

On the Theory of Type 1, 1-Operators

by

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Dissertation Preface

The following is identical to the dissertation submitted on November 1, 2010, except that the references [18] and [19] on page viii have been updated (and similarly in the bibliography).

Aalborg, 9 May 2011

Jon Johnsen

Scientific work of Jon Johnsen

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- [14] (with W. Sickel) *On the trace problem for Lizorkin–Triebel spaces with mixed norms*. *Math. Nachr.*, **281** (2008), 1–28.
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- [17] *Simple proofs of nowhere-differentiability for Weierstrass's function and cases of slow growth.* *J. Fourier Anal. Appl.* **16** (2010), 17–33.
- *[18] *Pointwise estimates of pseudo-differential operators.* *Journal of Pseudo-Differential Operators and Applications*, to appear. (Available as Tech. Report R-2010-12, Aalborg University, 2010: <http://vbn.aau.dk/files/38938927/R-2010-12.pdf>)
- *[19] *Type 1,1-operators on spaces of temperate distributions.* Tech. Report R-2010-13, Aalborg University, 2010.
(Available at <http://vbn.aau.dk/files/38938995/R-2010-13.pdf>)

The five entries marked by * in the above list constitute the author's doctoral dissertation.

CHAPTER 1

Introduction

In this presentation of the subject it is assumed that the reader is familiar with basic concepts of Schwartz' distribution theory; Section 2.1 below gives a summary of this and notation used throughout.

1.1. Basics

An operator of type $1, 1$ is a special example of a pseudo-differential operator, whereby the latter is the mapping $u \mapsto a(x, D)u$ defined on Schwartz functions $u(x)$, ie on the $u \in \mathcal{S}(\mathbb{R}^n)$, by the classical Fourier integral

$$a(x, D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} a(x, \eta) \hat{u}(\eta) d\eta. \quad (1.1)$$

Hereby its *symbol* $a(x, \eta)$ could in general be of type ρ, δ for $0 \leq \delta \leq \rho \leq 1$ and, say of order $d \in \mathbb{R}$. This means that $a(x, \eta)$ is in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies L. Hörmander's condition that for all multiindices $\alpha, \beta \in \mathbb{N}_0^n$ there is a constant $C_{\alpha, \beta}$ such that

$$|D_\eta^\alpha D_x^\beta a(x, \eta)| \leq C_{\alpha, \beta} (1 + |\eta|)^{m - \rho|\alpha| + \delta|\beta|}, \quad \text{for } x \in \mathbb{R}^n, \eta \in \mathbb{R}^n. \quad (1.2)$$

Such symbols constitute the Fréchet space $S_{\rho, \delta}^d(\mathbb{R}^n \times \mathbb{R}^n)$. The map $a(x, D)u$ is also written $\text{OP}(a(x, \eta))u$.

The classical case is $\rho = 1, \delta = 0$, that gives a framework for partial differential operators with bounded C^∞ coefficients on \mathbb{R}^n . For example, when

$$p(x, D) = \sum_{|\alpha| \leq d} a_\alpha(x) D^\alpha \quad (1.3)$$

is applied to $u = \mathcal{F}^{-1} \mathcal{F}u$, it is seen at once that $p(x, D)$ has symbol $p(x, \eta) = \sum_{|\alpha| \leq d} a_\alpha(x) \eta^\alpha$, which belongs to $S_{1, 0}^d(\mathbb{R}^n \times \mathbb{R}^n)$. It is well known that this allows inversion of $p(x, D)$ modulo smoothing operators if $p(x, \eta)$ is elliptic, ie if $|p(x, \eta)| \geq c|\eta|^d > 0$ for $|\eta| \geq 1$.

A *type 1, 1-operator* is the more general case with $\rho = 1, \delta = 1$ in (1.2). A basic example of such symbols is due to C. H. Ching [Chi72]; it results by taking a unit vector $\theta \in \mathbb{R}^n$ and some auxiliary function $A \in C_0^\infty(\mathbb{R}^n)$ for which $A(\eta) \neq 0$ only holds in the corona $\frac{3}{4} \leq |\eta| \leq \frac{5}{4}$ and setting

$$a_\theta(x, \eta) = \sum_{j=0}^{\infty} 2^{jd} \exp(-i2^j \theta \cdot x) A(2^{-j} \eta). \quad (1.4)$$

This symbol is C^∞ since there is at most one non-trivial term at each point (x, η) ; it belongs to $S_{1,1}^d$ because x -derivatives of the exponential function increases the order of growth with respect to η , since $|2^j \theta| \approx |\eta|$ on $\text{supp}A(2^{-j}\cdot)$.

Type 1, 1-operators are interesting because they have important applications to non-linear maps and non-linear partial differential operators, as indicated below, — but this is undoubtedly also the origin of this operator class's peculiar properties.

To give a glimpse of this, it is recalled that elementary estimates show that the mapping OP: $(a, u) \mapsto a(x, D)u$ in (1.1) is bilinear and continuous

$$S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n). \quad (1.5)$$

Beyond this, difficulties emerge when one tries to extend a given type 1, 1-operator $a(x, D)$ in a consistent way to $\mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{S}(\mathbb{R}^n)$. It is also a tricky task to determine the subspaces E with

$$\mathcal{S}(\mathbb{R}^n) \subset E \subset \mathcal{S}'(\mathbb{R}^n) \quad (1.6)$$

to which $a(x, D)$ extends. Conversely, already when E is fixed as $E = L_2(\mathbb{R}^n)$, there is no known characterisation of *symbols* of the type 1, 1-operators that extend to E .

Above all, the main technical difficulty of type 1, 1-operators is that they can change every frequency in $u(x)$, ie every $\eta \in \text{supp}\hat{u}$, to the frequency $\xi = 0$ — intuitively this can be understood from (1.4) because the factor $e^{-ix \cdot 2^j \theta}$ oscillates as much as $e^{ix \cdot \eta}$ in (1.1).

Consequently, at every singular point x_0 of u they may change the high frequencies causing the singularity, hence change its nature (known as non-preservation of wavefront sets). However, from this perspective it might seem surprising that they cannot *create* singularities; for open sets $\Omega \subset \mathbb{R}^n$ this means that

$$u \text{ is } C^\infty \text{ in } \Omega \quad \implies \quad a(x, D)u \text{ is } C^\infty \text{ in } \Omega. \quad (1.7)$$

(This is known as the *pseudo-local* property). As (1.7) obviously holds true whenever $a(x, D)$ in (1.1) is applied to a Schwartz function, cf (1.5), it is clear that (1.7) pertains to the $u \in \mathcal{S}' \setminus \mathcal{S}$ on which $a(x, D)$ can be defined, and that (1.1) alone is of little use in the proof of (1.7).

Besides the challenge of describing the unusual properties of type 1, 1-operators, they also have interesting applications as recalled in the next two sections.

1.2. The historic development

The review below is mainly cronological and deliberately brief, but hopefully it can serve the reader as a point of reference in Chapters 3–7. The author's contributions are given in footnotes where comparisons make sense (a thorough review will follow in Section 3.2 below).

Symbols of type ρ, δ were introduced in 1966 in a seminar on hypoelliptic equations by L. Hörmander [Hör67]. (Unlike the definition of $S_{\rho, \delta}^d(\mathbb{R}^n \times \mathbb{R}^n)$ in (1.2), the estimates were local in x as customary at that time.)

The pathologies of type 1, 1-operators were revealed around 1972–73 when C. H. Ching [Chi72] in his thesis gave examples of symbols $a_\theta(x, \eta)$ in $S_{1,1}^0$ for which the corresponding operators are unbounded in $L^2(\mathbb{R}^n)$. Essentially these symbols had the form in (1.4).

Moreover, E. M. Stein showed C_*^s -boundedness, $s > 0$, for all operators of order $d = 0$, in lecture notes from Princeton University (1972-73). This result is now available in [Ste93, VII.§1.3], albeit with a misprint in the reference to the lecture notes (as noticed in [Joh08b]).

