mentioned on p. 75 in [25] as resulting from an* even *function $\theta_1\in C_0^{\infty}(\mathbb{R})$ with $\theta_1\ge 0$ if*

$$\hat{\psi}(\xi) = \theta_1(\xi)e^{-\mathrm{i}\,\xi/2}, \tag{2.13}$$

provided that $\theta_1(\xi)\ne 0$ only holds for $2\pi/3\le|\xi|\le 8\pi/3$, and that

$$\theta_1(\xi)^2+\theta_1(2\xi)^2 = 1 \quad \text{for } 2\pi/3\le|\xi|\le 4\pi/3, \tag{2.14}$$

$$\theta_1(\xi)^2+\theta_1(4\pi-\xi)^2 = 1 \quad \text{for } 4\pi/3\le|\xi|\le 8\pi/3. \tag{2.15}$$

After a lengthy substantiation of this claim, Daubechies described in her book the construction as relying on "quasi-miraculous cancellation"; cf. [8, p. 119]. However, the cancellation simply comes from the orthogonality of even and odd functions on symmetric intervals $[-L,L]$:

Indeed, the possible θ_1 are parametrised by the odd *$\chi\in C^{\infty}(\mathbb{R},\mathbb{R})$ with $\chi\ge -1/2$ on $[0,\infty[$, $\chi\equiv 1/2$ on $[\pi/3,\infty[$, so that $\chi\equiv -1/2$ on $]-\infty,-\pi/3]$, by means of*

$$\theta_1(\xi)^2 = \frac{1}{2} + \begin{cases} \chi(\xi-\pi) & \text{for } 0<\xi\le 4\pi/3,\\ \chi(\pi-\xi/2) & \text{for } \xi>4\pi/3. \end{cases} \tag{2.16}$$

Here (2.14) *holds as χ is odd, whence we have* (2.5) *if we take $\varphi=\psi$ as in* (2.13).

The induced family $2^{j/2}\psi(2^jx-k)$ is orthonormal: for two such wavelets ψ', ψ'' with $j'=j''$ we use $k=k'-k''$ and properties of parity and periodicity to get

$$\begin{aligned} \langle\psi',\psi''\rangle &= 2\int_0^{\infty}\theta_1(\xi)^2\cos(k\xi)\,d\xi/(2\pi) \\ &= \delta_{k,0} + \int_{-\pi/3}^{\pi/3}\chi(\eta)(\cos(k\eta+k\pi)+\cos(2k\eta))\,d\eta/\pi = \delta_{k',k''}, \end{aligned} \tag{2.17}$$

since both cosines are even functions (the former is $-\cos(k\eta)$ for odd k). If, say $j' = j'' + 1$ a substitution gives for $k = k' - 2k''$, now because of the phase factor $e^{-\,\mathrm{i}\,\xi/2}$,

$$\begin{aligned}\langle\psi',\psi''\rangle &= 2\int_{2\pi/3}^{4\pi/3}\sqrt{2}\theta_1(\xi)\theta_1(2\xi)\cos((k-\frac{1}{2})\xi)\,d\xi/(2\pi)\\ &= \frac{\sqrt{2}}{\pi}\int_{-\pi/3}^{\pi/3}(\frac{1}{4}-\chi(\eta)^2)\sin((k-\frac{1}{2})\eta+k\pi)\,d\eta = 0,\end{aligned} \tag{2.18}$$

as $\frac{1}{4}-\chi^2$ is even and the sine is odd, also for odd k. The case $j' = j'' - 1$ is similar, and clearly $\langle\psi',\psi''\rangle = 0$ for $|j'-j''| \geq 2$ by the support condition on θ_1.

Remark 2.5. *Meyer's orthonormal wavelets basis for $L_2(\mathbb{R})$, which is recalled in Example 2.4, falls outside the framework of Frazier and Jawerth [11], and his ψ is not even admissible in the more general sense in Definition 2.1. In fact our set-up can never for $\psi = \varphi$ yield an orthonormal basis of $L_2(\mathbb{R})$, for the normalisation would mean that $2\pi = \int_{K_0\leq|\xi|\leq K^0}|\hat\varphi(\xi)|^2\,d\xi$, and the constraint $K^0 < \pi$ in* (2.4) *makes this impossible, as $|\hat\varphi(\xi)|^2 \leq 1$ holds by* (2.5).

2.3. **Maximal operators.** Let us recall some maximal inequalities pertaining to the Lebesgue space with mixed norm in (2.1).

A fundamental tool is the maximal operator M_k in the x_k-variable, $1 \leq k \leq n$, for which we write $x = (x', x_k, x'')$, whereby one of the groups $x' = (x_1,\dots,x_{k-1})$ and $x'' = (x_{k+1},\dots,x_n)$ can be empty, to define for $f(x)$ locally integrable

$$M_k f(x) = \sup_{I\in I_{x,k}}\frac{1}{|I|}\int_I |f(x',y_k,x'')|dy_k, \tag{2.19}$$

where $I_{x,k}$ is the set of all intervals in $\mathbb{R}_{x_k}$ containing x_k.

If R is a rectangle $R = I_1\times\cdots\times I_n$ it is easy to see that

$$\int_R |f(x)|dx \leq |R|\cdot M_n(\cdots(M_1 f)\cdots)(x), \quad \text{for every } x\in R. \tag{2.20}$$

Usually we omit the parentheses in the repeated use of $M_1,\dots,M_n$.

For the mixed norms we use the following version of the Fefferman–Stein vector-valued maximal inequality (cf. Stein [28], and Bagby [2] for the mixed-norms): if $\vec p = (p_1,\dots,p_n)$ for $0 < p_1,\dots,p_n < \infty$, $0 < q \leq \infty$ and $0 < t < \min(p_1,\dots,p_n,q)$ then

$$\Big\|\Big(\sum_{\nu\in\mathbb{Z}}\big(M_n\cdots M_1|f_\nu|^t\cdots\big)^{1/t}(\cdot)\big)^q\Big)^{1/q}\Big\|_{\vec p} \leq c\Big\|\Big(\sum_{\nu\in\mathbb{Z}}|f_\nu|^q\Big)^{1/q}\Big\|_{\vec p}. \tag{2.21}$$

In addition we need a well-known Peetre-type maximal inequality; cf. [18, 27]: for $t > 0$ there exists a constant $c_t > 0$, such that for every $f\in\mathcal{S}'$ satisfying $\operatorname{supp}\hat f\subset[-2^\nu,2^\nu]^n$ for some $\nu\in\mathbb{Z}$, it holds for $x\in\mathbb{R}^n$ and $\tau\geq n/t$ that

$$\sup_{y\in\mathbb{R}^n}\frac{|f(y)|}{(1+2^\nu|y-x|)^\tau} \leq c_t\Big(M_n\cdots M_1|f|^t\cdots\Big)^{1/t}(x). \tag{2.22}$$

As a digression, we note the novelty that the spectral condition on f is far from being necessary, at least if $\tau > n/t$. In fact, as a corollary to Appendix B, cf. Remark B.1, we have

Proposition 2.6. *When $\tau > n/t$, then Peetre's maximal inequality* (2.22) *is also valid for piecewise constant functions induced by a lattice of length 2^ν, $\nu\in\mathbb{Z}$ in all variables, that is, for functions having the form $f(x) = \sum_{\ell(P)=2^\nu} a_P \mathbb{1}_P(x)$ for $a_P\in\mathbb{C}$.*

3. The φ-transform

As a general framework we use the space $\mathcal{S}'/\mathcal{P}$, consisting of tempered distributions modulo polynomials. However, when considering $\mathcal{S}'/\mathcal{P}$ as a topological vector space we shall adopt the notation of Triebel [30, Ch. 5] and write $\mathcal{S}'/\mathcal{P}$ as $\mathcal{Z}'$, which is the dual space of

$$\mathcal{Z}(\mathbb{R}^n) = \Big\{\,\psi\in\mathcal{S}\mid\int_{\mathbb{R}^n}x^\alpha\psi(x)\,dx = 0 \text{ for all multiindices } \alpha\Big\}. \tag{3.1}$$

We recall that $\mathcal{Z}$ is closed in $\mathcal{S}$, hence a Fréchet space; and $\mathcal{Z}' = \mathcal{S}'/\mathcal{P}$ as $\mathcal{P} = \mathcal{Z}^\perp$. If $\mathscr{Q}\colon \mathcal{S}' \to \mathcal{S}'/\mathcal{P}$ denotes the quotient map, the w*-topology on $\mathcal{Z}'$ is induced by the seminorms $\mathscr{Q}f \mapsto |\langle f, \phi\rangle_{\mathcal{S}'\times\mathcal{S}}|$ parametrised (only) by $\phi \in \mathcal{Z}$. So trivially $\mathscr{Q}$ is a continuous linear operator:

$$\mathscr{Q}\colon \mathcal{S}' \to \mathcal{Z}'. \tag{3.2}$$

To recall the φ-transform S_φ (a discrete wavelet transform), we shall in the sequel adhere to the common practice of referring to any family $(a_Q)_{Q\in\mathcal{Q}}$ as a "sequence", even when the countable index set $\mathcal{Q}$ is considered without any numbering.

Definition 3.1. *Let φ be an admissible function. The φ-transform S_φ is the map sending each $f \in \mathcal{S}'/\mathcal{P}$ to the complex-valued sequence $S_\varphi f = \{(S_\varphi f)_Q\}_{Q\in\mathcal{Q}}$ with*

$$(S_\varphi f)_Q = \langle f, \varphi_Q\rangle \quad \textit{for all } Q \in \mathcal{Q}.$$

When ψ is admissible, T_ψ is the linear operator defined *tentatively* on sequences $a = \{a_Q\}_{Q\in\mathcal{Q}}$ having finite support (i.e. $a_Q \neq 0$ only for finitely many $Q \in \mathcal{Q}$) by

$$T_\psi a = \sum_{Q\in\mathcal{Q}} a_Q \psi_Q. \tag{3.3}$$

T_ψ is the so-called inverse φ-transform when φ, ψ are admissible and fulfil (2.5). We sometimes refer to T_ψ as the *synthesis* operator, and to S_φ as the analysis operator.

Furthermore, we need to make sense of $T_\psi a$ in a concise way for a variety of sequences a without finite support. In this case the summation in (3.3) has no a priori meaning, despite the countability of $\mathcal{Q}$. Indeed, the *counting measure* τ_{1+n} on $\mathcal{Q} \simeq \mathbb{Z}\times\mathbb{Z}^n$ does not suffice alone, since the sum in (3.3) should be a distribution.

Now each ψ_Q can be identified with an element of $\mathcal{Z}'$, for the quotient operator $\mathscr{Q}$ in (3.2) is injective on the subset of admissible functions, as e.g. $\hat{\psi}_Q(\xi) = 0$ in a neighbourhood of $\xi = 0$. Therefore our aim is to make sense of $T_\psi a$ in $\mathcal{Z}'$.

To sum the values of $Q \mapsto a_Q\psi_Q$ we take recourse to integration with respect to τ_{1+n} in a weak sense. More precisely, we shall use the notion of a Pettis integral (or weak Bochner integral) of a vector function $f\colon X \to F'$ with respect to a measure μ on a σ-algebra $\mathbb{E}$ in a set X; and F' being the dual of some Fréchet space F.

Namely, such f is said to be *Pettis integrable* (or weakly integrable) if the scalar function $x \mapsto \langle f, v\rangle$ is in the Lebesgue space $L^1(\mu)$ for every vector $v \in F$ and, moreover, the dual space F' contains some vector written $\int_X f\,d\mu$ such that

$$\Big\langle \int_X f\,d\mu, v\Big\rangle = \int_X \langle f, v\rangle\,d\mu \qquad \text{for all } v \in F. \tag{3.4}$$

In general it is not easy to give sufficient conditions for the existence of the Pettis integral (its uniqueness is obvious). But as we show below, it is manageable in case of $T_\psi a$.

We consider $f(Q) = a_Q\psi_Q$ defined on $X = \mathcal{Q}$, with $\mu = \tau_{1+n}$ and $F = \mathcal{Z}$. Then the basic criterion for Pettis integrability of f is that $\int_{\mathcal{Q}} |a_Q||\langle\psi_Q, \phi\rangle|\,d\tau_{1+n} < \infty$ for arbitrary $\phi \in \mathcal{Z}$, where $\langle\psi_Q, \phi\rangle$ stands for the action of $\psi_Q \in \mathcal{Z}'$ on ϕ. We denote by $L_1(\mathcal{Q}, \langle\psi_Q, \phi\rangle\,d\tau_{1+n})$ the space of such sequences, and find the condition $a \in L_1(\mathcal{Q}, \langle\psi_Q, \phi\rangle\,d\tau_{1+n})$ for all $\phi \in \mathcal{Z}$.