Afterwards C. Parenti and L. Rodino [PR78] discovered that some type 1,1-operators do not preserve wavefront sets.¹ As the background for this, the pseudo-local property of type 1,1-operators was anticipated in [PR78] with an incomplete argument.²

Around 1980, Y. Meyer [Mey81a, Mey81b] obtained the fundamental property that a composition operator $u \mapsto F(u)$, for a fixed C^∞ -function F with $F(0) = 0$, acting on $u \in \bigcup_{s > n/p} H_p^s(\mathbb{R}^n)$, can be written

$$F(u) = a_u(x, D)u \quad (1.8)$$

for a specific u -dependent symbol $a_u \in S_{1,1}^0$. Namely, when $1 = \sum_{j=0}^\infty \Phi_j$ is a Littlewood–Paley partition of unity, then $a_u(x, \eta)$ is an elementary symbol in the sense of R. R. Coifman and Y. Meyer [CM78], ie it is given by the formula

$$a_u(x, \eta) = \sum_{j=0}^\infty m_j(x) \Phi_j(\eta) \quad (1.9)$$

with the smooth multipliers

$$m_j(x) = \int_0^1 F' \left(\sum_{k < j} \Phi_k(D)u(x) + t \Phi_j(D)u(x) \right) dt. \quad (1.10)$$

This gave a convenient proof of the fact that the non-linear map $u \mapsto F(u)$ sends $H_p^s(\mathbb{R}^n)$ into itself for $s > n/p$. Indeed, this follows as Y. Meyer for general $a \in S_{1,1}^d$, using reduction to elementary symbols, established continuity

$$H_r^{t+d}(\mathbb{R}^n) \xrightarrow{a(x,D)} H_r^t(\mathbb{R}^n) \quad \text{for } t > 0, 1 < r < \infty. \quad (1.11)$$

So for $a = a_u$ and $t = s$, $r = p$ this yields at once that $F(u) = a_u(x, D)u$ also belongs to H_p^s . For integer s this could also be seen directly by calculating derivatives up to order s of $F(u)$, but for non-integer $s > n/p$, this use of pseudo-differential operators is a particularly elegant proof method.³

It was also realised then that type 1,1-operators show up in J.-M. Bony's paradifferential calculus [Bon81] and microlocal inversion together with propagations of singularities for non-linear partial differential equations of the form $F(x, u(x), \dots, \partial_x^\alpha u(x)) = 0$.

In the wake of this, in 1983, G. Bourdaud proved boundedness on the Besov space $B_{p,q}^s(\mathbb{R}^n)$ for $s > 0$, $p, q \in [1, \infty]$ in his thesis, cf [Bou83, Bou88a]. He also gave a simplified proof of (1.11), and noted that by duality and interpolation every type 1,1-operator

$$a(x, D): C_0^\infty(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n) \quad (1.12)$$

¹This is extended to all $d \in \mathbb{R}$, $n \in \mathbb{N}$ in [Joh08b, Sect. 3.2] with exact formulae for the wavefront sets.

²The first full proof appeared in [Joh08b, Thm. 6.4].

³In [Joh08b, Sect. 9] these results are deduced from the precise definition of type 1,1-operators in [Joh08b], together with a straightforward proof of continuity on H_p^s of $u \mapsto F \circ u$ in Theorem 9.4 there.

with $d = 0$ is bounded on $H_p^s(\mathbb{R}^n)$ for all real s , $1 < p < \infty$, in particular on L_2 , if its adjoint $a(x, D)^* : C_0^\infty(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ is also of type 1, 1.

Denoting this subclass of symbols by $\tilde{S}_{1,1}^0$, or more generally

$$\text{OP}(\tilde{S}_{1,1}^d) = \text{OP}(S_{1,1}^d) \cap \text{OP}(S_{1,1}^d)^*, \quad (1.13)$$

he proved that $\text{OP}(\tilde{S}_{1,1}^0)$ is a maximal self-adjoint subalgebra of $\mathbb{B}(L_2(\mathbb{R}^n)) \cap \text{OP}(S_{1,1}^0)$. Hence self-adjointness suffices for L_2 -boundedness, but it is *not* necessary:

G. Bourdaud also showed that the auxiliary function A in Ching's counter-example can be chosen for $n = 1$ so that $a_\theta(x, D)$ *does* belong to $\mathbb{B}(L_2) \cap \text{OP}(S_{1,1}^0)$ even though neither $a_\theta(x, D)^*$ nor $a_\theta(x, D)^2$ is of type 1, 1.

In addition G. Bourdaud analysed the borderline $s = 0$ and showed that every $a(x, D)$ of order 0 is bounded $B_{p,1}^0(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n)$ for all $p \in [1, \infty]$, where the Besov space $B_{p,1}^0$ is slightly smaller than L_p ; whilst $a_\theta(x, D)$ was proven unbounded on $B_{2,1}^0$.⁴

In their fundamental paper on the $T1$ -theorem G. David and J.-L. Journé [DJ84] concluded that $T = a(x, D) \in \text{OP}(S_{1,1}^0)$ is bounded on L_2 if and only if $T^*(1) \in \text{BMO}(\mathbb{R}^n)$, the space of functions (modulo constants) of bounded mean oscillation. (Formally this condition is weaker than G. Bourdaud's $T^* \in \text{OP}(S_{1,1}^0)$; but none of these are expressed in terms of the symbol.) Inspired by this, G. Bourdaud [Bou88a] noted that certain singular integral operators and hence every $a(x, D) \in \text{OP}(S_{1,1}^0)$ extends to a map $\mathcal{O}_M(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$, where \mathcal{O}_M denotes the space of C^∞ -functions of polynomial growth.⁵

Concerning L_p -estimates, T. Runst [Run85b] treated continuity in the more general Besov spaces $B_{p,q}^s$ for $p \in]0, \infty[$ and in Lizorkin–Triebel spaces $F_{p,q}^s$ for $p \in]0, \infty[$, although the necessary control of the frequency changes created by $a(x, D)$ was not quite achieved in [Run85b].⁶ J. Marschall [Mar91] worked on further generalisations to the weighted, anisotropic cases.⁷

L. Hörmander treated type 1, 1-operators four times, first in lecture notes [Hör] from University of Lund (1986–87); the results appeared in [Hör88] with important improvements in [Hör89] the year after. When the notes were published after a decade [Hör97], the chapter on type 1, 1-operators was rewritten with a new presentation including the results from [Hör89] and a few additional conclusions.

⁴In [Joh05] this was sharpened in an optimal way to continuity $F_{p,1}^0 \rightarrow L_p$, where the Lizorkin–Triebel space $F_{p,1}^0$ fulfils $B_{p,1}^0 \subset F_{p,1}^0 \subset L_p$ with strict inclusions for $1 < p < \infty$.

⁵In [Joh10c, Thm. 2.6] this was generalised to a map from the maximal space of smooth functions, more precisely to a map $C^\infty \cap \mathcal{S}' \rightarrow C^\infty$ that moreover leaves \mathcal{O}_M invariant.

⁶This flaw was explained and remedied in [Joh05, Rem. 5.1] and supplemented by $F_{p,q}^s$ and $B_{p,q}^s$ continuity results for operators fulfilling L. Hörmander's twisted diagonal condition; with a further extension to operators in the self-adjoint subclass $\text{OP}(\tilde{S}_{1,1}^d)$ to follow in [Joh10c].

⁷[Mar91] contains flaws similar to [Run85b] as explained in [Joh05, Rem. 5.1]; [Joh05, Rem. 4.2] also pertains to [Mar91].