Theorem 3.2. *When ψ is admissible, then the operator $a \mapsto T_\psi a$ in* (3.3) *has an extension to a linear map*

$$\bigcap_{\phi\in\mathcal{Z}} L_1(\mathcal{Q}, \langle\psi_Q, \phi\rangle\,d\tau_{1+n}) \xrightarrow{T_\psi} \mathcal{Z}'(\mathbb{R}^n), \tag{3.5}$$

which on each such sequence $a = \{a_Q\}_Q$ is given as a distribution in $\mathcal{Z}'(\mathbb{R}^n)$ by the formula, where $\phi \in \mathcal{Z}(\mathbb{R}^n)$,

$$\langle T_\psi a, \phi\rangle = \int_{\mathcal{Q}} a_Q \langle\psi_Q, \phi\rangle\,d\tau_{1+n}. \tag{3.6}$$

Moreover, $T_\psi a$ equals the w*-*limit in $\mathcal{Z}'(\mathbb{R}^n)$ resulting from arbitrary approximation of $a = \{a_Q\}_Q$ by truncation to sequences having finite, increasing and exhausting supports. This property determines the extension T_ψ uniquely.*

Proof. Obviously $\int_{\mathcal{Q}} a_Q \langle \psi_Q, \phi\rangle \, d\tau_{1+n}$ is well defined for every sequence a belonging to the intersection in (3.5) and every $\phi \in \mathcal{Z}$. For any numbering $Q_1, Q_2, \dots$ of the dyadic cubes, dominated convergence gives for $N \to \infty$,

$$
\begin{aligned}
\Big\langle \sum_{j=1}^{N} a_{Q_j}\psi_{Q_j}, \phi \Big\rangle &= \sum_{j=1}^{N} a_{Q_j} \langle \psi_{Q_j}, \phi\rangle \\
&= \int_{\mathcal{Q}} \mathbb{1}_{\{Q_1,\dots,Q_N\}} a_Q \langle \psi_Q, \phi\rangle \, d\tau_{1+n} \to \int_{\mathcal{Q}} a_Q \langle \psi_Q, \phi\rangle \, d\tau_{1+n}.
\end{aligned} \tag{3.7}
$$

So according to the Banach–Steinhaus theorem for the Fréchet space $\mathcal{Z}$, the linear functionals $\varphi \mapsto \langle \sum_{j=1}^{N} a_{Q_j}\psi_{Q_j}, \overline{\varphi}\rangle$ are equicontinuous $\mathcal{Z} \to \mathbb{C}$. In terms of increasing seminorms $p_n(\phi)$ inducing the topology on $\mathcal{Z}$, this means that for some $\delta > 0$, $N \in \mathbb{N}$ the 0-neighbourhood $\{\, \phi \mid p_N(\phi) < \delta \,\}$ is mapped into the unit ball in $\mathbb{R}$; so the functional Λ defined as the above limit satisfies $|\Lambda(\phi)| \le \delta^{-1} p_N(\phi)$ for $\phi \in \mathcal{Z}$. Hence Λ is continuous, i.e. $\Lambda \in \mathcal{Z}'(\mathbb{R}^n)$, and

$$
\langle \Lambda, \phi\rangle = \lim_{N\to\infty} \langle \sum_{j=1}^{N} a_{Q_j}\psi_{Q_j}, \phi\rangle = \int_{\mathcal{Q}} a_Q \langle \psi_Q, \phi\rangle \, d\tau. \tag{3.8}
$$

As the right hand side is independent of the numbering, so is Λ, i.e. Λ depends only on a and ψ. Setting $T_\psi a = \Lambda$ we obtain (3.6).

Clearly $a \mapsto T_\psi a$ is a linear map, and if a has finite support a little algebra as in (3.7) shows that the finite sum $\sum_Q a_Q \psi_Q$ equals $T_\psi a$; so T_ψ extends the map (3.3).

Now let a sequence a, given in the intersection in (3.5), be approximated by sequences of finite support by truncation; this may be written in terms of characteristic functions of finite sets $\mathcal{Q}_m \subset \mathcal{Q}$ as

$$
a^{(m)} = \mathbb{1}_{\mathcal{Q}_m} a, \quad \text{for } m \in \mathbb{N}. \tag{3.9}
$$

If $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \dots$ and $\bigcup_m \mathcal{Q}_m = \mathbb{R}^n$, the supports of the $a^{(m)}$ are increasing and they exhaust $\operatorname{supp} a$. Then it follows analogously to (3.7) that $T_\psi a^{(m)} = \sum_{Q\in\mathcal{Q}_m} a_Q\psi_Q$ converges in $\mathcal{Z}'(\mathbb{R}^n)$ to the distribution $T_\psi a$; cf. (3.6).

Finally, whenever $\tilde{T}_\psi$ extends the map in (3.3) to an operator as in (3.5) having the property just obtained for T_ψ, for some approximation $a^{(m)}$ with the properties above, then $\tilde{T}_\psi a = \lim_m \tilde{T}_\psi a^{(m)} = \lim_m T_\psi a^{(m)} = T_\psi a$ in $\mathcal{Z}'$. □

Notice that $T_\psi a = \sum_Q a_Q \psi_Q$ whenever (a_Q) has finite support. In general the sum must be understood as the Pettis integral in (3.6), although the latter looks less intuitive. We proceed to show that it has a number of desired properties of T_ψ.

In practice the convergence questions related to application of T_ψ may often be handled via the classical estimate for $N > n$,

$$
\sum_{k\in\mathbb{Z}^n} (1+|k|)^{-N} \le \sum_{k_1,\dots,k_n\in\mathbb{Z}} \prod_{j=1}^{n} (1+|k_j|)^{-\frac{N}{n}} \le \Big(1 + 2\sum_{m=1}^{\infty} m^{-\frac{N}{n}}\Big)^n < \infty. \tag{3.10}
$$

This is useful e.g. for the basic result that T_ψ always is defined on a sequence of wavelet coefficients:

Proposition 3.3. *When a sequence $a = S_\varphi f$ for some $f \in \mathcal{S}'/\mathcal{P}$, then the general synthesis operator T_ψ in Theorem 3.2, with any admissible ψ, is defined on a and*

$$
\langle T_\psi(S_\varphi f), \phi\rangle = \sum_{\nu=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} (S_\varphi f)_{Q_{\nu k}} \langle \psi_{Q_{\nu k}}, \phi\rangle \qquad \text{for } \phi \in \mathcal{Z}, \tag{3.11}
$$

where the terms are τ_{1+n}-integrable on $\mathcal{Q}$. In short, $R(S_\varphi) \subset D(T_\psi)$.

Proof. To verify the L_1-condition in (3.5) we shall prove $\sum_Q |\langle f,\varphi_Q\rangle||\langle\psi_Q,\phi\rangle|$ finite for every $\phi\in\mathcal{Z}$. Since f is temperate we have for some $d>0$, if $Q=Q_{\mu k}$,

$$\begin{aligned}|\langle f,\varphi_Q\rangle| &\le c\sum_{|\alpha|\le d}\sup_x |(1+|x|)^d D^\alpha(2^{n\mu/2}\varphi(2^\mu x-k)|\\ &\le c\sum_{|\alpha|\le d}\sup_y |(1+2^{-\mu}|y+k|)^d 2^{\mu(|\alpha|+n/2)}D^\alpha\varphi(y)|\\ &\le c(\varphi,d)2^{d|\mu|+\mu n/2}(1+|k|)^d.\end{aligned} \tag{3.12}$$

To invoke Lemma 2.3, we observe that $\langle\psi_Q,\phi\rangle=\psi_Q*\tilde\phi(0)$, so the estimates there apply for $\nu=0$ and $x=0$. Thus $(1+2^\mu|x-x_Q|)^N=(1+|k|)^N$ for our cube Q. We may therefore apply (3.10) for $N=n+1$ if we take $N=d+n+1$ in Lemma 2.3.

Indeed, we can make a crude estimate thus: for $\mu\ge 0$ we may use (2.11) for any $M>0$ as ψ is admissible, or for $\mu<0$ note that ϕ as a member of $\mathcal{Z}$ fulfils (2.12) for every $M>0$, to get

$$|\langle\psi_Q,\phi\rangle|\le C2^{-|\mu|(M-d-1-n/2)}(1+|k|)^{-d-n-1}. \tag{3.13}$$

So using Tonelli's theorem to pick a summation order for $Q=Q_{\mu k}$, and by taking e.g. $M>2d+1+n$, we have $\sum_{\mu\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}|\langle f,\varphi_Q\rangle||\langle\psi_Q,\phi\rangle|<\infty$. By Fubini's theorem the integral in (3.6) therefore equals the iterated sum in (3.11). □

As a virtue of the proposition, it makes sense to study $T_\psi(S_\varphi f)$ and to derive from (2.5) the wavelet decomposition formula

$$T_\psi(S_\varphi f)=f \quad \text{for all } f\in\mathcal{S}'/\mathcal{P}. \tag{3.14}$$

In two steps we give a proof based on our explicit definition of T_ψ in Theorem 3.2.

The first step is to subject the f in (3.14) to Peetre's homogeneous Littlewood–Paley decomposition, which we have derived in a precise version in Appendix C: when (2.5) is satisfied by φ, ψ, then $\hat\phi:=\mathcal{F}\psi\overline{\mathcal{F}\varphi}=\mathcal{F}\psi\mathcal{F}\tilde\varphi$ obviously fulfils (C.1), so by Proposition C.5 there is, for each $f\in\mathcal{S}'$, specific polynomials $P_{m,N}$ of degree m fulfilling

$$f=\sum_{\nu=-N}^{\infty}\psi_\nu*\tilde\varphi_\nu*f+P_{m,N}+R_m \qquad \text{in} \quad \mathcal{S}'. \tag{3.15}$$

Here the remainder fulfils $\langle R_m,\omega\rangle=\mathcal{O}(2^{-N(n+m+1-d)})$ for every $\omega\in\mathcal{S}$, with $d\ge 0$ denoting the $\mathcal{S}'$-order of f; cf. (C.3) for the notion. For $d\le n$ this estimate yields exponentially fast convergence to f in $\mathcal{S}'$ for $N\to\infty$; the full statement in Proposition C.5 moreover shows that $P_{m,N}$'s individual terms $c_{\alpha,N}x^\alpha$ are $\mathcal{O}(2^{-N(n+|\alpha|-d)})$ in $\mathcal{S}'$-seminorm, so that in case $d<n$ even $P_{m,N}=\mathcal{O}(2^{-N})\to 0$ for $N\to\infty$.

The polynomials $P_{m,N}$ are asymptotically uniquely given (as the Taylor polynomials of a convolution $2^{-nN}\Phi(2^{-N}\cdot)*f$, cf. (C.4)), but since the degree m is at our disposal, we can for a general temperate order d of f arrange that $n+m+1>d$, so that at least the remainder R_m converges to 0 in $\mathcal{S}'$ for $N\to\infty$. However, the terms of $P_{m,N}$ with $|\alpha|\le d-n$ do not necessarily go to 0, so the convergence of $\sum_{\nu\ge -N}\psi_\nu*\tilde\varphi_\nu*f$ to f for $N\to\infty$ is only obtained in $\mathcal{Z}'=\mathcal{S}'/\mathcal{P}$.

To convert (3.15) into a summation over $\mathcal{Q}$, we need a convenient result asserting that certain convolution integrals over $\mathbb{R}^n$ can be replaced by convolution over a discrete subgroup. This fact is not surprising, and we state the result in Lemma 3.4 below providing both a pointwise limit and convergence in $\mathcal{S}'$ of the sum (3.16). The result of the lemma was used implicitly in [6, 10].

Lemma 3.4. *If $\phi\in\mathcal{S}$ and $g\in\mathcal{S}'$ satisfy that* $\operatorname{supp}\hat\phi$ *and* $\operatorname{supp}\hat g$ *are subsets of* $]-L,L[^n$ *for some* $L>0$*, then*

$$\phi*g(x)=\Big(\frac{\pi}{L}\Big)^n\sum_{k\in\mathbb{Z}^n}\phi\Big(x-\frac{\pi}{L}k\Big)\,g\Big(\frac{\pi}{L}k\Big). \tag{3.16}$$

The sum converges absolutely and unconditionally in the Fréchet space $C^\infty(\mathbb{R}^n)$ and in $\mathcal{S}'(\mathbb{R}^n)$.

The optimality of the constant π/L was amply elucidated by Meyer [25, Thm.1.1].

Proof. If we apply Poisson's summation formula to the Schwartz function $\phi(x-\cdot)g$, cf. [13, (7.2.1)'], we have for any $a>0$ (with pointwise convergence for $x\in\mathbb{R}^n$)

$$(\frac{2\pi}{a})^n \sum_{k\in\mathbb{Z}^n} \phi(x-\frac{2\pi}{a}k)g(\frac{2\pi}{a}k) = \sum_{k\in\mathbb{Z}^n} \mathcal{F}(\phi(x-\cdot)g)(ak). \tag{3.17}$$

Here the Fourier transformed product is supported in $]-2L,2L[^n$ by the support rule of convolutions. So for $a=2L$ the sum on the right-hand side is trivial for $k\neq 0$; i.e. the sum equals $\mathcal{F}(\phi(x-\cdot)g)(0)=\int_{\mathbb{R}^n}\phi(x-y)g(y)\,dy=\phi*g(x)$.

For the absolute convergence we shall prove $\sum_{k\in\mathbb{Z}^n}\sup_{x\in K}|D^\alpha\phi(x-\frac{\pi}{L}k)g(\frac{\pi}{L}k)|$ finite for arbitrary multiindices α and compact sets $K\subset\mathbb{R}^n$. Since g is a slowly increasing function we have for some $N>0$,

$$\Big|g(\frac{\pi}{L}k)\Big| \le c\frac{(1+|x-\frac{\pi}{L}k|)^{N+n+1}(1+|x|)^{N+n+1}}{(1+|\frac{\pi}{L}k|)^{n+1}}. \tag{3.18}$$

Here $(1+|x|)^{N+n+1}$ is bounded on K, and so is $(1+|x-\frac{\pi}{L}k|)^{N+n+1}$ times the Schwartz function $D^\alpha\phi(x-\frac{\pi}{L}k)$ on $K\times\mathbb{Z}^n$, so finiteness results at once from (3.10).

In view of this, any numbering of the $k\in\mathbb{Z}^n$ induces a Cauchy sequence in $C^\infty(\mathbb{R}^n)$, where the limit must equal $\phi*g$ by the first part of the proof. Thus it is independent of the numbering.

For $\mathcal{S}'$ the statement also boils down to (3.10), for when $\psi\in\mathcal{S}$ the finiteness of $\sum_{k\in\mathbb{Z}^n}|\langle\phi(x-\frac{\pi}{L}k)g(\frac{\pi}{L}k),\psi\rangle|$ follows from a uniform bound, which via (3.18) is reduced to a test of the fixed function $(1+|x|)^{N+n+1}$ against $2^p\psi(x)\phi(x-\frac{\pi}{L}k)(1+|x-\frac{\pi}{L}k|^2)^p$, $p=(N+n+1)/2$, which by virtue of ψ runs through a bounded set in $\mathcal{S}$ as k varies in $\mathbb{Z}^n$. Now the limit theorem for $\mathcal{S}'$ yields (unconditional) convergence to some $u\in\mathcal{S}'$, which in $\mathcal{D}'$ coincides with the limit in C^∞, so $u=\phi*g$. □

As the second step towards (3.14) we may apply Lemma 3.4 to $g=\tilde{\varphi}_\nu*f$ and $\phi=\psi_\nu$, taking $L=2^\nu\pi$ to get a clean formulation; cf. (2.7) and the constraint $K^0<\pi$ in (2.4). Then (3.15) gives

$$f=\sum_{\nu=-N}^{\infty}\big(2^{-n\nu}\sum_{k\in\mathbb{Z}^n}\psi_\nu(x-2^{-\nu}k)\tilde{\varphi}_\nu*f(2^{-\nu}k)\big)+P_{m,N}+R_m. \tag{3.19}$$

Here $2^{-n\nu}=(|Q|^{1/2})^2$ is a product of two normalisation factors, one of which yields the factor $\psi_Q(x)$ in the sum, cf. (2.6), and since we use sesqui-linear pairings,

$$2^{-n\nu/2}\psi_\nu(x-2^{-\nu}k)=\psi_Q(x) \tag{3.20}$$

$$2^{-n\nu/2}\tilde{\varphi}_\nu*f(2^{-\nu}k)=\langle f,|Q|^{1/2}\varphi_\nu(\cdot-2^{-\nu}k)\rangle=\langle f,\varphi_Q\rangle. \tag{3.21}$$

The latter expression equals $S_\varphi\mathscr{Q}f$, but we usually just write $S_\varphi f$, as any polynomial clearly can be added to f without changing its wavelet coefficients.

We now obtain the decomposition of each $f\in\mathcal{Z}'$ as a "linear combination" of the wavelets $\psi_Q(x)$. More precisely, it is a Pettis integral of the building blocks $\langle f,\varphi_Q\rangle\psi_Q(x)$, although for convenience we simply write it as a sum.

Proposition 3.5. *When φ,ψ are admissible and satisfy* (2.5)*, then one has for every $f\in\mathcal{S}'/\mathcal{P}$ that, with convergence in $\mathcal{Z}'(\mathbb{R}^n)$,*

$$f(x)=\sum_Q\langle f,\varphi_Q\rangle\psi_Q(x)=T_\psi\circ S_\varphi f(x). \tag{3.22}$$

More precisely, there is $\mathcal{S}'$-convergence to f of the iterated sum in (3.19) *with the identifications* (3.20)–(3.21)*; and error term $R_m=\mathcal{O}(2^{-N(n+m+1-d)})$, cf.* (3.15).

Proof. In (3.19) we take m so large that $n+m+1>d$, whence $R_m\to 0$ for $N\to\infty$ as discussed after (3.15). Then the outer sum in (3.19) converges in $\mathcal{S}'$ for $N\to\infty$, cf. (3.15) or the appendix. And the inner sum converges in $\mathcal{S}'$ to a temperate distribution, according to the last statement of

Lemma 3.4, so the quotient operator $\mathscr{Q}$ commutes with both summations for $f \in \mathcal{S}'$ because of its continuity in (3.2). So (3.21) gives

$$\mathscr{Q}f = \sum_{\nu=-\infty}^{\infty} \mathscr{Q}\big(\sum_{k\in\mathbb{Z}^n} \langle f, \varphi_Q\rangle \psi_Q\big) = \sum_{\nu=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} (S_\varphi f)_Q \mathscr{Q}\psi_Q \qquad \text{in } \mathcal{Z}'. \tag{3.23}$$

Thus we get the first formula in (3.22) for the elements of $\mathcal{S}'/\mathcal{P}$ (as $\mathscr{Q}\psi_Q$ identifies with ψ_Q), if we just write a sum with respect to Q. Applying the continuous functional $\langle \cdot, \phi\rangle$ on both sides of (3.23) for an arbitrary $\phi \in \mathcal{Z}$, Proposition 3.3 gives that $\langle \mathscr{Q}f, \phi\rangle = \langle T_\psi(S_\varphi f), \phi\rangle$; whence the second formula in (3.22). The final remarks on $\mathcal{S}'$-convergence and R_m was seen prior to the statement. □

As a corollary to the proof, note that when $\langle \cdot, g\rangle$ is applied to both sides of (3.23), since $\langle \psi_Q, g\rangle = \int \psi_Q \overline{g}\, dx = \overline{(S_\psi g)_Q}$, one obtains for all $f \in \mathcal{Z}'$, $g \in \mathcal{Z}$,

$$\langle f, g\rangle = \sum_{\nu=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} (S_\varphi f)_Q \overline{(S_\psi g)_Q}. \tag{3.24}$$

More intuitively, one could obtain this by noting that both sides of (3.24) equal the following, whenever $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \dots$ are finite subsets fulfilling $\mathcal{Q} = \bigcup_m \mathcal{Q}_m$,

$$\lim_{m\to\infty} \sum_{Q\in\mathcal{Q}_m} \langle (S_\varphi f)_Q \psi_Q, g\rangle. \tag{3.25}$$

This follows on the left of (3.24) from insertion of $f = T_\psi(S_\varphi f)$ and use of the w*-approximation property in Theorem 3.2; on the right from the τ_{1+n}-integrability in Proposition 3.3 and dominated convergence.

To simplify (3.24) one may invoke the scalar product $\langle s, t\rangle = \sum_{Q\in\mathcal{Q}} s_Q \overline{t_Q}$ defined for sequences $s = (s_Q)$ and $t = (t_Q)$ for which the product $(s_Q\overline{t_Q})_Q$ is in $\ell_1(\mathcal{Q}, \tau_{1+n})$: by the abovementioned integrability, (2.5) now more simply implies

$$\langle f, g\rangle = \langle S_\varphi f, S_\psi g\rangle \qquad \text{for all } f \in \mathcal{Z}',\ g \in \mathcal{Z}. \tag{3.26}$$

This justifies the "Parseval identity" alluded to by Frazier and Jawerth [11], and these formulas can now be further specialised as done in [10, 11].

It is also obvious that $P = S_\varphi \circ T_\psi$ is a projection, or more precisely an idempotent when (2.5) holds, as $T_\psi \circ S_\varphi = I$ then; cf. Proposition 3.5. In fact, $P = I$ holds precisely when the wavelets φ_Q, ψ_Q form a biorthogonal system, i.e.

$$\langle \psi_Q, \varphi_J\rangle = \int_{\mathbb{R}^n} \psi_Q(x)\overline{\varphi_J(x)}\, dx = \delta_{Q,J} \qquad \text{(Kronecker delta)}. \tag{3.27}$$

Although this is known in other contexts, it is perhaps instructive to note how nicely a formal proof fits with the definition of T_ψ by the Pettis integral:

Proposition 3.6. *If* (2.5) *holds, the identity* $S_\varphi \circ T_\psi = I$ *is equivalent to* (3.27)

Proof. By definition of S_φ and T_ψ, any a in $D(T_\psi)$ satisfies

$$(S_\varphi(T_\psi a))_J = \langle T_\psi a, \varphi_J\rangle = \int_{\mathcal{Q}} a_Q \langle \psi_Q, \varphi_J\rangle\, d\tau_{1+n} \qquad \text{for all } J \in \mathcal{Q}. \tag{3.28}$$

When $S_\varphi \circ T_\psi = I$ holds, then insertion of the sequence $a = (\delta_{Q,J_0})_{Q\in\mathcal{Q}}$ from $D(T_\psi)$, cf. (3.5), shows that $\delta_{J,J_0} = \int \delta_{Q,J_0}\langle \psi_Q, \varphi_J\rangle\, d\tau_{1+n} = \langle \psi_{J_0}, \varphi_J\rangle_{\mathcal{S}',\mathcal{S}}$. The converse is clear from (3.28). □

In the present set-up, biorthogonality is also equivalent to the property that the wavelets constitute a basis of $\mathcal{Z}'$; and this is always unconditional:

Theorem 3.7. *Let* φ, $\psi \in \mathcal{S}$ *be admissible and fulfil the reconstruction identity* (2.5). *If the family* $\psi_Q(x) = 2^{n\nu/2}\psi(2^\nu x - k)$ *through any numbering* $Q_1, Q_2, \dots$ *of the dyadic cubes form a basis of* $\mathcal{Z}'$*, then it satisfies the biorthogonality condition* (3.27). *Conversely,* (3.27) *implies that every such numbering gives an unconditional basis satisfying* $f = \sum_{j=1}^{\infty} (S_\varphi f)_{Q_j} \psi_{Q_j}$ *for all* $f \in \mathcal{Z}'$.

Let us first mention that *any* numbering $Q_1, Q_2, \dots$ of the dyadic cubes yields $f = \sum_{j=1}^{\infty} a_{Q_j}\psi_{Q_j}$ in $\mathcal{Z}'$ for $a = S_\varphi f$. This results from the w*-approximation property in Theorem 3.2 (cf. its proof for the notation) by taking $\mathcal{Q}_m = \{Q_1, \dots, Q_m\}$, which yields for arbitrary $g \in \mathcal{Z}$,

$$\langle f, g\rangle = \langle T_\psi a, g\rangle = \lim_{m\to\infty} \langle \sum_{j=1}^{m} a_{Q_j}\psi_{Q_j}, g\rangle = \langle \sum_{j=1}^{\infty} a_{Q_j}\psi_{Q_j}, g\rangle. \tag{3.29}$$

More precisely, the Banach–Steinhaus theorem gives the existence of a distribution $\sum_{j=1}^{\infty} a_{Q_j}\psi_{Q_j}$ in $\mathcal{Z}'$ satisfying the last identity (cf. the proof of Theorem 3.2), whence f is the sum of the series.

Proof. If a numbering $\psi_{Q_1}, \psi_{Q_2}, \dots$ gives a basis, there is to each $f \in \mathcal{Z}'$ unique scalars c_j such that $f = \sum_{j=1}^{\infty} c_j\psi_{Q_j}$. Now also $f = \sum_{j=1}^{\infty} a_{Q_j}\psi_{Q_j}$ holds in $\mathcal{Z}'$, cf. (3.29), so $c_j = a_{Q_j} = \langle f, \varphi_{Q_j}\rangle$. For $f = \psi_Q$ it is obvious that $c_j = \delta_{Q_j,Q}$ whilst $\langle f, \varphi_{Q_j}\rangle = \langle \psi_Q, \varphi_{Q_j}\rangle$, whence (3.27).