He sharpened G. Bourdaud's analysis of $a_\theta(x, D)$ by proving that continuity $H^s \rightarrow \mathcal{D}'$ for $s \leq 0$ only holds if $s > -r$ where r is the order of the zero of the auxiliary function A at the point θ on the unit sphere.⁸

Moreover, L. Hörmander characterised the $s \in \mathbb{R}$ (except for a limit point s_0) for which a given $a(x, D) \in \text{OP}(S_{1,1}^d)$ extends by continuity to a bounded operator $H^{s+d} \rightarrow H^s$. More precisely he obtained a largest interval $]s_0, \infty[\ni s$ together with constants C_s such that

$$\|a(x, D)u\|_{H^s} \leq C_s \|u\|_{H^{s+d}} \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^n); \quad (1.14)$$

and conversely that existence of such a C_s implies $s \geq s_0$.

In order to give conditions in terms of the symbols, L. Hörmander introduced, as a novelty in the analysis of pseudo-differential operators, the *twisted diagonal*

$$\mathcal{T} = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi + \eta = 0\}. \quad (1.15)$$

This was shown to play an important role, for if eg the partially Fourier transformed symbol $\hat{a}(\xi, \eta) := \mathcal{F}_{x \rightarrow \xi} a(x, \eta)$ vanishes in a conical neighbourhood of a non-compact part of \mathcal{T} , that is, if for some $B \geq 1$,

$$B(|\xi + \eta| + 1) < |\eta| \implies \hat{a}(x, \eta) = 0, \quad (1.16)$$

then $a(x, D): H^{s+d} \rightarrow H^s$ is continuous for every $s \in \mathbb{R}$ (ie $s_0 = -\infty$).

Moreover, continuity for all $s > s_0$ was shown in [Hör89] to be equivalent to the twisted diagonal condition of order $\sigma = s_0$, which is a specific asymptotic behaviour of $\hat{a}(\xi, \eta)$ at \mathcal{T} . This is formulated in the style of the fundamental Mihlin–Hörmander multiplier theorem: there is a constant $c_{\alpha, \sigma}$ such that for $0 < \varepsilon < 1$,

$$\sup_{R>0, x \in \mathbb{R}^n} R^{-d} \left(\int_{R \leq |\eta| \leq 2R} |R^{|\alpha|} D_\eta^\alpha a_{\chi, \varepsilon}(x, \eta)|^2 \frac{d\eta}{R^n} \right)^{1/2} \leq c_{\alpha, \sigma} \varepsilon^{\sigma+n/2-|\alpha|}. \quad (1.17)$$

Hereby $a_{\chi, \varepsilon}(x, \eta)$ denotes a specific localisation of $a(x, \eta)$ to a conical neighbourhood of \mathcal{T} . Cf Section 6.3 below.

L. Hörmander also characterised the case $s_0 = -\infty$ as the one with symbol in the class $\tilde{S}_{1,1}^d$ and as the one where (1.17) holds for all $\sigma \in \mathbb{R}$; roughly speaking such symbols vanish to infinite order at \mathcal{T} . A concise presentation was given in [Hör97, Thm. 9.4.2].

For operators with additional properties, a symbolic calculus was also developed together with microlocal regularity results at non-characteristic points as well as a sharp Gårding inequality. Although important for the general theory of type 1, 1-operators, this is, however an area adjacent to the present one. So is Chapter 10–11 in [Hör, Hör97] where the paradifferential calculus, linearisation and propagation of singularities of J.-M. Bony [Bon81] is exposed with consistent use of type 1, 1-operators. (A partly similar approach was used by M. Taylor [Tay91] and in the treatment of P. Auscher and M. Taylor [AT95] of commutator estimates by paradifferential operators.)

⁸ $a_\theta(x, D)$ can moreover be taken *unclosable* in \mathcal{S}' , cf [Joh08b, Sect. 3.1], where it was also shown that extension to $d \in \mathbb{R}$ and $\theta \neq 0$ was useful for a precise version of the non-preservation of wavefront sets observed in [PR78].

Shortly after [Hör88, Hör89], R. Torres [Tor90] also estimated $a(x, D)u$ for $u \in \mathcal{S}(\mathbb{R}^n)$, using the atoms and molecules of M. Frazier and B. Jawerth [FJ85, FJ90]. This gave unique extensions by continuity to maps $A: F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n)$ for all s so large that, for all multi-indices γ ,

$$0 \leq |\gamma| < \max\left(0, \frac{n}{p} - n, \frac{n}{q} - n\right) - s \implies \text{supp } \mathcal{F}(a(x, D)^* x^\gamma) \in \mathcal{E}'(\mathbb{R}^n). \quad (1.18)$$

Obviously this refers to the adjoint $a(x, D)^*: \mathcal{S}' \rightarrow \mathcal{S}'$, which in general is an even less understood operator than those of type 1, 1. However, as noted in [Tor90], this implies vanishing of $D_\xi^\gamma \hat{a}(\xi, -\xi)$ for large ξ if the symbol has compact support in x . L. Grafakos and R. Torres [GT99] made a similar study in corresponding homogeneous Besov and Lizorkin–Triebel spaces, using symbols in the homogeneous symbol class $\dot{S}_{1,1}^d$, defined by removing “1+” from (1.2) for $a(x, \eta) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$.

G. Garello [Gar94, Gar98] worked on an anisotropic version of the results in [PR78, Hör88, Hör89] for locally estimated symbols, although with flawed arguments for the non-preservation of wavefront sets.⁹

A. Boulkhemair [Bou95, Bou99] worked (in a general context) on the use of symbols $a \in S_{1,1}^d$ in the Weyl calculus, ie in $\text{Op}^W(a) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\eta} a\left(\frac{x+y}{2}, \eta\right) u(y) dy d\eta$. It was shown for Ching’s symbol a_θ with $d = 0$, cf (1.4), that when $A(\theta) = 1$ also the operator $\text{Op}^W(a_\theta)$ is bounded on H^s if and only if $s > 0$. In addition it was observed that Weyl operators are worse for type 1, 1-symbols since certain $b(x, D) \in \text{Op}^W(S_{1,1}^0)$ are unbounded on H^s for every $s \in \mathbb{R}$; as noted with credit to J. M. Bony, this results for $b = \text{Re} a_\theta$ or $b = \text{Im} a_\theta$ because $\text{Op}^W(b)^* = \text{Op}^W(\bar{b})$. Condition (1.16) was shown to split into two similar conditions (pertaining to $\eta \pm \frac{1}{2}\xi = 0$) that give boundedness in H^s for $s > 0$ and $s < 0$, hence for all s when both hold.

Very recently, J. Hounie and R. A. dos Santos Kapp [HdSK09] utilised atomic decompositions of the local Hardy space $h_p(\mathbb{R}^n)$, which identifies with $F_{p,2}^0$ for $0 < p < \infty$, to derive existence of h_p -bounded extensions of $a(x, D)$ in the self-adjoint subclass of order $d = 0$ from the L_2 -estimates of L. Hörmander [Hör89, Hör97].¹⁰

The above review summarises the scientific contributions, which resulted from the author’s search in the literature for works devoted to type 1, 1-operators.

The review is intended to be complete, and the contributions of the author from 2004-2009 [Joh04, Joh05, Joh08b, Joh10a, Joh10c] are described accordingly.

It is clear (from the review) that a general definition of $a(x, D)u$ for a given symbol $a \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ has not been described in the previous literature. The estimates of L. Hörmander [Hör88, Hör89], cf (1.14), gave a uniquely defined bounded operator $A: H^{s+d} \rightarrow H^s$; and an

⁹This was noted in [Joh08b, p. 214].

¹⁰As a special case of [Joh10c, Thm. 7.9] it was shown that every $a(x, D)$ in $\text{OP}(\dot{S}_{1,1}^0)$ is continuous

$$h_p(\mathbb{R}^n) \rightarrow F_{p,2}^{s'}(\mathbb{R}^n)$$

for every $s' < 0$ if $0 < p \leq 1$. For $1 < p < \infty$ this was also shown for $s' = 0$ in [Joh10c, Thm. 7.5].

extension of A to $\bigcup_{s>s_0} H^{s+d}(\mathbb{R}^n)$ for some limit s_0 or possibly even $s_0 = -\infty$, depending on $a(x, \eta)$. Similarly the approach of R. Torres could at most define A on $\bigcup F_{p,q}^s(\mathbb{R}^n)$.