When the biorthogonality (3.27) is satisfied, then we observe that if $\sum_{j=1}^{m} b_{Q_j}\psi_{Q_j}$ converges to $f = T_\psi a$, the continuous functionals $\langle\cdot, \varphi_{Q_k}\rangle$ on $\mathcal{Z}'$ yield

$$b_{Q_k} = \langle T_\psi a, \varphi_{Q_k}\rangle = (S_\varphi(T_\psi a))_{Q_k} = a_{Q_k} \quad \text{for any } k, \tag{3.30}$$

since $S_\varphi \circ T_\psi = I$ holds by Proposition 3.6. Also $f \mapsto \langle f, \varphi_{Q_j}\rangle = a_{Q_j}$ is continuous, so (ψ_{Q_j}) is a Schauder basis for $\mathcal{Z}'$; even an unconditional basis, since any bijection $p\colon \mathbb{N} \to \mathbb{N}$ via $\mathcal{Q}_m = \{Q_{p(1)}, \dots, Q_{p(m)}\}$ in the w*-approximation property gives $\sum_{j=1}^{\infty} a_{Q_{p(j)}}\psi_{Q_{p(j)}} = T_\psi a = f$. □

Remark 3.8. *In view of Theorem 3.7 it would be interesting to know whether or not there exists wavelets in our framework that fulfil the biorthogonality condition* (3.27)*. We envisage that an explicit construction would require more than a single generator for $n > 1$, but to keep the presentation simple, we have left this aspect to the future.*

Remark 3.9. *It is noteworthy that simplification of $T_\psi a$ to the sum $\sum_{j=1}^{\infty} a_{Q_j}\psi_{Q_j}$ is obtained if and only if the biorthogonality* (3.27) *is fulfilled (as* (2.5) *always is assumed). So in general $T_\psi a$ must be understood as the Pettis integral in Theorem 3.2.*

Remark 3.10. *The explanation of Frazier and Jawerth [10,11] left quite a burden with the reader; e.g. no argument was given for the (heuristically obvious) identification of the sum over Q in* (3.22) *with $T_\psi \circ S_\varphi$. They did account for pointwise convergence in* (3.16)*, but the $\mathcal{S}'$-convergence in Proposition 3.4 was first stated by Bownik and Ho [6]. As the formula $T_\psi(S_\varphi f) = f$ only makes sense if $R(S_\varphi) \subset D(T_\psi)$, let us also point out that Proposition 3.3 seems to be a novelty—and so is the unified result in Theorem 3.7 that wavelets form unconditional bases in the topological vector space $\mathcal{Z}'$ if and only if they are biorthogonal.*

4. Mixed-norm Triebel–Lizorkin spaces

We are now ready to introduce the homogeneous Triebel–Lizorkin spaces based on the mixed norms in (2.1). The mixed-norm homogeneous Triebel–Lizorkin spaces generalize the classical homogeneous Triebel–Lizorkin spaces (cf. [30, 31] or [11]). In fact, the classical homogeneous Triebel–Lizorkin can easily be recovered from the special choice $\vec{p} = (p, \dots, p)$, for $0 < p < \infty$, in the following definition.

Definition 4.1. *For $s \in \mathbb{R}$, $\vec{p} = (p_1, \dots, p_n)$, with $0 < p_1, \dots, p_n < \infty$, $0 < q \le \infty$ and φ admissible, the homogeneous mixed-norm Triebel–Lizorkin space $\dot{F}^s_{\vec{p},q}$ is the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that*

$$\|f\|_{\dot{F}^s_{\vec{p},q}} := \Big\| \Big(\sum_{\nu\in\mathbb{Z}} (2^{\nu s}|\varphi_\nu * f|)^q \Big)^{1/q} \Big\|_{\vec{p}} < \infty, \tag{4.1}$$

with the l_q-norm replaced by $\sup_\nu$ for $q = \infty$.

At this point the reader should just consider $\dot{F}^s_{\vec{p},q}$ as a quasi-normed space. In the end of the section we shall obtain φ-independence of the above space and equivalent quasi-norms, subject to (2.4), together with its completeness.

For now we mention the basic embeddings, where $\hookrightarrow$ is understood to mean linear continuous injection.

Proposition 4.2. *For $s \in \mathbb{R}$, $p_1, \dots, p_n \in]0,\infty[$ and $q \in]0,\infty]$, while $u \in [q,\infty]$,*

$$\dot{F}^s_{\vec{p},q} \hookrightarrow \dot{F}^s_{\vec{p},u}, \tag{4.2}$$

$$\mathcal{Z} \hookrightarrow \dot{F}^s_{\vec{p},q} \hookrightarrow \mathcal{Z}'. \tag{4.3}$$

Proof. The first line is trivial since $\ell_q \hookrightarrow \ell_u$. Continuity of $\mathcal{Z} \hookrightarrow \dot{F}^s_{\vec{p},q}$ follows from Lemma 2.3 by taking $\mu = 0$ and $x_J = 0$ there: if $\psi \in \mathcal{Z}$ then $(1+|x|)^{-N}$ is in $L_{\vec{p}}$ for some $N > n$ (say $p_j N/n > 1$, all j); and the number of vanishing moments M in (2.11) can be taken so large that $-s+N-n-(M+1) < 0$, which applies for $\nu < 0$ in (4.1); whilst for $\nu \geq 0$ one can invoke (2.12) for M so large that $s-(M+1) < 0$, as φ is admissible. This way $\|\psi\|_{\dot{F}^s_{\vec{p},q}} \leq c(C'_{N,M} + C''_{N,M})$, where both $\psi \mapsto C'_{N,M}$ and $\psi \mapsto C''_{N,M}$ are continuous seminorms on $\mathcal{S}$; cf. the proof of Lemma 2.3.

Now $\dot{F}^s_{\vec{p},q} \hookrightarrow \mathcal{Z}'$ follows from Proposition C.5 by setting $\hat{\phi} = \overline{\hat{\psi}}\hat{\varphi}$ with ψ as in (2.5), for if $f \in \dot{F}^s_{\vec{p},q}$ and $g \in \mathcal{Z}$,

$$|\langle f, g\rangle| = |\langle \sum_{\nu\in\mathbb{Z}} \tilde{\psi}_\nu * \varphi_\nu * f, g\rangle| \leq \sum_{\nu\in\mathbb{Z}} \|\varphi_\nu * f\|_\infty \|\psi_\nu * g\|_1. \tag{4.4}$$

Here $\|\varphi_\nu * f\|_\infty \leq c 2^{\nu(\frac{1}{p_1}+\dots+\frac{1}{p_n})} \|\varphi_\nu * f\|_{\vec{p}}$, since $\varphi_\nu * f$ has spectral radius $K^0 2^\nu$; cf. [17, Prop. 4]. By adding more terms in the $L_{\vec{p}}$-norm,

$$|\langle f, g\rangle| \leq c\|f\|_{\dot{F}^s_{\vec{p},q}} \sum_{\nu\in\mathbb{Z}} 2^{\nu(\frac{1}{p_1}+\dots+\frac{1}{p_n}-s)} \|\psi_\nu * g\|_1. \tag{4.5}$$

The final sum is finite, in fact bounded from above by a Schwartz seminorm on g as one can see by adapting the above proof of $\mathcal{Z} \hookrightarrow \dot{F}^s_{\vec{p},q}$. Hence $|\langle f, g\rangle| \leq c\|f\|_{\dot{F}^s_{\vec{p},q}}$. ☐

Proposition 4.3. *There are Sobolev embeddings in the $\dot{F}^s_{\vec{p},q}$-scale, namely*

$$\|f\|_{\dot{F}^t_{\vec{r}q}} \leq c\|f\|_{\dot{F}^s_{\vec{p}\infty}} \quad \text{for} \quad t < s \tag{4.6}$$
$$\text{and} \quad t - \frac{1}{r_1} - \dots - \frac{1}{r_n} = s - \frac{1}{p_1} - \dots - \frac{1}{p_n}, \quad r_1 \geq p_1, \dots, r_n \geq p_n.$$

This inequality may be obtained from the arguments given for the inhomogeneous spaces in [17]. Indeed, this proof requires no essential changes if one only observes the interpolation inequality, for $-\infty < s_1 < s_2 < \infty$, $0 < \theta < 1$, $0 < q < \infty$,

$$\|2^{(s_0\theta + s_2(1-\theta))j} x_j\|_{\ell_q} \leq c\|2^{s_1 j} x_j\|^\theta_{\ell_\infty} \|2^{s_2 j} x_j\|^{1-\theta}_{\ell_\infty}. \tag{4.7}$$

This was proved for sequences (x_j) with $j \in \mathbb{N}$ by Brezis and Mironescu [7], but their argument based on monotonicity extends verbatim to sequences having $j \in \mathbb{Z}$.

Following Franke [9] we establish the Fatou property, namely that the centered balls of $\dot{F}^s_{\vec{p},q}$ are stable under sequential convergence in $\mathcal{Z}'$:

Lemma 4.4. *If $f^{(m)}$, $m \in \mathbb{N}$, satisfy that $f^{(m)} \to f$ in the w^*-sense in $\mathcal{Z}'$, then*

$$\|f\|_{\dot{F}^s_{\vec{p},q}} \leq \liminf_m \|f^{(m)}\|_{\dot{F}^s_{\vec{p},q}}. \tag{4.8}$$

Proof. Using φ_ν from the $\dot{F}^s_{\vec{p},q}$-norm we set $f^{(m)}_\nu(x) = \langle f^{(m)}, \overline{\varphi}_\nu(x-\cdot)\rangle$ and define $f_\nu = \varphi_\nu * f(x)$ analogously. Since φ is in $\mathcal{Z} \subset \mathcal{S}$, this can be read as a scalar product on $\mathcal{S}' \times \mathcal{S}$, so $f^{(m)}_\nu$ and f_ν are C^∞-functions by the Paley–Wiener–Schwartz Theorem. Clearly $f^{(m)}_\nu(x) \to f_\nu(x)$ pointwise for $m \to \infty$, so we obtain

$$\begin{aligned} \|f\|_{\dot{F}^s_{\vec{p},q}} = \|\liminf_m f^{(m)}\|_{\dot{F}^s_{\vec{p},q}} &\leq \|\liminf_m (\sum_\nu 2^{s\nu q} |f^{(m)}_\nu(x)|^q)^{1/q}\|_{\vec{p}} \\ &\leq \liminf_m \|(\sum_\nu 2^{s\nu q} |f^{(m)}_\nu(x)|^q)^{1/q}\|_{\vec{p}} \end{aligned} \tag{4.9}$$

by using Fatous's lemma for the counting measure on $\mathbb{Z}$ and n times for the Lebesgue measure (and that $(\liminf_m |x_m|)^t = \liminf_m |x_m|^t$ for $t > 0$). ☐

The discrete analogue of Triebel–Lizorkin spaces is the space of sequences which we introduce immediately.

Definition 4.5. *For $s \in \mathbb{R}$, $\vec{p} = (p_1, \dots, p_n)$, with $0 < p_1, \dots, p_n < \infty$, $0 < q \le \infty$, the sequence space $\dot{f}^s_{\vec{p},q}$ consists of all complex-valued $a = \{a_Q\}_{Q\in\mathcal{Q}}$ such that*

$$\|a\|_{\dot{f}^s_{\vec{p},q}} := \Big\| \Big(\sum_{Q\in\mathcal{Q}} \big(|Q|^{-s/n} |a_Q| \widetilde{\mathbb{1}}_Q\big)^q \Big)^{1/q} \Big\|_{\vec{p}} < \infty, \tag{4.10}$$

where $\widetilde{\mathbb{1}}_Q = |Q|^{-1/2}\mathbb{1}_Q$, with $\mathbb{1}_Q$ denoting the characteristic function of the cube Q.

The sum over $Q \in \mathcal{Q}$ should be understood as the unambiguous expression

$$\Big(\sum_{\nu\in\mathbb{Z}} \sum_{\ell(Q)=2^{-\nu}} \big(\frac{|a_Q|}{|Q|^{s/n}} \widetilde{\mathbb{1}}_Q \big)^q \Big)^{1/q} = \Big(\sum_{\nu\in\mathbb{Z}} \big(\sum_{\ell(Q)=2^{-\nu}} \frac{|a_Q|}{|Q|^{s/n}} \widetilde{\mathbb{1}}_Q \big)^q \Big)^{1/q}. \tag{4.11}$$

Indeed, the inner sum is just a convenient notation for a piecewise constant function on $\mathbb{R}^n$, equal to $|a_Q| 2^{\nu(s+n/2)}$ in each Q; the identity is due to the disjoint cubes. Accordingly, for $q = \infty$ the ℓ_q-norm is replaced by the supremum over ν only. For $q = 2$ the quantity (4.11) is known as the discrete Littlewood-Paley expression.

At the level of the sequence space, completeness is rather easily obtained:

Lemma 4.6. *The sequence space $\dot{f}^s_{\vec{p},q}$ is a quasi-Banach space, and for $q < \infty$ the sequences of finite support form a dense subspace.*

Proof. Given a Cauchy sequence $a^{(k)}$ of elements in $\dot{f}^s_{\vec{p},q}$ there is to $\varepsilon > 0$ some K such that for $k, m \ge K$,

$$\Big\| \Big(\sum_{Q\in\mathcal{Q}} \big(|Q|^{-s/n} |a^{(k)}_Q - a^{(m)}_Q| \widetilde{\mathbb{1}}_Q\big)^q \Big)^{1/q} \Big\|_{\vec{p}} < \varepsilon. \tag{4.12}$$

Keeping a single summand indexed by Q_0 yields $|a^{(k)}_{Q_0} - a^{(m)}_{Q_0}| \|\widetilde{\mathbb{1}}_{Q_0}\|_{\vec{p}} < \varepsilon$, so it is seen that $a^{(k)}_{Q_0}$ converges to some $a_{Q_0} \in \mathbb{C}$ for $k \to \infty$. By $(n+1)$-fold application of Fatou's lemma to the limit $m \to \infty$ in the above, one finds that $\|a^{(k)} - a\|_{\dot{f}^s_{\vec{p},q}} \le \varepsilon$ for $k \ge K$; whence $a \in \dot{f}^s_{\vec{p},q}$ and completeness follows.

When $q < \infty$ and $a \in \dot{f}^s_{\vec{p},q}$, then any chain of finite subsets $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \dots$ such that $\mathcal{Q} = \bigcup_N \mathcal{Q}_N$ gives sequences $a\mathbb{1}_{\mathcal{Q}_N}$ of finite support. That $a\mathbb{1}_{\mathcal{Q}_N} - a \to 0$ in $\dot{f}^s_{\vec{p},q}$ for $N \to \infty$ follows by repeated dominated convergence. □

For completeness of the function space $\dot{F}^s_{\vec{p},q}$ the reader is referred to Corollary 5.7, where this property is carried over from Lemma 4.6 as an easy consequence of the main theorem.

As a comment on the sequences in the space $\dot{f}^s_{\vec{p},q}$, note that by dropping all terms but one in the norm we get $|a_Q| |Q|^{-s/n-1/2} \|\mathbb{1}_Q\|_{\vec{p}} \le \|a\|_{\dot{f}^s_{\vec{p},q}} =: C_a$. Thus we obtain the crude estimate

$$|a_Q| \le C_a |Q|^{s/n+1/2} \ell(Q)^{-\frac{1}{p_1} - \dots - \frac{1}{p_n}} = C_a 2^{-\nu(s+n/2-(\frac{1}{p_1}+\dots+\frac{1}{p_n}))}. \tag{4.13}$$

For $\nu \to \infty$ this is only a decay estimate in case $s + n/2 > \frac{1}{p_1} + \dots + \frac{1}{p_n}$; whereas for $\nu \to -\infty$ it is only for $s + n/2 < \frac{1}{p_1} + \dots + \frac{1}{p_n}$ that the above gives decay.

5. Proof of the Main Result

In this section we derive the main theorem on the boundedness of S_φ and T_ψ together with the identity $T_\psi \circ S_\varphi = I$. Namely, we prove the following:

Theorem 5.1. *Let $s \in \mathbb{R}$ with $0 < p_1, \dots, p_n < \infty$ and $0 < q \le \infty$. For admissible functions φ, ψ the φ-transform $S_\varphi \colon \dot{F}^s_{\vec{p},q} \to \dot{f}^s_{\vec{p},q}$ and the inverse φ-transform $T_\psi \colon \dot{f}^s_{\vec{p},q} \to \dot{F}^s_{\vec{p},q}$ are bounded operators.*

Furthermore, when the reconstruction identity (2.5) is satisfied by φ and ψ, then

$$T_\psi \circ S_\varphi f = f \qquad \text{for every } f \in \dot{F}^s_{\vec{p},q}. \tag{5.1}$$

In particular $\|\cdot\|_{\dot F^s_{\vec p,q}} \approx \|S_\varphi(\cdot)\|_{\dot f^s_{\vec p,q}}$ *and* $S_\varphi(\dot F^s_{\vec p,q})$ *is complemented; cf.* (5.25).

To avoid an excess of concurrent estimates, and to crystallise results of independent interest, we will split the proof of Theorem 5.1 into a number of steps.

5.1. **The synthesis operator T_ψ.** First we show that the crude estimates (4.13) suffice for the following basic result which extends Proposition 3.3 on $D(T_\psi)$ to the full sequence space:

Proposition 5.2. *For any admissible* $\psi \in \mathcal{S}$ *the synthesis operator* T_ψ *from Theorem 3.2 is defined on* $\dot f^s_{\vec p,q}$ *for any* $s\in\mathbb{R}$, $p_1,\dots,p_n \in]0,\infty[$ *and* $0<q\le\infty$.

Proof. To verify the integrability condition in Theorem 3.2 for a given $a\in \dot f^s_{\vec p,q}$, we shall show that the series $\sum_Q |a_Q\langle\psi_Q,\phi\rangle|$ is finite for arbitrary $\phi\in\mathcal{Z}$. We invoke (4.13) to make a comparison with $S_+ + S_-$, whereby

$$S_\pm = \sum_{\mu\gtrless 0}\sum_{k\in\mathbb{Z}^n} |\langle\psi_Q,\phi\rangle| C_a 2^{-\mu(s+n/2-(\frac{1}{p_1}+\dots+\frac{1}{p_n}))}. \tag{5.2}$$

Here we use that $\langle\psi_Q,\phi\rangle = \psi_Q * \tilde\phi(0)$, where the normalisation by $|Q|^{-1/2}$ in ψ_Q is cancelled by the factor $2^{-n\mu/2} = |Q|^{1/2}$ above. Now Lemma 2.3 applies for $\nu=0$, and for $\mu\ge 0$ the first estimate (2.10) gives the comparison

$$S_+ \le \sum_{\mu\ge 0}\sum_{k\in\mathbb{Z}^n} C_a 2^{-\mu(s-(\frac{1}{p_1}+\dots+\frac{1}{p_n}))} C_N 2^{(N-n)\mu}(1+2^\mu|x_Q|)^{-N}. \tag{5.3}$$

Since $2^\mu x_Q = k$ in this sum, we use (3.10) for $N>n$. So in case $s>\frac{1}{p_1}+\dots+\frac{1}{p_n}$ we obtain for sufficiently small $N>n$ that $S_+<\infty$.

For $s\le \frac{1}{p_1}+\dots+\frac{1}{p_n}$ there is a reinforcement in terms of the estimate (2.11), which we may apply as ψ is admissible, hence fulfils the moment condition of any order $M\in\mathbb{N}_0$. Therefore one can replace $C_N 2^{(N-n)\mu}$ in (5.3) by $C'_N 2^{(N-n-M)\mu}$ for M so large that $s-\frac{1}{p_1}-\dots-\frac{1}{p_n}-(N-n)+M>0$, implying that S_+ is finite.

For S_- the argument leading to inequality (5.3) gives a slightly simpler estimate, since (2.10) now applies for $\mu<0=\nu$; namely

$$S_- \le \sum_{\mu=-\infty}^{-1} C_a 2^{-\mu(s-(\frac{1}{p_1}+\dots+\frac{1}{p_n}))} C_N \sum_{k\in\mathbb{Z}^n}(1+|k|)^{-N}. \tag{5.4}$$

Clearly the right-hand side is finite for $s-\frac{1}{p_1}-\dots-\frac{1}{p_n}<0$ and $N>n$; cf. (3.10).

In the complementing region with $s\ge\frac{1}{p_1}+\dots+\frac{1}{p_n}$ we note that ϕ as a member of $\mathcal{Z}$ also fulfils the moment condition of any order $M\in\mathbb{N}_0$, so (2.12) allows us to replace C_N in (5.4) by $C''_N 2^{M\mu}$ for some $N>n$ and $M\in\mathbb{N}$ so large that $s-\frac{1}{p_1}-\dots-\frac{1}{p_n}-M<0$. Whence $S_-<\infty$ also in this case.

Altogether this shows that every $a\in\dot f^s_{\vec p,q}$ belongs to $\bigcap_{\phi\in\mathcal{Z}} L_1(\mathcal{Q},\langle\psi_Q,\phi\rangle d\tau_{1+n})$, i.e. to the domain of T_ψ. □

To obtain a more refined estimate, we will need the following estimate, closely related to the Peetre-type maximal inequality (2.22); cf. Appendix B for a proof.

Lemma 5.3. *Let* $0<t\le 1$, $\tau>n/t$ *and* $\mu\in\mathbb{Z}$. *For any sequence* $(a_P)_{P\in\mathcal{Q}}$ *we have*

$$\sum_{\ell(P)=2^{-\mu}} \frac{|a_P|}{\Big(1+2^{\min(\mu,\nu)}|x_P-x_Q|\Big)^\tau} \le c2^{\frac{n}{t}(\mu-\nu)_+}\Big(M_n\cdots M_1\big(\sum_{\ell(P)=2^{-\mu}}|a_P|\mathbb{1}_P\big)^t\cdots\Big)^{\frac1t}(x), \tag{5.5}$$

whenever $x\in Q$ *with* $\ell(Q)=2^{-\nu}$ *for some* $\nu\in\mathbb{Z}$.

We note a few consequences of Lemma 5.3 for later use. They concern the transformation of a given sequence a into $a^* = \{a^*_Q\}_Q$ given by

$$a^*_Q = \sum_{\ell(P)=2^{-\nu}} |a_P|(1+2^\nu|x_P-x_Q|)^{-\tau}, \tag{5.6}$$

for every $Q \in \mathcal{Q}$ with $\ell(Q) = 2^{-\nu}$, $\nu \in \mathbb{Z}$, and some $\tau > n/t$, for $0 < t \le 1$. Here Lemma 5.3 applied for $\mu = \nu$ gives, for every $x \in Q$,

$$a_Q^* \le c\Big(M_n \cdots M_1\big(\sum_{\ell(P)=\ell(Q)} |a_P|\mathbb{1}_P\big)^t \cdots\Big)^{1/t}(x). \tag{5.7}$$

Since the set of all dyadic cubes with the same side-length is a disjoint partition of $\mathbb{R}^n$, it is clear from (5.7) that on $\mathbb{R}^n$,

$$\sum_{\ell(Q)=2^{-\nu}} a_Q^*\mathbb{1}_Q(x) \le c\Big(M_n \cdots M_1\big(\sum_{\ell(P)=2^{-\nu}} |a_P|\mathbb{1}_P\big)^t \cdots\Big)^{1/t}(x). \tag{5.8}$$

We can now prove the following refinement of Proposition 5.2:

Proposition 5.4. *When ψ is admissible, then the synthesis operator T_ψ from Theorem 3.2 is a bounded linear map $\dot{f}^s_{\vec{p},q} \to \dot{F}^s_{\vec{p},q}$.*

Proof. Let $a = \{a_Q\}$ be a sequence in $\dot{f}^s_{\vec{p},q}$ having finite support and $f := T_\psi a = \sum_Q a_Q\psi_Q$. By the finiteness and (2.7), we obtain with $h = \log_2(K^0/K_0)$

$$\varphi_\nu * f(x) = \sum_J a_J\varphi_\nu * \psi_J = \sum_{\nu-h\le\mu\le\nu+h}\ \sum_{\ell(J)=2^{-\mu}} a_J\varphi_\nu * \psi_J(x) \tag{5.9}$$

for any $x \in \mathbb{R}^n$, $\nu \in \mathbb{Z}$. Using the basic estimate (2.10) from Lemma 2.3, and the support relation (2.8), we see that for $\tau > 0$

$$|\varphi_\nu * \psi_J(x)| \le c|J|^{-1/2}(1 + 2^\mu|x - x_J|)^{-\tau}. \tag{5.10}$$

Now we fix τ so large that $\tau > n/t$ and $0 < t < \min(1, p_1, \dots, p_n, q)$. Hence

$$|Q|^{-s/n}|\varphi_\nu * f(x)| \le c \sum_{|\mu-\nu|\le h}\ \sum_{\ell(J)=2^{-\mu}} |a_J||J|^{-s/n-1/2}(1 + 2^\mu|x - x_J|)^{-\tau}. \tag{5.11}$$

For every $x \in \mathbb{R}^n$ and $\mu \in \mathbb{Z}$, there exists a unique $P_0 \in \mathcal{Q}$ such that $x \in P_0$ and $\ell(P_0) = 2^{-\mu}$. Then it holds that $1 + 2^\mu|x_{P_0} - x_J| \le 1 + 2^\mu(|x - x_{P_0}| + |x - x_J|) \le 1 + \sqrt{n} + 2^\mu|x - x_J| \le c_n(1 + 2^\mu|x - x_J|)$ and thus from (5.11)

$$|Q|^{-s/n}|\varphi_\nu * f(x)| \le c \sum_{|\mu-\nu|\le h}\ \sum_{\ell(J)=2^{-\mu}} |a_J||J|^{-s/n-1/2}(1 + 2^\mu|x_{P_0} - x_J|)^{-\tau}. \tag{5.12}$$

For simplicity we introduce $b = \{b_J\}$ with $b_J = a_J|J|^{-s/n-1/2}$ for every $J \in \mathcal{Q}$, and then using (5.12), (5.6) and (5.8) we derive

$$\begin{aligned} 2^{\nu s}|\varphi_\nu * f(x)| &\le c \sum_{|\mu-\nu|\le h} b^*_{P_0}\mathbb{1}_{P_0}(x) \le c \sum_{\mu=\nu-1}^{\nu+1}\ \sum_{\ell(P)=2^{-\mu}} b^*_P\mathbb{1}_P(x) \\ &\le c \sum_{|\mu-\nu|\le h} \Big(M_n \cdots M_1\big(\sum_{\ell(P)=2^{-\mu}} |b_P|\mathbb{1}_P\big)^t \cdots\Big)^{1/t}(x). \end{aligned} \tag{5.13}$$

Applying (5.13) in the $\dot{F}^s_{\vec{p},q}$-norm of f, the quasi-triangle inequality gives

$$\begin{aligned} \|f\|_{F^s_{\vec{p},q}} &= \Big\|\Big(\sum_{\nu\in\mathbb{Z}} \Big(2^{\nu s}|\varphi_\nu * f(\cdot)|\Big)^q\Big)^{\frac{1}{q}}\Big\|_{\vec{p}} \\ &\le c\Big\|\Big(\sum_{\nu\in\mathbb{Z}}\Big(\sum_{|\mu-\nu|\le h}\Big(M_n \cdots M_1\big(\sum_{\ell(P)=2^{-\mu}} |b_P|\mathbb{1}_P\big)^t \cdots\Big)^{\frac{1}{t}}\Big)^q\Big)^{\frac{1}{q}}\Big\|_{\vec{p}} \\ &\le c(1+2h)\Big\|\Big(\sum_{\mu\in\mathbb{Z}}\Big(\Big(M_n \cdots M_1\big(\sum_{\ell(P)=2^{-\mu}} |b_P|\mathbb{1}_P\big)^t \cdots\Big)^{\frac{1}{t}}\Big)^q\Big)^{\frac{1}{q}}\Big\|_{\vec{p}}. \end{aligned} \tag{5.14}$$