Later elementary arguments in [Joh05, Prop. 1] gave that every type 1, 1-operator is defined on $\mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n)$, and even on $C^\infty \cap \mathcal{S}'$. These spaces clearly contain all polynomials $\sum_{|\alpha|\leq k} c_\alpha x^\alpha$ that do not belong to $\bigcup H^s$, nor to $\bigcup F_{p,q}^s$.

This development therefore only emphasises the need for a unifying point of view, that is, a general definition of type 1, 1-operators without reference to spaces other than $\mathcal{S}'(\mathbb{R}^n)$.

1.3. Application to non-linear boundary value problems

In addition to the applications developed by Y. Meyer [Mey81a, Mey81b] and J.-M. Bony [Bon81], type 1, 1-operators were recently used by the author in the analysis of semi-linear boundary problems [Joh08a]. More precisely, their pseudo-local property was shown to be useful for the derivation of local regularity improvements.

To explain this, one can as a typical example consider a perturbed k -harmonic Dirichlét problem in a bounded C^∞ -region $\Omega \subset \mathbb{R}^n$,

$$\begin{aligned} (-\Delta)^k u + u^2 &= f & \text{in } \Omega, \\ \gamma_0 u &= \varphi_0 & \text{on } \partial\Omega, \\ & \vdots \\ \gamma_{k-1} u &= \varphi_{k-1} & \text{on } \partial\Omega. \end{aligned} \tag{1.19}$$

Here $\Delta = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$ denotes the Laplacian while γ_j stands for the normal derivative of order j at the boundary.

For such problems the parametrix construction of [Joh08a] yields the solution formula

$$u = P_u^{(N)}(R_k f + K_0 \varphi_0 + \dots + K_{k-1} \varphi_{k-1}) + (R_k L_u)^N u, \tag{1.20}$$

where the parametrix $P_u^{(N)}$ is the u -dependent linear operator

$$P_u^{(N)} = I + R_k L_u + \dots + (R_k L_u)^{N-1}. \tag{1.21}$$

Here it was a crucial point of [Joh08a] to use the so-called *exact parilinearisation* L_u of u^2 as a main ingredient. In effect this means that L_u is a localised type 1, 1-operator, as reviewed in (1.23) below. (This is a result from [Joh08a, Thm. 5.15], but it would lead too far to explain its deduction from the rather technical parilinearisation.) With a convenient sign convention L_u fulfils $-L_u(u) = u^2$.

Moreover, the other terms R_k, K_0, \dots, K_{k-1} in the formula are the solution operators of the linear problem.¹¹ It is perhaps instructive to reduce to the linear case by formally setting $L_u \equiv 0$ above: this shows that the parametrix $P_u^{(N)}$ and the remainder $(R_k L_u)^N$ simply modify u in the presence of the non-linear term.

¹¹The operators R_k, K_0, \dots, K_{k-1} can be explicitly described in local coordinates at the boundary $\partial\Omega$. This is the subject of the calculus of L. Boutet de Monvel [BdM71] of pseudo-differential boundary operators; it has been amply described eg in works of G. Grubb [Gru91, Gru96, Gru97, Gru09]. The calculus was exploited in [Joh08a] but details are left out here because it would be too far from the topic of type 1, 1-operators.

Formula (1.20) also has the merit of showing *directly* that the regularity of u will be uninfluenced by the non-linear term u^2 . Or more precisely, u will belong to the same Sobolev space H_p^s as the corresponding linear problem's solution v , ie

$$v = R_k f + K_0 \varphi_0 + \cdots + K_{k-1} \varphi_{k-1}. \quad (1.22)$$

Indeed, in (1.20) the parametrix $P_u^{(N)}$ is applied to v , but it is of order 0 for every N , hence sends each Sobolev space H_p^s into itself; while the remainder $(R_m L_u)^N u$ will be in $C^k(\overline{\Omega}) \subset H_p^s(\overline{\Omega})$ for some fixed k if N is taken large enough (in both cases because $R_k L_u$ will have negative order if the given solution u a priori meets a rather weak regularity assumption; cf (1.23) below). These inferences may be justified using parameter domains as in [Joh08a], to keep track of the spaces on which various steps are valid.

Moreover, to explain the usefulness of type 1, 1-operators here, it is noted that in subregions $\Xi \Subset \Omega$, extra regularity properties of f carry over to u . Eg, if $f|_{\Xi}$ is C^∞ so is $u|_{\Xi}$. Other examples involve improvements in Ξ of eg the Sobolev space regularity.

Such local properties can also be deduced from formula (1.20), because L_u factors through a specific type 1, 1-operator A_u (this is in itself a minor novelty, because of the boundary). That is, when r_Ω denotes restriction to Ω and ℓ_Ω stands for a linear extension operator from Ω , then

$$L_u = r_\Omega \circ A_u \circ \ell_\Omega, \quad A_u \in \text{OP}(S_{1,1}^d); \quad (1.23)$$

here the order $d \geq (\frac{n}{p_0} - s_0)_+$ if u is given in $H_{p_0}^{s_0}$, though with strict inequality if $s_0 = n/p_0$.

To exploit this, one may simply take cut-off functions $\psi, \chi \in C_0^\infty(\Xi)$ with $\chi = 1$ around support ψ . Insertion of these into (1.20), cf [Joh08a, Thm. 7.8], gives

$$\begin{aligned} \psi u &= \psi P_u^{(N)}(R_k(\chi f)) + \psi P_u^{(N)}(R_k((1-\chi)f)) \\ &\quad + \psi P_u^{(N)}(K_0 \varphi_0 + \cdots + K_{k-1} \varphi_{k-1}) + \psi (R_k L_u)^N u. \end{aligned} \quad (1.24)$$

As desired ψu has the same regularity as the *first* term on the right-hand side. Indeed, the last term has the same regularity as the first if N is large, and—since the set of pseudo-local operators is invariant under sum and composition, so that pseudo-locality of A_u by (1.23) carries over to $P^{(N)}$ —the disjoint supports of ψ and $1-\chi$ will imply that the second term is C^∞ ; the $K_j \varphi_j$ always contribute C^∞ -functions in the interior, to which set ψ localises while $P^{(N)}$ is pseudo-local.

Therefore the *pseudo-local* property of A_u will lead easily to improved regularity of u locally in Ξ , to the extent this is permitted by the data f . Hence it was a serious drawback that the literature had not established pseudo-locality in the 1, 1-context.

But motivated by the above application in (1.24), the pseudo-local property of general type 1, 1-operators was proved recently by the author in [Joh08b]. The only previous work mentioning this subject was that of C. Parenti and L. Rodino [PR78], who three decades ago anticipated the result but merely gave an incomplete argument, partly because they did not assign a specific meaning to $a(x, D)u$ for $u \in \mathcal{S}' \setminus C_0^\infty$.

1.4. The definition of type 1,1-operators

As seen at the end of the last two sections, it will be well motivated to introduce a general definition of type 1,1 *operators*.

This was first done rigorously in [Joh08b], taking into account that in some cases they can only be defined on proper subspaces $E \subset \mathcal{S}'(\mathbb{R}^n)$. Indeed, it was proposed to stipulate that u belongs to the domain $D(a(x,D))$ and to set

$$a(x,D)u := \lim_{m \rightarrow \infty} \text{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u \quad (1.25)$$

if this limit exists, say in $\mathcal{D}'(\mathbb{R}^n)$, for all the $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi = 1$ in a neighbourhood of the origin and if it does not depend on such ψ .

The definition, its consequences and the techniques developed are discussed in the author's contributions [Joh04, Joh05, Joh08b, Joh10a, Joh10c], where the first is an early announcement of the results in the second. These works are summarised in Chapter 3.

CHAPTER 2

Preliminaries

2.1. Notions and notation

As usual $t_{\pm} = \max(0, \pm t)$ will denote the positive and negative part of $t \in \mathbb{R}$; and $[t]$ will stand for the largest integer $k \in \mathbb{Z}$ such that $k \leq t$. The characteristic function of a set $M \subset \mathbb{R}^n$ is denoted 1_M ; by $M \Subset \mathbb{R}^n$ it is indicated that the subset M is precompact.

The Lebesgue spaces $L_p(\mathbb{R}^n)$ with $0 < p \leq \infty$ consist of the (equivalence classes of) measurable functions having finite (quasi-)norm $\|f\|_p = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$ for $0 < p < \infty$, respectively $\|f\|_{\infty} = \text{ess sup}_{\mathbb{R}^n} |f|$.

In general $\|f + g\|_p \leq 2^{(\frac{1}{p}-1)_+} (\|f\|_p + \|g\|_p)$ for $0 < p < \infty$. Hence for $0 < p < 1$ the map $f \mapsto \|f\|_p$ is only a quasi-norm, but it does have a subadditive power as $\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$ for $0 < p < 1$.

For every multiindex $\alpha \in \mathbb{N}_0^n$ it is convenient to set $x^{\alpha} = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and to introduce the differential operator $D^{\alpha} = (-i)^{|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ where $|\alpha| = \alpha_1 + \dots + \alpha_n$.

The space of smooth functions with compact support is denoted by $C_0^{\infty}(\Omega)$ or $\mathcal{D}(\Omega)$, when $\Omega \subset \mathbb{R}^n$ is open; $\mathcal{D}'(\Omega)$ is the dual space of distributions on Ω . Throughout $\langle u, \varphi \rangle$ denotes the action of $u \in \mathcal{D}'(\Omega)$ on $\varphi \in C_0^{\infty}(\Omega)$. Therefore $\langle \cdot, \cdot \rangle$ is a bilinear form; the sesquilinear form $(\cdot | \cdot)$ is used for the action of conjugate linear functionals on C_0^{∞} and \mathcal{S} , consistently with the *inner* product on the Hilbert space $L_2(\mathbb{R}^n)$ (both $\langle \cdot, \cdot \rangle$ and $(\cdot | \cdot)$ are called scalar products for convenience).

The space of slowly increasing functions, ie C^{∞} -functions f fulfilling $|D^{\alpha} f(x)| \leq c_{\alpha} \langle x \rangle^{N_{\alpha}}$ for all multindices α is written $\mathcal{O}_M(\mathbb{R}^n)$; hereby $\langle x \rangle = (1 + |x|^2)^{1/2}$.

The Schwartz space of rapidly decreasing C^{∞} -functions is written \mathcal{S} or $\mathcal{S}(\mathbb{R}^n)$, while its dual space $\mathcal{S}'(\mathbb{R}^n)$ constitutes the space of tempered distributions. The Fourier transformation of u is denoted by $\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx$, with inverse $\mathcal{F}^{-1}v(x) = \check{v}(x)$.

The subspace $\mathcal{E}'(\mathbb{R}^n)$ consists of the distributions of compact support; it is the dual of $C^{\infty}(\mathbb{R}^n)$. The *spectrum* of $u \in \mathcal{S}'$ is by definition $\text{supp } \mathcal{F}u$; hence $\mathcal{F}^{-1}(\mathcal{E}')$ is the space of distributions with compact spectrum (though it equals $\mathcal{F}(\mathcal{E}')$ as a set, the slightly more pedantic $\mathcal{F}^{-1}\mathcal{E}'$ is preferred to emphasize the role of the Fourier transformation).

Pseudo-differential operators are given on $\mathcal{S}(\mathbb{R}^n)$ by (1.1), with symbols fulfilling (1.2). On $S_{\rho, \delta}^d = S_{\rho, \delta}^d(\mathbb{R}^n \times \mathbb{R}^n)$ the Fréchet topology is defined by a family of seminorms $p_{\alpha, \beta}(a)$, that are given as the smallest possible constants $C_{\alpha, \beta}$ in (1.2). For short $S_{\rho, \delta}^{\infty} := \bigcup_{d \in \mathbb{R}} S_{\rho, \delta}^d$ is used for the set of all symbols (of type ρ, δ). The symbol class $S^{-\infty} := \bigcap_d S_{1, 0}^d = \bigcap_{d, \rho, \delta} S_{\rho, \delta}^d$ defines the smoothing operators; they are bounded $H^s \rightarrow H^t$ for all $s, t \in \mathbb{R}$.

The pseudo-differential operators $a(x, D)$ are in bijective correspondence with their distribution kernels, that are given by

$$K(x, y) = \mathcal{F}_{\eta \rightarrow x-y}^{-1} a(x, \eta). \quad (2.1)$$

By definition the kernel satisfies the kernel relation

$$\langle a(x, D)\psi, \varphi \rangle = \langle K, \varphi \otimes \psi \rangle \quad \text{for all } \varphi, \psi \in C_0^\infty(\mathbb{R}^n). \quad (2.2)$$

As customary, the support $\text{supp} K \subset \mathbb{R}^n \times \mathbb{R}^n$ is seen as a relation mapping sets in \mathbb{R}_y^n to other sets in \mathbb{R}_x^n . More precisely, each subset $M \subset \mathbb{R}_y^n$ is mapped to

$$\text{supp} K \circ M = \{x \in \mathbb{R}^n \mid \exists y \in M : (x, y) \in \text{supp} K\}. \quad (2.3)$$

The singular support of $u \in \mathcal{D}'$, denoted $\text{sing supp } u$, is the complement of the largest open set on which u acts a C^∞ -function. The wavefront set $\text{WF}(u)$ is the complement of those $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ for which $\mathcal{F}(\varphi u)$ decays rapidly in a conical neighbourhood of ξ for some $\varphi \in C_0^\infty$ for which $\varphi(x) \neq 0$.

Every pseudo-differential operator considered here is continuous $a(x, D): \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$, hence has a continuous adjoint $a(x, D)^*: \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ with respect to the scalar product $(\cdot | \cdot)$; this fulfils

$$(a(x, D)^* \varphi | \psi) = (\varphi | a(x, D)\psi), \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^n). \quad (2.4)$$

Its restriction $a(x, D)^*: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is also continuous, hence is a pseudo-differential operator by Schwartz' kernel theorem; cf [Hör85, 18.1]. More precisely,

$$a(x, D)^* = \text{OP}(b(x, \eta)) \quad \text{for } b(x, \eta) = e^{iD_x \cdot D_\eta} \bar{b}(x, \eta). \quad (2.5)$$

The adjoint symbol $e^{iD_x \cdot D_\eta} \bar{b}(x, \eta)$ is also written $a^*(x, \eta)$, so $\text{OP}(a(x, \eta))^* = \text{OP}(a^*(x, \eta))$.

2.2. Scales of function spaces

The Sobolev spaces $H_p^s(\mathbb{R}^n)$ are defined for $s \in \mathbb{R}$ and $1 < p < \infty$ as $\text{OP}(\langle \xi \rangle^{-s})(L_p)$, with $\|f\|_{H_p^s} = \|\text{OP}(\langle \xi \rangle^{-s})f\|_p$. The special case $p = 2$ is written as $H^s(\mathbb{R}^n)$ or H^s for simplicity.

The Hölder class $C^s(\mathbb{R}^n)$ is for non-integer $s > 0$ defined as the functions $f \in C^{[s]}(\mathbb{R}^n)$ having finite norm

$$|f|_s = \sum_{|\alpha| \leq [s]} \|D^\alpha f\|_\infty + \sum_{|\alpha| = [s]} \sup_{x \neq y} |D^\alpha f(x) - D^\alpha f(y)| |x - y|^{[s] - s}. \quad (2.6)$$

To get an interpolation invariant half-scale $C_*^s(\mathbb{R}^n)$, $s > 0$, it is well known that one should fill in for $s \in \mathbb{N}$ by means of the Zygmund condition. Eg the space C_*^1 consists of the $f \in C(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ for which

$$|f|_1 = \|f\|_\infty + \sup_{y \neq 0} \sup_{x \in \mathbb{R}^n} |f(x+y) + f(x-y) - 2f(x)| / |y| < \infty. \quad (2.7)$$

These spaces appear naturally as a part of a full scale of Hölder–Zygmund spaces $C_*^s(\mathbb{R}^n)$ defined for $s \in \mathbb{R}$; as explained in eg [Hör97, Sc. 8.6].