So by invoking the maximal inequality (2.21), the definition of b_P above, and that the sum over P contains a single term at each fixed x, respectively, we obtain

$$\begin{aligned}\|f\|_{F^s_{\vec p,q}} &\le c\Big\|\Big(\sum_{\mu\in\mathbb{Z}}\Big(\sum_{\ell(P)=2^{-\mu}}|b_P|\mathbb{1}_P\Big)^q\Big)^{1/q}\Big\|_{\vec p}\\ &= c\Big\|\Big(\sum_{\mu\in\mathbb{Z}}\Big(\sum_{\ell(P)=2^{-\mu}}|P|^{-s/n}|a_P|\widetilde{\mathbb{1}}_P\Big)^q\Big)^{1/q}\Big\|_{\vec p}\le c\|a\|_{\dot f^s_{\vec p,q}}.\end{aligned}\tag{5.15}$$

Thus $T_\psi\colon \dot f^s_{\vec p,q}\to \dot F^s_{\vec p,q}$ has been shown to be bounded on the subspace of sequences of finite support. For any given sequence a in $\dot f^s_{\vec p,q}$ we recall from Proposition 5.2 that $T_\psi a$ is defined. This is exploited by choosing approximations $a^{(m)}$ having finite, increasing and exhausting supports, so that by the last part of Theorem 3.2 we have in the w*-topology of $\mathcal{Z}'$ that

$$T_\psi a=\lim_{m\to\infty}T_\psi a^{(m)}.\tag{5.16}$$

Moreover, the boundedness and truncation give the uniform bound

$$\|T_\psi a^{(m)}\|_{\dot F^s_{\vec p,q}}\le c\|a^{(m)}\|_{\dot f^s_{\vec p,q}}\le c\|a\|_{\dot f^s_{\vec p,q}}.\tag{5.17}$$

In view of these facts, the Fatou property yields that also $\|T_\psi a\|_{\dot F^s_{\vec p,q}}\le c\|a\|_{\dot f^s_{\vec p,q}}$; cf. Lemma 4.4. Hence T_ψ is bounded on the full sequence space $\dot f^s_{\vec p,q}$. □

5.2. **The analysis operator S_φ.** For the analysis operator S_φ we adopt a 2-step procedure. For clarity we write $\dot F^s_{\vec p,q}(\varphi)$ to emphasize that the Triebel–Lizorkin space, or its norm, has been defined by means of the admissible function φ.

Proposition 5.5. *If φ is admissible, then S_φ has the property of boundedness*

$$\|S_\varphi f\|_{\dot f^s_{\vec p,q}}\le c\|f\|_{\dot F^s_{\vec p,q}(\tilde\varphi)}\qquad\text{for every } f\in\dot F^s_{\vec p,q}(\tilde\varphi).\tag{5.18}$$

Proof. Let f be arbitrary in $\dot F^s_{\vec p,q}(\tilde\varphi)$ for the given admissible function $\tilde\varphi$. For $Q\in\mathcal{Q}$ with $\ell(Q)=2^{-\nu}$, $\nu\in\mathbb{Z}$, we obtain as in (3.21) that

$$(S_\varphi f)_Q=\langle f,\varphi_Q\rangle=|Q|^{1/2}\tilde\varphi_\nu * f(x_Q).$$

Therefore we crudely get for any $t>0$, since $1+2^\nu|x_Q-x|\le 1+\sqrt{n}$ for $x\in Q$,

$$\begin{aligned}\sum_{\ell(Q)=2^{-\nu}}\frac{|(S_\varphi f)_Q|^q}{|Q|^{sq/n}}\widetilde{\mathbb{1}}_Q(x)&\le c\sum_{\ell(Q)=2^{-\nu}}\frac{2^{\nu qs}|\tilde\varphi_\nu * f(x_Q)|^q}{(1+2^\nu|x_Q-x|)^{nq/t}}\mathbb{1}_Q(x)\\ &\le c\sum_{\ell(Q)=2^{-\nu}}\Big(\sup_{y\in\mathbb{R}^n}\frac{2^{\nu s}|\tilde\varphi_\nu * f(y)|}{(1+2^\nu|y-x|)^{n/t}}\Big)^q\mathbb{1}_Q(x).\end{aligned}\tag{5.19}$$

Fixing $t<\min(p_1,\dots,p_n,q)$ we may apply the maximal inequality (2.22), as (2.7) entails that $\operatorname{supp}(\widehat{\tilde\varphi_\nu * f})=\operatorname{supp}(\widehat{\tilde\varphi_\nu}\hat f)\subset[-K^0 2^\nu,K^0 2^\nu]^n$, which is contained in $[-2^{v'},2^{v'}]^n$ for some $\nu'>\nu$. Thus we get, for $x\in\mathbb{R}^n$,

$$\sum_{\ell(Q)=2^{-\nu}}\frac{|(S_\varphi f)_Q|^q}{|Q|^{sq/n}}\widetilde{\mathbb{1}}_Q(x)\le c\Big(\Big(M_n\cdots M_1\big(2^{\nu s}|\tilde\varphi_\nu * f|\big)^t\cdots\Big)^{1/t}(x)\Big)^q\mathbb{1}_{\mathbb{R}^n}(x).\tag{5.20}$$

We pass to the discrete Triebel–Lizorkin norm of $S_\varphi f$ by calculating the norm of ℓ_q with respect to $\nu\in\mathbb{Z}$ and that of $L_{\vec p}(\mathbb{R}^n)$ on both sides of (5.20). So by using the maximal inequality (2.21) we obtain

$$\|S_\varphi f\|_{f^s_{\vec p,q}}\le c\Big\|\Big(\sum_{\nu\in\mathbb{Z}}\Big(M_n\cdots M_1\big(2^{\nu s}|\tilde\varphi_\nu * f|\big)^t(\cdot)\Big)^{q/t}\Big)^{1/q}\Big\|_{\vec p}\le c\|f\|_{\dot F^s_{\vec p,q}(\tilde\varphi)}.\tag{5.21}$$

This proves the stated inequality for S_φ. □

The above result could be improved, since it would be natural to replace $\tilde{\varphi}$ by φ in the inequality—or indeed to replace it by an arbitrary admissible function Φ, so that boundedness of S_φ would be decoupled from the choice of norm on $\dot{F}^s_{\vec{p},q}$.

The remedy lies in a classical argument from the φ-transform theory. But it is a main point that heuristic use of $T_\psi a$ as a "sum" should be replaced by rigorous reference to the definition by the Pettis integral, so we proceed with diligence:

Let φ, Φ be two arbitrary admissible functions. Then there are admissible functions ψ, Ψ such that each couple (φ,ψ), (Φ,Ψ) satisfies (2.5). This implies that

$$\|f\|_{\dot{F}^s_{\vec{p},q}(\varphi)} = \|\sum_Q (S_\Phi f)_Q \Psi_Q\|_{\dot{F}^s_{\vec{p},q}(\varphi)} \le c\|S_\Phi f\|_{\dot{f}^s_{\vec{p},q}} \le c\|f\|_{\dot{F}^s_{\vec{p},q}(\tilde{\Phi})}. \tag{5.22}$$

Indeed, we may substitute $f=\sum_Q (S_\Phi f)_Q\Psi_Q$ in the first norm, since we proved in all details that T_Ψ is a left-inverse of S_φ on $\mathcal{S}'/\mathcal{P}$; cf. Proposition 3.5. And the first inequality above holds, since T_Ψ is everywhere defined and bounded according to Proposition 5.4 (no connection between the two admissible functions φ, Ψ is required). Finally Proposition 5.5 suffices for the last inequality.

Substituting by the admissible functions $\tilde{\Phi}$ and $\tilde{\varphi}$, it is seen at once that also $\|f\|_{\dot{F}^s_{\vec{p},q}(\tilde{\Phi})} \le c\|f\|_{\dot{F}^s_{\vec{p},q}(\varphi)}$ holds. Consequently either both or none of the Triebel–Lizorkin norms are finite on any given $f\in\mathcal{S}'/\mathcal{P}$. Therefore $\dot{F}^s_{\vec{p},q}(\varphi)$ equals the space $\dot{F}^s_{\vec{p},q}(\tilde{\Phi})$; and their norms are equivalent in view of the just shown inequalities.

Hence Proposition 5.5 can be sharpended to boundedness of S_φ with respect to any norm on $\dot{F}^s_{\vec{p},q}$. Thus we have completed the 2-step procedure; the outcome may be stated as follows:

Proposition 5.6. *When φ, Φ are admissible for the same set of constants in* (2.2)–(2.4)*, then the induced Triebel–Lizorkin spaces coincide and the corresponding norms are equivalent, i.e.*

$$\|\cdot\|_{\dot{F}^s_{\vec{p},q}(\varphi)} \approx \|\cdot\|_{\dot{F}^s_{\vec{p},q}(\Phi)}. \tag{5.23}$$

Moreover, $S_\varphi\colon \dot{F}^s_{\vec{p},q}\to \dot{f}^s_{\vec{p},q}$ is a bounded operator.

5.3. **Proof of Theorem 5.1.** The boundedness of S_φ and T_ψ has been obtained in Proposition 5.6 and 5.4, respectively.

Combining the boundedness and the identity $T_\psi\circ S_\varphi = I$, cf. Proposition 3.5, one gets at once that $\|S_\varphi f\|_{\dot{f}^s_{\vec{p},q}}$ is equivalent to the norm on $\dot{F}^s_{\vec{p},q}$: that is, for certain constants $B\ge 1\ge A>0$ we have the classical inequalities

$$A\|f\|_{\dot{F}^s_{\vec{p},q}} \le \|S_\varphi f\|_{\dot{f}^s_{\vec{p},q}} \le B\|f\|_{\dot{F}^s_{\vec{p},q}}. \tag{5.24}$$

Secondly, $P := S_\varphi\circ T_\psi$ is a continuous idempotent, and as such projects onto its range $R(S_\varphi)$ along the nullspace of T_ψ; and the range is closed (cf. the proof of Corollary 5.7 below). More precisely, $S_\varphi(\dot{F}^s_{\vec{p},q})$ is a complemented subspace, i.e. with a direct sum of (quasi-)Banach spaces,

$$\dot{f}^s_{\vec{p},q} = S_\varphi(\dot{F}^s_{\vec{p},q}) \oplus \{\, a \mid T_\psi a = 0\,\}. \tag{5.25}$$

This concludes the proof of Theorem 5.1.

Now it is straightforward to derive additional properties, as expected.

Corollary 5.7. *$\dot{F}^s_{\vec{p},q}(\mathbb{R}^n)$ is complete and the range $S_\varphi(\dot{F}^s_{\vec{p},q})$ is closed in $\dot{f}^s_{\vec{p},q}$.*

Proof. Every Cauchy sequence f_k in $\dot{F}^s_{\vec{p},q}$ is sent into another Cauchy sequence $S_\varphi f_k$ by the bounded map S_φ; this has a limit $a=\{a_Q\}$ by the completeness of the sequence space shown in Lemma 4.6. Using the boundedness of T_ψ and that S_φ is a right-inverse, cf. Theorem 5.1, one obtains with limits in $\dot{F}^s_{\vec{p},q}$ that

$$T_\psi a = \lim_k T_\psi(S_\varphi f_k) = \lim_k f_k. \tag{5.26}$$

This shows completeness. That S_φ has closed range can be shown analogously, if one concludes by applying S_φ to the above equation. □

Corollary 5.8. *If φ, ψ are admissible and fulfil the reconstruction identity* (2.5) *and the biorthogonality condition* (3.27)*, then the wavelets ψ_Q give, through any numbering of the cubes $Q \in \mathcal{Q}$, an unconditional basis for every $\dot{F}^s_{\vec{p},q}$ having $q < \infty$.*

Proof. According to Theorem 3.7 we have $f = \sum_{j=1}^{\infty} a_{Q_j} \psi_{Q_j}$ for $a = S_\varphi f$ in $\mathcal{Z}'$, with unique scalars; by the continuous injection in Proposition 4.2, the $f \mapsto a_{Q_j}$ are also continuous on $\dot{F}^s_{\vec{p},q}$. To show convergence in the topology of $\dot{F}^s_{\vec{p},q}$, we introduce sequences of finite support $a^{(m)} = a\mathbb{1}_{\{Q_1,\dots,Q_m\}}$, that converge to a in $\dot{f}^s_{\vec{p},q}$ for $q < \infty$; cf. the proof of Lemma 4.6. Now, by the continuity of T_ψ in Theorem 5.1, $\sum_{j=1}^{m} a_{Q_j}\psi_{Q_j} = T_\psi a^{(m)}$ converges for $m \to \infty$ to $T_\psi a = T_\psi(S_\varphi f) = f$ in $\dot{F}^s_{\vec{p},q}$. Any other numbering gives the same result in view of Theorem 3.7. □

Remark 5.9. *With Corollary 5.8 we just want to indicate how useful the rigorous definition of T_ψ is in the discussion. Unconditional bases have also been emphasized by Triebel in a space-by-space approach in his works on wavelets, cf. [32, 3.1.3] or [33, Thm. 1.20], but without explicit proofs.*

Appendix A. Convolution estimates

The proof of Lemma 2.3 can be conducted as follows. In the convolution integral one can exploit the classical estimate

$$(1 + 2^\mu|x - x_J|)^N \le (1 + 2^\mu|x - x_J - y|)^N (1 + 2^\mu|y|)^N, \tag{A.1}$$

which yields at once that the left-hand side of (2.10) at most equals $\sup(1 + |\cdot|)^N|\psi|$ times the integral $\int (1 + 2^\mu|y|)^N |2^{n\nu}\varphi(2^\nu y)|\,dy$. So for $\mu \le \nu$ one may set $y = 2^{-\nu}w$ to get (2.10) with $C_N = \sup(1 + |\cdot|)^N|\psi| \cdot \int_{\mathbb{R}^n} |\varphi(w)|(1 + |w|)^N\,dw$.