However, all the H_p^s and C_*^s spaces are contained in two more general scales, namely the Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ and Lizorkin–Triebel spaces $F_{p,q}^s(\mathbb{R}^n)$, that are well adapted to harmonic analysis. They are recalled below.

First a Littlewood–Paley decomposition is constructed using a function $\tilde{\Psi}$ in $C^\infty(\mathbb{R})$ for which $\tilde{\Psi}(t) \equiv 0$ and $\tilde{\Psi}(t) \equiv 1$ holds for $t \geq 2$ and $t \leq 1$, respectively; then $\Psi(\xi) = \tilde{\Psi}(|\xi|)$ and $\Phi = \Psi - \Psi(2\cdot)$ gives the partition of unity $1 = \Psi(\xi) + \sum_{j=1}^{\infty} \Phi(2^{-j}\xi)$. For brevity it is here convenient to set $\Phi_0 = \Psi$ and $\Phi_j = \Phi(2^{-j}\cdot)$ for $j \geq 1$.

Then, for a *smoothness indices* $s \in \mathbb{R}$, *integral-exponent* $p \in]0, \infty]$ and *sum-exponent* $q \in]0, \infty]$, the Besov space $B_{p,q}^s(\mathbb{R}^n)$ is defined to consist of the $u \in \mathcal{S}'(\mathbb{R}^n)$ for which

$$\|u\|_{B_{p,q}^s} := \left(\sum_{j=0}^{\infty} 2^{sjq} \left(\int_{\mathbb{R}^n} |\Phi_j(D)u(x)|^p dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty. \quad (2.8)$$

(As usual the norm in ℓ_q should be replaced by the supremum over $j \in \mathbb{N}_0$ in case $q = \infty$.)

Similarly the Lizorkin–Triebel space $F_{p,q}^s(\mathbb{R}^n)$ is defined as the $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{F_{p,q}^s} := \left(\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{sjq} |\Phi_j(D)u(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty. \quad (2.9)$$

Throughout it will be tacitly understood that $p < \infty$ whenever Lizorkin–Triebel spaces are under consideration.

The spaces are described in eg [RS96, Tri83, Tri92, Yam86a]. They are quasi-Banach spaces with the quasi-norms given by the finite expressions in (2.8) and (2.9); and Banach spaces if both $p \geq 1$ and $q \geq 1$.

In general $u \mapsto \|u\|^\lambda$ is subadditive for $\lambda \leq \min(1, p, q)$, so $\|f - g\|^\lambda$ is a metric on each space in the $B_{p,q}^s$ - and $F_{p,q}^s$ -scales.

There are a number of embeddings of these spaces, like the simple ones $F_{p,q}^s \hookrightarrow F_{p,q}^{s-\varepsilon}$ for $\varepsilon > 0$ and $F_{p,q}^s \hookrightarrow F_{p,r}^s$ for $q \leq r$. The Sobolev embedding theorem takes the form

$$F_{p_0,q_0}^{s_0} \hookrightarrow F_{p_1,q_1}^{s_1} \quad \text{for } s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}, \quad p_0 < p_1. \quad (2.10)$$

The analogous results are valid for the $B_{p,q}^s$ spaces, provided that $q_0 \leq q_1$. Moreover,

$$B_{p,\min(p,q)}^s \hookrightarrow F_{p,q}^s \hookrightarrow B_{p,\max(p,q)}^s. \quad (2.11)$$

Among the well-known identifications it should be mentioned that

$$H_p^s = F_{p,2}^s \quad \text{for } s \in \mathbb{R}, \quad 1 < p < \infty, \quad (2.12)$$

$$C_*^s = B_{\infty,\infty}^s \quad \text{for } s \in \mathbb{R}. \quad (2.13)$$

In particular this means that

$$H^s = F_{2,2}^s = B_{2,2}^s \quad \text{for } s \in \mathbb{R}. \quad (2.14)$$

One interest of this is that statements proved for all $B_{p,q}^s$ are automatically valid for the Sobolev spaces H^s by specialising to $p = q = 2$, as well as for the Hölder–Zygmund spaces C_*^s by setting $p = q = \infty$. (Much of the literature on partial differential equations has focused on these two scales, with two rather different types of arguments.)

Among the other relations, it could be mentioned that $F_{p,2}^0(\mathbb{R}^n)$ equals the local Hardy space $h_p(\mathbb{R}^n)$ for $0 < p < \infty$. [Tri92] has ample information on these identifications, and also on the extension of $F_{p,q}^s$ to $p = \infty$; this is not considered here.

REMARK 2.2.1. The quasi-norms of $B_{p,q}^s$ and $F_{p,q}^s$ depend of course on the choice of the Littlewood–Paley decomposition; cf (2.8) and (2.9). It is well known that different choices yield equivalent quasi-norms, which may be seen with a multiplier argument. However, a slight extension of this shows that the above assumption on $\tilde{\Psi}(t)$ can be completely weakened, that is, any $\tilde{\Psi} \in C_0^\infty(\mathbb{R})$ equalling 1 around $t = 0$ will lead to an equivalent quasi-norm (cf the framework for Littlewood–Paley decompositions in Section 6.1 below). This is convenient for the treatment of type 1, 1-operators in $B_{p,q}^s$ and $F_{p,q}^s$ spaces.

The general definition of type 1, 1-operators

This section gives a brief description of the author’s contributions; for the sake of readability, the statements will occasionally only address the main cases. A more detailed account can be found in the subsequent sections (and in the papers, of course).

3.1. Definition by vanishing frequency modulation

As the background for Definition 3.1.2 below, it is recalled that the very first result on type 1, 1-operators was the counter-example by C. H. Ching [Chi72], who showed that there exists $a_\theta(x, \eta)$ in $S_{1,1}^0$, cf (1.4), for which the operator $a_\theta(x, D)$ does not have a continuous extension to L_2 .

For later reference, this is now explicated with a refined version of order d .

LEMMA 3.1.1 ([Joh08b, Lem. 3.2]). *Let $a_\theta(x, \eta)$ be given as in (1.4) for $d \in \mathbb{R}$ and with $|\theta| = 1$ and $A = 1$ on the ball $B(\theta, \frac{1}{10})$. Taking $v \in \mathcal{S}(\mathbb{R}^n)$ with $\emptyset \neq \text{supp } \hat{v} \subset B(0, \frac{1}{20})$, then*

$$v_N = v(x) \sum_{j=N}^{N^2} \frac{e^{i2^j x \cdot \theta}}{j 2^{jd} \log N} \quad (3.1)$$

defines a sequence of Schwartz functions with the properties

$$\begin{aligned} \|v_N\|_{H^d} &\leq c \|v\|_2 \left(\sum_{j=N}^{\infty} j^{-2} \right)^{1/2} \searrow 0, \\ a_\theta(x, D)v_N(x) &= \frac{1}{\log N} \left(\frac{1}{N} + \frac{1}{N+1} + \dots + \frac{1}{N^2} \right) v(x) \xrightarrow{N \rightarrow \infty} v(x) \quad \text{in } \mathcal{S}(\mathbb{R}^n). \end{aligned} \quad (3.2)$$

Consequently $a_\theta(x, D)$ is unbounded $H^d \rightarrow L_2$ and unclosable in $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n)$.

Later in 1983, G. Bourdaud [Bou83] showed in his doctoral dissertation that every $a(x, D) \in \text{OP}(S_{1,1}^0)$ is bounded on $L_2(\mathbb{R}^n)$ if also its adjoint $a(x, D)^*$ is of type 1, 1. Hence $a_\theta(x, D)$ above fulfils $a_\theta(x, D)^* \notin \text{OP}(S_{1,1}^0)$, so this adjoint need not send $\mathcal{S}(\mathbb{R}^n)$ into itself.

This has two important consequences: first of all, while $a(x, D)$ as usual does have the “double” adjoint $a^*(x, D)^* = \text{OP}(a^*(x, \eta))^*$ as an extension, the latter is not necessarily defined on the entire space $\mathcal{S}'(\mathbb{R}^n)$ when $a(x, \eta)$ is of type 1, 1. In fact, already for $a_\theta(x, D)^*$ it can be shown explicitly that its image of $\mathcal{S}(\mathbb{R}^n)$ contains functions in $\mathcal{S}' \setminus \mathcal{S}$ (see eg [Joh08b, (3.4),(3.9)]), whence $a^*(x, D)^*$ is defined on a proper subspace of \mathcal{S}' .

Secondly, if one tries to see $u \in \mathcal{S}'(\mathbb{R}^n)$ as a limit $u = \lim_{k \rightarrow \infty} u_k$ for Schwartz functions u_k , one cannot hope to get a useful definition by setting

$$a(x, D)u = \lim_{k \rightarrow \infty} \text{OP}(a)u_k. \quad (3.3)$$

Indeed, this would not always give a linear operator, as $a_\theta(x, D)$ is *unclosable*; cf Lemma 3.1.1. This is obviously important also because it shows that a type 1, 1-operator cannot be given an extended definition just by closing its graph $G(a(x, D))$ as a subset of $\mathcal{S}' \times \mathcal{D}'$ — and nor can one hope to give a definition by other means and obtain a closed operator in general.

In view of this, and especially in comparison with (3.3), it is perhaps not surprising that [Joh08b] proposes a regularisation of the symbol instead:

$$a(x, D)u(x) = \lim_{m \rightarrow \infty} \text{OP}(b_m(x, \eta))u(x). \quad (3.4)$$

However, the precise choice of the approximating symbol $b_m(x, \eta)$ is decisive here.

To prepare for the formal definition, a *modulation* function ψ will in the sequel mean an arbitrary $\psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighbourhood of the origin. Then, after setting $\hat{a}(\xi, \eta) = \mathcal{F}_{x \rightarrow \xi} a(x, \eta)$ for symbols, the following notation is used throughout

$$a^m(x, \eta) = \mathcal{F}_{\xi \rightarrow x}^{-1}[\psi(2^{-m}\xi)\hat{a}(\xi, \eta)]. \quad (3.5)$$

One can then take $b_m(x, \eta) = a^m(x, \eta)\psi(2^{-m}\eta)$, which is in $S^{-\infty}$, so that $b_m(x, D)u$ is defined for every $u \in \mathcal{S}'$. It is easy to see that if $a \in S_{1,1}^d$ then $b_m \rightarrow a$ in $S_{1,1}^{d+1}$ for $m \rightarrow \infty$; cf [Joh08b, Lem. 2.1].

To make the dependence on ψ explicit, set

$$a_\psi(x, D)u = \lim_{m \rightarrow \infty} \text{OP}(a^m(x, \eta)\psi(2^{-m}\eta))u. \quad (3.6)$$

DEFINITION 3.1.2. For every symbol $a \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ the distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the domain $D(a(x, D))$ if the above limit $a_\psi(x, D)u$ exists in $\mathcal{D}'(\mathbb{R}^n)$ for every modulation function ψ and if, in addition, this limit is independent of such ψ . In this case

$$a(x, D)u = a_\psi(x, D)u. \quad (3.7)$$

In [Joh08b] this was termed the definition of $a(x, D)$ by *vanishing frequency modulation*, since all high frequencies are cut off, both in $u(y)$ and in the symbol's dependence on x .

To explain the notation, note first that D appears in two meanings when the domain is denoted by $D(a(x, D))$. Moreover, (1.1) may be written out as

$$a(x, D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\eta} a(x, \eta)u(y) dy d\eta. \quad (3.8)$$

Here u is seen as a function of y ; accordingly the dual variable is denoted by η . Clearly $a(x, D)u(x)$ depends on x , whence its Fourier transform is written as a function of $\xi \in \mathbb{R}^n$. Likewise, when $\mathcal{F}_{x \rightarrow \xi}$ is applied to $a(x, \eta)$, one obtains $\hat{a}(\xi, \eta)$.

The modulation parameter is throughout denoted by $m \in \mathbb{N}$. The modulation function is denoted by ψ , or Ψ if more than one is considered simultaneously. Moreover, with $u^m =$

$\psi(2^{-m}D)u$ and $a^m(x, \eta)$ as defined above, Definition 3.1.2 is for convenience often expressed in short form as

$$a(x, D)u = \lim_{m \rightarrow \infty} a^m(x, D)u^m. \quad (3.9)$$

This may look self-contradicting, however, for $a^m(x, D)$ is just another type 1, 1-operator. But as $u^m \in \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n)$, it will be clear below (from the general extension to $\mathcal{F}^{-1}\mathcal{E}'$) that $a^m(x, D)u^m$ is defined and equals $\text{OP}(a^m(x, \eta)\psi(2^{-m}\eta))u$.

Definition 3.1.2 is actually just a rewriting of the usual one, which is suitable for type 1, 1-symbols as a point of departure, for if $u \in \mathcal{S}$ it follows from the continuity in (1.5) that $a(x, D)u = \text{OP}(a(x, \eta))u$. It also gives back the usual operator $\text{OP}(a(x, \eta))u$ on \mathcal{S}' whenever $a \in S_{1,0}^d$, for it is well known that this is equal to the limit $a_\psi(x, D)u$.

Formally Definition 3.1.2 is reminiscent of oscillatory integrals, as exposed by for example X. St.-Raymond [SR91], now with the natural proviso (as $\delta = 1$) that $u \in D(a(x, D))$ when the regularisation yields a limit independent of the integration factor.

Of course, $a(\cdot, \eta)$ is not modified here with an integration factor proper, but rather with the Fourier multiplier $\psi(2^{-m}D_x)$. This obvious difference is emphasized because $\psi(2^{-m}D_x)$ later gives easy access to Littlewood–Paley analysis of $a(x, D)$.

For other remarks on the feasibility of the frequency modulation, in particular the relation to pointwise multiplication, the reader may refer to [Joh08b, Sect. 1.2].

3.2. Consequences for type 1, 1-operators

Although the definition by vanishing frequency modulation is rather unusual (which is unavoidable), it does have a dozen important properties:

- (I) Definition 3.1.2 unifies 4 previous extensions of type 1, 1-operators.
- (II) The resulting densely defined map $a(x, D): \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ is *maximal* among the extensions $\widetilde{\text{OP}}(a(x, \eta))$ that are stable under vanishing frequency modulation as well as compatible with $\text{OP}(S^{-\infty})$.
- (III) Every operator $a(x, D)$ of type 1, 1 restricts to a map

$$a(x, D): C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n), \quad (3.10)$$

where $C^\infty \cap \mathcal{S}'$ is the maximal subspace of smooth functions. Moreover, $\mathcal{O}_M(\mathbb{R}^n)$ is invariant under $a(x, D)$.

- (IV) Every operator $a(x, D)$ of type 1, 1 is pseudo-local.
- (V) Some type 1, 1-operators do not preserve wavefront sets, eg (1.4) gives

$$\text{WF}(u) = \mathbb{R}^n \times \mathbb{R}_+\theta \quad (3.11)$$

$$\text{WF}(a_{2\theta}(x, D)u) = \mathbb{R}^n \times \mathbb{R}_+(-\theta) \quad (3.12)$$

for $|\theta| = 1$, $A(\eta) = 1$ around $\eta = \theta$ and a product $u(x) = v(x)f(\theta \cdot x)$ with a suitable $v \in \mathcal{F}^{-1}C_0^\infty$ and an oscillating factor $f(t) = \sum_{j=0}^\infty 2^{-jd}e^{i2^jt}$, which for $0 < d \leq 1$ is Weierstrass's continuous nowhere differentiable function.