For $\mu > \nu$ the convolution is written as $\int \psi(z)\varphi(2^\nu(x - x_J - 2^{-\mu}z))2^{n(\nu-\mu)}\,dz$. So by taking $y = 2^{-\mu}z$ in the above inequality, where $1 \le 2^{\mu-\nu}$, one finds (2.10) with the extra factor $2^{(N-n)(\mu-\nu)}$ for $C_N = \sup(1 + |\cdot|)^N|\varphi| \cdot \int_{\mathbb{R}^n} |\psi(w)|(1 + |w|)^N\,dw$.

Elaborating on this, the moment condition yields that $\psi(z)$ vanishes by integration against the Taylor polynomial of order M of φ, whence

$$\begin{aligned} \psi(2^\mu(\cdot - x_J)) * \varphi_\nu(x) &= \sum_{|\alpha|=M+1} \int 2^{n(\nu-\mu)}\psi(z)(-2^{\nu-\mu}z)^\alpha \varphi^{(\alpha)}(z)\,dz \\ \text{for} \quad \varphi^{(\alpha)}(z) &= \frac{M+1}{\alpha!}\int_0^1 (1-\theta)^M \partial^\alpha\varphi(2^\nu(x - x_J) - \theta 2^{\nu-\mu}z)\,d\theta. \end{aligned} \tag{A.2}$$

Now $y = \theta 2^{-\mu}z$ in the above inequality shows that one can take $C'_{N,M} = C'_{N,M}(\varphi,\psi)$ to be $\sum_{|\alpha|=M+1} \sup(1 + |\cdot|)^N|\partial^\alpha\varphi|/\alpha!$ times $\int |\psi|(1 + |\cdot|)^{N+M+1}\,dz$ for $\mu > \nu$. (Note that (2.11) is identical to (2.10) for $\mu \le \nu$.)

Finally, it is analogous to derive (2.12) by letting ψ and φ change roles, beginning with the convolution in the form $\int \psi(2^\mu(x - x_J - 2^{-\nu}z))\varphi(z)\,dz$. This gives $C''_{N,M} = C'_{N,M}(\psi,\varphi)$. The proof is complete.

Remark A.1. *The proof is valid* verbatim *for $\mu, \nu \in \mathbb{R}$, i.e. for dilation by $s = 2^\nu > 0$, $t = 2^\mu > 0$.*

Appendix B. Proof of Lemma 5.3

It suffices to prove the statement in Lemma 5.3 for any $\tau > n$ and $t = 1$, for one can just replace $|a_P|$ by $|a_P|^t$ and raise to the power $1/t$: on the left-hand side the fact that $\|\cdot\|_{\ell_1} \le \|\cdot\|_{\ell_t}$ for $0 < t \le 1$ gives the rest as $\tau/t > n/t$.

It also suffices to cover the case $\ell(Q) \ge \ell(P)$, i.e. $2^{-\nu} \ge 2^{-\mu}$ or $\nu \le \mu$. In fact, given $x \in Q$ for $\ell(Q) < \ell(P)$, then x also belongs to a cube $J \in \mathcal{Q}$ with $x_J = x_Q$ and $\ell(J) = \ell(P)$, for which one then arrives at the inequality stated for Q.

We split the set of P's as $\bigcup_{k\in\mathbb{N}_0} \Omega_k$, whereby

$$\Omega_0 = \{\, P \in \mathcal{Q} \mid \ell(P) = 2^{-\mu} \text{ and } |x_P - x_Q| \le 2^{-\nu} \,\}, \tag{B.1}$$

$$\Omega_k = \{\, P \in \mathcal{Q} \mid \ell(P) = 2^{-\mu} \text{ and } 2^{k-1-\nu} < |x_P - x_Q| \le 2^{k-\nu} \,\}, \quad k \ge 1. \tag{B.2}$$

When $P \in \Omega_k$ we have $1 + 2^\nu |x_P - x_Q| > 2^{k-1}$, so

$$\sum_{\ell(P)=2^{-\mu}} |a_P| \Big(1 + \frac{|x_P - x_Q|}{\ell(Q)}\Big)^{-\tau} \leq 2^\tau \sum_{k=0}^{\infty} 2^{-k\tau} \sum_{P\in\Omega_k} |a_P|. \tag{B.3}$$

Because the P in Ω_k are disjoint, and since $|P| = 2^{-\mu n}$,

$$\sum_{P\in\Omega_k} |a_P| = \int_R \Big(\sum_{P\in\Omega_k} |a_P| 2^{n\mu} \mathbb{1}_P(y)\Big)\, dy. \tag{B.4}$$

Indeed, $\bigcup_{\Omega_k} P$ is contained in $R = x_Q + [-2^{k-\nu+1}, 2^{k-\nu+1}]^n$, for if $(y_1, \dots, y_n) \in P$

$$|y_j - (x_Q)_j| \leq |y_j - (x_P)_j| + |(x_P)_j - (x_Q)_j| \leq 2^{-\nu} + 2^{k-\nu} \leq 2^{k-\nu+1}. \tag{B.5}$$

Since the side length of R is $2^{k-\nu+2}$, every $x \in Q$ is in R, so by (2.20)

$$\sum_{P\in\Omega_k} |a_P| \leq 4^n 2^{(k-\nu+\mu)n} (M_n \cdots M_1 \big(\sum_{\ell(P)=2^{-\mu}} |a_P| \mathbb{1}_P\big) \dots)(x). \tag{B.6}$$

Inserting (B.6) in (B.3), it is straightforward to sum over k and obtain the inequality stated in Lemma 5.3, since we assumed that $\tau > n$.

Remark B.1. *As a corollary to the above proof, one finds the extension of Peetre's maximal inequality to the functions in Proposition 2.6 (by the Paley–Wiener–Schwartz Theorem such functions are constant if they have compact spectrum as in* (2.22)*). Indeed, in* (2.22) *each y is in some P, and given $x \in Q$, the triangle inequality yields $1 + 2^\nu |x_P - x_Q| \leq (1 + 2\sqrt{n})(1 + 2^\nu |x - y|)$, hence an estimate from above by means of $\sum_{\ell(P)=2^{-\mu}} |a_P| \big(1 + 2^\nu |x_P - x_Q|\big)^{-\tau}$ for $\tau > n/t$. Taking $\mu = \nu$ in Lemma 5.3 there is a further estimate in terms of $c(M_n \dots M_1 |f|^t)^{1/t}(x)$, as claimed.*

Appendix C. Homogeneous Littlewood–Paley decompositions

It is known that when $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\hat{\phi} \not\ni 0$, so that $\operatorname{supp}\hat{\phi}$ is contained in an annulus $0 < C_0 \leq |\xi| \leq C^0$, and ϕ fulfils

$$\sum_{\nu=-\infty}^{\infty} \hat{\phi}_\nu(\xi) = 1 \qquad \text{for } \xi \neq 0, \tag{C.1}$$

then there is a kind of Littlewood–Paley decomposition of every $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$f = \sum_{\nu=-\infty}^{\infty} \phi_\nu * f \qquad \text{in} \quad \mathcal{S}'/\mathcal{P}. \tag{C.2}$$

The fact that the limit $\nu \to -\infty$ in general gives difficulties was first pointed out by Peetre [26, p. 52–54], who also gave the above remedy that all terms on the two sides must be understood modulo polynomials (cf. Remark C.8).

However, the statement can be made rather more precise, and we also attempt to explain the situation from a natural point of view. To achieve this, we say for the sake of precision that $f \in \mathcal{S}'$ has *temperate* order d, or $\mathcal{S}'$-order d, when d is the smallest integer such that f is estimated in terms of the seminorm p_d:

$$|\langle f, \psi\rangle| \leq c p_d(\psi), \qquad \text{for} \quad p_d(\psi) = \sup_{|\alpha|\leq d} \|(1 + |x|)^d D^\alpha \psi\|_\infty, \quad \psi \in \mathcal{S}. \tag{C.3}$$

The reader may recall that $\Lambda = e^x \cos e^x$ has order 0 in $\mathcal{D}'(\mathbb{R})$; but Λ is temperate with $\langle \Lambda, \psi\rangle = \int_{\mathbb{R}} -\psi'(x) \sin e^x \, dx$ for $\psi \in \mathcal{S}(\mathbb{R})$, so Λ has $\mathcal{S}'$-order $d \geq 1$.

For the convergence question in (C.2) we may conveniently depart from a convolution

$$f - \sum_{\nu=-N}^{\infty} \phi_\nu * f = \Phi_{-N} * f = 2^{-Nn} \Phi(2^{-N} \cdot) * f, \tag{C.4}$$

$$\text{for} \qquad \hat{\Phi}_{-N}(\xi) = 1 - \sum_{\nu=-N}^{\infty} \hat{\phi}(2^{-\nu}\xi), \tag{C.5}$$

whereby $\hat{\Phi}_{-N}$ is C^∞ and supported in the ball $|\xi| \le C^0/2^{N+1}$. Since $\hat{\Phi}_{-N}(\xi) = \hat{\Phi}(2^N\xi)$ holds by inspection if $\Phi := \Phi_0$, this is consistent with our notation for dilations.

As a simple example, for $f \in L_1(\mathbb{R}^n)$ the right-hand side of (C.4) tends to 0. In fact, it is $\mathcal{O}(2^{-nN})$ as the mere convolution $\Phi(2^{-N}\cdot) * f$ converges to $\Phi(0)\int f\,dx$. So addition of polynomials in (C.2) is unnecessary for such f.

Before analysing (C.4) for general $f \in \mathcal{S}'$, we first recall that for any $f \in \mathcal{S}'$, $\Phi \in \mathcal{S}$ the convolution $t^n\Phi(t\cdot) * f$ converges for $t \to \infty$ to cf where $c = \int \Phi\,dx = \hat{\Phi}(0)$. Moreover, for $c = 0$ the number of vanishing moments of Φ determines the leading terms and rate of convergence to 0 for $t \to \infty$, since by Taylor expansion of $\hat{\Phi}$

$$t^n\Phi(t\cdot) * f = \hat{\Phi}(0)f + \frac{1}{t}\sum_{|\alpha|=1}\partial^\alpha\hat{\Phi}(0)D^\alpha f + \dots + \frac{1}{t^M}\sum_{|\alpha|=M}\frac{\partial^\alpha\hat{\Phi}(0)}{\alpha!}D^\alpha f + R_M. \tag{C.6}$$

In the "wrong" limit $t \to 0^+$ the situation is radically different, as convergence cannot be expected (cf. $d > n$ below). Nevertheless there is an optimal asymptotics formula obtained from the Taylor polynomial $P_m(x)$ of the C^∞-function $t^n\Phi(t\cdot) * f$ itself:

$$t^n\Phi(t\cdot) * f = P_m + R_m = \sum_{|\alpha|\le m}\partial^\alpha(t^n\Phi(t\cdot) * f)(0)\frac{x^\alpha}{\alpha!} + R_m. \tag{C.7}$$

This asymptotics is elementary in nature, and it may well be folklore. But in lack of a reference we give a proof of the formula and the optimality. It is convenient first to observe the following decomposition of Schwartz functions.

Lemma C.1. *When $\psi \in \mathcal{S}$ has a trivial Taylor polynomial of degree $M \ge 0$ at $x = 0$, then there are other functions $\Psi_\gamma \in \mathcal{S}$ such that $\psi(x) = \sum_{|\gamma|=M+1} x^\gamma\Psi_\gamma(x)$.*

Proof. For $\psi \in C_0^\infty(\mathbb{R}^n)$ the claim follows at once from Taylor's formula by multiplying both sides by a cut-off function χ equal to 1 around $\operatorname{supp}\psi$. One can reduce to this case by means of a partition of unity, for when $0 \notin \operatorname{supp}\psi$ the multinomial formula shows that one can take $\Psi_\gamma(x) = (M+1)!x^\gamma\psi(x)|x|^{-2(M+1)}/\gamma! \in \mathcal{S}$. □

Optimality of (C.7) is obtained even among polynomials Q with t-dependent degrees:

Proposition C.2. *If $f \in \mathcal{S}'$ is of $\mathcal{S}'$-order $d \ge 0$ and $\Phi \in \mathcal{S}$, the Taylor polynomial $P_m(x)$ of degree $m \ge -1$ satisfies the asymptotics formula* (C.7) *with terms that are $\mathcal{O}(t^{n+|\alpha|-d})$ in $\mathcal{S}'$-seminorm for $t \to 0^+$, whilst the remainder term similarly is*

$$R_m = \mathcal{O}(t^{n+m+1-d}). \tag{C.8}$$

Any polynomial $Q(x) = \sum_{|\alpha|\le m(t)} c_\alpha(t)x^\alpha$ fulfilling (C.7) *for a remainder $R = o(1)$ in $\mathcal{S}'$-seminorm is given by $c_\alpha(t) = \frac{1}{\alpha!}\partial^\alpha(t^n\Phi(t\cdot) * f)(0) + o(1)$, where $c_\alpha(t) = o(1)$ for $|\alpha| > d - n$.*

Remark C.3. *The Taylor polynomial P_m is well defined even for $\Phi \in \mathcal{S}$, due to the well-known fact that the convolution $f \mapsto \langle f, \partial^\alpha\Phi(t(x-\cdot))\rangle$ is continuous $\mathcal{S}' \to C^\infty$.*

Proof. Here $\langle\cdot,\cdot\rangle$ denotes the bilinear form; we take $0 < t < 1$ and set $\Phi_t = t^n\Phi(t\cdot)$.

If $m = -1$, i.e. $P_m \equiv 0$, the statement is just $\Phi_t * f = \mathcal{O}(t^{n-d})$, but clearly $|\langle f, \overline{\tilde{\Phi}_t * \psi}\rangle|$ is less than $t^{n-d}c\|(1+|t\cdot|)^d\tilde{\Phi}(t\cdot)\|_\infty\sum_{|\beta|\le d}\int(1+|\cdot|)^d|D^\beta\psi|\,dx$.

For general $m \ge 0$ we estimate $|\langle R_m, \bar{\psi}\rangle|$ by moving the Taylor expansion to the test function $\psi \in \mathcal{S}$, as the formulas $I = (2\pi)^{-n}\bar{\mathcal{F}}\mathcal{F}$ and $\mathcal{F}1 = (2\pi)^n\delta_0$ give

$$\begin{aligned}(2\pi)^n\langle R_m, \bar{\psi}\rangle &= \langle\hat{\Phi}(\cdot/t)\hat{f}, \overline{\hat{\psi}}\rangle - \sum_{|\alpha|\le m}\frac{1}{\alpha!}\langle\delta_0, \partial^\alpha\Phi_t * f\rangle_{\mathcal{E}'\times C^\infty}\langle\mathcal{F}1, \overline{(\mathrm{i}\,\partial)^\alpha\hat{\psi}}\rangle\\ &= \langle\hat{f}, \hat{\Phi}(\cdot/t)\overline{\hat{\psi}}\rangle - \sum_{|\alpha|\le m}\langle\hat{f}, \hat{\Phi}(\cdot/t)(\mathrm{i}\,\xi)^\alpha/\alpha!\rangle\overline{(-\mathrm{i}\,\partial)^\alpha\hat{\psi}(0)}\\ &= \langle\hat{f}, \hat{\Phi}(\cdot/t)\Big(\hat{\psi} - \sum_{|\alpha|\le m}\partial^\alpha\hat{\psi}(0)\xi^\alpha/\alpha!\Big)^-\rangle.\end{aligned} \tag{C.9}$$

Indeed, the second line is seen at once for a Schwartz function f, and it extends to all $f \in \mathcal{S}'$ by density and the continuity in Remark C.3.

In (C.9) it follows from Lemma C.1 that the last difference for certain $\hat{\psi}_\alpha$ in $\mathcal{S}$ has the form $\sum_{|\alpha|=m+1} \xi^\alpha \hat{\psi}_\alpha(\xi)$. As the $\mathcal{S}'$-order of f is d, we get from (C.3)

$$|\langle R_m, \psi\rangle| \le c \sum_{|\alpha|=m+1} \sum_{|\beta|\le d} \|(1+|\cdot|)^d \tilde{\Phi}_t * D^{\alpha+\beta}\psi_\alpha\|_\infty. \tag{C.10}$$

Because of the ξ^α, each $D^{\alpha+\beta}\psi_\alpha$ has vanishing moments at least up to order m, so if we use the uniform estimates in Lemma 2.3 for $t = 2^\mu$, cf. Remark A.1,

$$\begin{aligned} |(1+|x|)^d t^n \tilde{\Phi}(t\cdot) * D^{\alpha+\beta}\psi_\alpha(x)| &\le t^{n-d} |(1+|tx|)^d \tilde{\Phi}(t\cdot) * D^{\alpha+\beta}\psi_\alpha(x)| \\ &\le C''_{d,m} t^{n-d+(m+1)}. \end{aligned} \tag{C.11}$$

By summing over $|\alpha| \le m$ it follows that $R_m = \mathcal{O}(t^{n+m+1-d})$ in seminorm.

Concerning $\langle \partial^\alpha \Phi_t * f(0) x^\alpha, \overline{\psi}\rangle$ for $|\alpha| \le m$ we may follow this term through (C.9) yielding simply $\partial^\alpha \hat{\psi}(0)/\alpha!$ times $\langle f, \overline{t^n D^\alpha(\tilde{\Phi}(t\cdot))}\rangle = t^{n+|\alpha|} \langle f, \overline{D^\alpha \tilde{\Phi}(t\cdot)}\rangle$. Using (C.3) directly and handling t as in (C.11), all such terms are seen to be $\mathcal{O}(t^{n+|\alpha|-d})$.

If $\Phi_t * f = Q + o(1)$ for a t-dependent polynomial $Q = \sum_{|\alpha|\le m(t)} c_\alpha(t) x^\alpha$, we fix α and take $M \ge |\alpha|$ such that $n+M \ge d$, whence the above estimate of R_M gives $P_M - Q = O(t) - o(1) = o(1)$ in seminorm for $t \to 0^+$. With $\chi \in C_0^\infty$ equal to 1 around $\xi = 0$, we deduce from $\langle \partial^\beta \delta_0, \xi^\alpha \chi\rangle = \alpha! \delta_{\alpha,\beta}$ that

$$o(1) = \langle \hat{P}_M - \hat{Q}, \xi^\alpha \chi\rangle = (2\pi)^n (-\mathrm{i})^{|\alpha|} (\partial^\alpha \Phi_t * f(0) - c_\alpha). \tag{C.12}$$

Thus the coefficient $c_\alpha(t) = \frac{1}{\alpha!}\partial^\alpha \Phi_t * f(0) + o(1)$, i.e. it must behave asymptotically for $t \to 0^+$ as that of an indiviual term of the Taylor polynomial P_M at 0 of $\Phi_t * f$, hence by the previous part of the proof be $o(1)$ if $n + |\alpha| - d > 0$. □

The asymptotic uniqueness of $Q(x)$ in Proposition C.2 yields

$$\lim_{t\to 0} \partial^\alpha t^n \Phi(t\cdot) * f(0) \ne 0 \implies \lim_{t\to 0} c_\alpha(t) \ne 0, \tag{C.13}$$

so even by accepting error terms as vague as $R = o(1)$ there is for general f no hope to have an *approximating* polynomial Q of degree $m < d - n$.

It should be observed that the estimate $R_m = \mathcal{O}(t^{n+m+1-d})$ shows that in (C.7) the approximation by $P_m(x)$ gets increasingly better for $t \to 0^+$ if the degree m is fixed so large that $n + m \ge d$. For $m = -1$ the convolution itself is $\mathcal{O}(t^{n-d})$, and goes to 0 in $\mathcal{S}'$ if $d < n$, in particular it is $\mathcal{O}(t^n)$ for $d = 0$—which for $f \in L_1$ is immediate (cf. (C.4)). For $d < n$ even all terms in $P_m(x)$ go to 0 in $\mathcal{S}'$ for $t \to 0^+$.

Furthermore, the convergence rate improves with many vanishing moments:

Corollary C.4. *When the seminorm $|\langle\cdot, \psi\rangle|$ in Proposition C.2 is given by a $\psi \in \mathcal{S}$ having vanishing moments of order $M \ge 0$, then*

$$\langle R_m, \psi\rangle = \mathcal{O}(t^{n+\max(M,m)+1-d}) \tag{C.14}$$

and each term in P_m has $\langle \partial^\alpha t^n \Phi(t\cdot) * f, \psi\rangle = 0$ for $|\alpha| \le M$ and else is $\mathcal{O}(t^{n+|\alpha|-d})$.

Proof. If $m < M$ the last difference in (C.9) just equals $\hat{\psi}$, which by Lemma C.1 is a sum of terms $\xi^\gamma \hat{\psi}_\gamma(\xi)$ for $|\gamma| = M+1$. Hence the α in (C.10)–(C.11) should be of length $M+1$, which as before gives $\mathcal{O}(t^{n+M+1-d})$. The treatment of the individual terms is unchanged, but they clearly vanish for $|\alpha| \le M$ as $\partial^\alpha \hat{\psi}(0) = 0$. □

Now, if we return to (C.4), and take advantage of the fact that Φ_{-N} is a dilation as observed there, and if we set

$$P_{m,N}(x) = \sum_{|\alpha|\le m} c_{\alpha,N} x^\alpha \qquad \text{for } c_{\alpha,N} = \frac{1}{\alpha!}\partial^\alpha 2^{-nN}\Phi(2^{-N}\cdot) * f(0), \tag{C.15}$$

then Proposition C.2 gives at once that for each $N \in \mathbb{N}$,

$$f = \sum_{\nu=-N}^{\infty} \phi_\nu * f + P_{m,N} + \mathcal{O}(2^{-N(n+m+1-d)}). \tag{C.16}$$

For $N \to \infty$ we get for $m = d$ the well-known result of Peetre in (C.2), although the resulting error term $\mathcal{O}(2^{-N(n+1)})$ here yields an exponentially fast convergence of the sum $\sum_{\nu=-N}^{\infty} \phi_\nu * f$ for $N \to \infty$. This seems to be a novelty in the context.

For $m \neq d$ we furthermore obtain from formula (C.16) a general, but sharp version of the homogeneous Littlewood–Paley decomposition:

Proposition C.5. *If ϕ is admissible and fulfils* (C.1) *and $f \in \mathcal{S}'$ is of temperate order d, the polynomials $P_{m,N}$ in* (C.15) *satisfy* (C.16)*. Moreover any polynomial $P_{m,N}(x) = \sum_{|\alpha| \le m} c_{\alpha,N} x^\alpha$ will fulfil* (C.16) *with an $o(1)$-error if and only if its coefficients for $N \to \infty$ satisfy*

$$c_{\alpha,N} = \frac{1}{\alpha!} \partial^\alpha (2^{-nN} \Phi(2^{-N} \cdot) * f)(0) + o(1), \tag{C.17}$$

where the leading term also is given by $c_{\alpha,N} = \langle \hat{f}, \frac{(\mathrm{i}\,\xi)^\alpha}{\alpha!}(1 - \sum_{v=-N}^{\infty} \hat{\phi}_\nu)\rangle/(2\pi)^n$.

If desired, one may appeal to the continuity of D^α to obtain, for $m + n \ge d$,

$$D^\alpha f = \lim_{N \to \infty} (\sum_{\nu=-N}^{\infty} \phi_\nu * D^\alpha f - D^\alpha P_{m,N}). \tag{C.18}$$

Remark C.6. *Kyriazis [20] made a study of the Littlewood–Paley decomposition* (C.2) *in the space $(\mathcal{S}_M)'$, i.e. the dual of $\mathcal{S}_M = \{\, \psi \in \mathcal{S} \mid \int x^\alpha \psi \, dx = 0$ for $|\alpha| \le M \,\}$. Inspired by this, let us note that by testing against such special ψ, the remainder term in* (C.16) *improves in view of Corollary C.4 to $\mathcal{O}(2^{-N(n+\max(M,m)+1-d)})$.*

Remark C.7. *Kyriazis [20] gave an example of distributions f_s, $s > 0$, in $\mathcal{S}'(\mathbb{R})$ for which the series* (C.16) *only converges for $m \ge 0$; i.e. addition of at least constants is necessary to have convergence to f_s in $\mathcal{S}'(\mathbb{R})$.*

Remark C.8. *Peetre [26, p. 54] treated convergence in $\mathcal{S}'$ of the Littlewood–Paley decomposition* (C.2)*, which he essentially stated with unspecified polynomials P and P_N in the form*

$$f - P = \lim_{N \to \infty} (\sum_{\nu=-N}^{\infty} \phi_\nu * f - P_N). \tag{C.19}$$

According to Proposition C.5 subtraction of P can be avoided (of course P can also be added on both sides, replacing P_N by $P_N - P$, to have only f on the left). Peetre sketched a proof based on the obvious convergence of the differentiated series $\sum_\nu \phi_\nu * D^\alpha f$ for $|\alpha| = d+1$, where polynomials of degree d form the common null space of these D^α, leading to the P_N and P—but no details were given on convergence in (C.19)*. Frazier and Jawerth [10] claimed restrictions on the degrees of P_N and P. Later Kyriazis [20] gave a full proof of* (C.19)*, and so did Bownik and Ho [6]. Our Proposition C.5 presents an alternative approach with Taylor polynomials $P_{m,N}$ of an* arbitrary *degree $m \ge -1$, which by the asymptotic uniqueness in Proposition C.2 yields that the above P_N and P must be* interrelated *by $P_N - P = P_{m,N} + o(1)$. It also provides a comprehensive error analysis, entailing that the band-limited series $\sum_{\nu \ge -N} \phi_\nu * f$ plus $P_{m,N}$ converges to f itself in the topology of $\mathcal{S}'$ whenever $m \ge d - n$.*

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