(VI) The operators satisfy the *support rule*, respectively the *spectral support rule*,

$$\text{supp } a(x, D)u \subset \text{supp } K \circ \text{supp } u, \quad (3.13)$$

$$\text{supp } \mathcal{F}a(x, D)u \subset \text{supp } \mathcal{K} \circ \text{supp } \mathcal{F}u, \quad (3.14)$$

where K is the distribution kernel of $a(x, D)$, whereas \mathcal{K} is that of $\mathcal{F}a(x, D)\mathcal{F}^{-1}$.

(VII) The auxiliary function ψ in Definition 3.1.2 allows a direct transition to Littlewood–Paley analysis of $a(x, D)u$, which in particular gives the well-known paradifferential decomposition, cf (6.12),

$$a(x, D)u = a^{(1)}(x, D)u + a^{(2)}(x, D)u + a^{(3)}(x, D)u. \quad (3.15)$$

(VIII) The operator $a(x, D)$ is everywhere defined and continuous

$$a(x, D): \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n) \quad (3.16)$$

if $a(x, \eta)$ satisfies Hörmander's *twisted diagonal condition*; ie, if for some $B \geq 1$

$$\hat{a}(\xi, \eta) = 0 \quad \text{whenever} \quad B(|\xi + \eta| + 1) < |\eta|. \quad (3.17)$$

(IX) The continuity in (3.16) more generally holds in the self-adjoint subclass $\text{OP}(\tilde{S}_{1,1}^\infty)$, ie if $a(x, D)$ fulfils Hörmander's twisted diagonal condition of order σ for every $\sigma \in \mathbb{R}$.

(X) Every $a(x, D)$ of order d is for $p \in [1, \infty[$ and $q \leq 1$ a continuous map

$$a(x, D): F_{p,q}^d(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n); \quad (3.18)$$

for $a_\theta(x, D)$ from (1.4) this is optimal within the scales $B_{p,q}^s$ and $F_{p,q}^s$ of Besov and Lizorkin–Triebel spaces. (These contain C^s and H_p^s , respectively.)

(XI) Every $a(x, \eta)$ in $S_{1,1}^d$ is continuous, for $s > \max(0, \frac{n}{p} - n)$, $0 < p < \infty$, $0 < q \leq \infty$,

$$a(x, D): F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,r}^s(\mathbb{R}^n) \quad \text{if} \quad r \geq q, \quad r > n/(n+s). \quad (3.19)$$

This holds for all $s \in \mathbb{R}$ and $r = q$ when $a(x, \eta)$ fulfils the twisted diagonal condition (3.17), and if $p > 1$, $q > 1$ also when $a(x, \eta) \in \tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$.

These properties extend to the scale $B_{p,q}^s(\mathbb{R}^n)$ for $0 < p \leq \infty$, $r = q$.

(XII) When $a(x, \eta)$ is in $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$, cf (IX), and $0 < p \leq 1$, $0 < q \leq \infty$,

$$a(x, D): F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,q}^{s'}(\mathbb{R}^n) \quad \text{for arbitrary} \quad s' < s \leq \frac{n}{p} - n. \quad (3.20)$$

This extends verbatim to the $B_{p,q}^s$ -scale.

Definition 3.1.2 together with the properties (I)–(XII) constitute the author's main contribution to the theory of type 1, 1-operators.

Among the above items, (V) and (X) amount to sharpenings of results in the existing literature. The other ten results are rather more substantial, as eg both (I)–(II) and the \mathcal{S}' -continuity in (VIII)–(IX) have not been treated at all hitherto.

Further comments on (I)–(XII) follow below. For convenience the properties (I)–(VI) will be reviewed in Chapter 5 in corresponding sections 5.1–5.6, whereas the more technical results in (VII)–(XII) are described separately in Chapter 6.

Behind the type $1, 1$ -results (I)–(XII) above, there are at least three new *techniques*:

- (i) Pointwise estimates of pseudo-differential operators.
- (ii) The spectral support rule of pseudo-differential operators.
- (iii) Stability of extended distributions under regular convergence.

These tools are useful already for classical pseudo-differential operators, so they are reviewed first, in the next chapter.

CHAPTER 4

Techniques for pseudo-differential operators

The results in this chapter are interesting already for a classical symbol, ie for $a(x, \eta)$ in $S_{1,0}^d(\mathbb{R}^n \times \mathbb{R}^n)$, to which the reader may specialise if desired. However, it is convenient to state them for symbols in $S_{1,\delta}^d$ with $0 \leq \delta < 1$, ie when

$$|D_x^\beta D_\eta^\alpha a(x, \eta)| \leq C_{\alpha,\beta} (1 + |\eta|)^{d - |\alpha| + \delta|\beta|}. \quad (4.1)$$

In this way, the extra precaution that would be needed for $\delta = 1$ is unnecessary here, although the results extend directly to type 1, 1-operators, unless otherwise is mentioned.

4.1. Pointwise estimates of pseudo-differential operators

It seems to be a new observation, that the value of $a(x, D)u(x)$ can be estimated at each point $x \in \mathbb{R}^n$ thus:

$$|a(x, D)u(x)| \leq cu^*(x) \quad \text{when} \quad \text{supp } \hat{u} \in \mathbb{R}^n. \quad (4.2)$$

Hereby u^* is the maximal function of Peetre–Fefferman–Stein type; that is,

$$u^*(x) = u^*(N, R; x) = \sup_{y \in \mathbb{R}^n} \frac{|u(x-y)|}{(1 + R|y|)^N} \quad (4.3)$$

with $R > 0$ chosen so that $\text{supp } \hat{u}$ is contained in the closed ball $\bar{B}(0, R)$. The parameter $N > 0$ can eg be larger than the order of \hat{u} , so that $u^*(x) < \infty$ holds by the Paley–Wiener–Schwartz Theorem.

The above inequality is really a consequence of the following *factorisation inequality*, shown in [Joh10a, Thm. 4.1]. This involves a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ that should equal 1 in a neighbourhood of $\text{supp } \hat{u} \in \mathbb{R}^n$:

$$|a(x, D)u(x)| \leq F_a(N, R; x) \cdot u^*(N, R; x) \quad (4.4)$$

$$F_a(N, R; x) = \int_{\mathbb{R}^n} (1 + R|y|)^N |\mathcal{F}_{\eta \rightarrow y}^{-1}(a(x, \eta)\chi(\eta))| dy \quad (4.5)$$

This simply means that the action of $a(x, D)$ on u can be decomposed, at the unimportant price of an estimate, into a product where the *entire* dependence on the symbol lies in the “ a -factor” $F_a(N, R; x)$, also called the symbol factor.

The symbol factor F_a only depends vaguely on u through N and R . (Eg $N = [n/2] + 1$ works for all $u \in \bigcup H^s(\mathbb{R}^n)$, so then N plays no role.) Formula (4.5) shows that F_a is a weighted L_1 -norm of a regularisation of the distribution kernel K . In general $F_a \in C^0 \cap L_\infty(\mathbb{R}^n)$, so together (4.4)–(4.5) yield (4.2).

In the exploitation of (4.4), it is rather straightforward to control the maximal function $u^*(x)$ with polynomial bounds. Eg, if N is greater than the order of \hat{u} , the Paley–Wiener–Schwartz Theorem gives $|u(y)| \leq c(1+|y|)^N \leq (1+|x|)^N(1+R|x-y|)^N$ when $R \geq 1$, so that in this case

$$u^*(N, R; x) \leq c(1+|x|)^N. \quad (4.6)$$

Moreover, the maximal operator $u \mapsto u^*$ is bounded with respect to the L_p -norm on $L_p \cap \mathcal{F}^{-1}\mathcal{E}'$,

$$\int_{\mathbb{R}^n} u^*(N, R; x)^p dx \leq C_p \int_{\mathbb{R}^n} |u(x)|^p dx, \quad 0 < p \leq \infty, \quad N > n/p. \quad (4.7)$$

Consequently the ‘trilogy’ (4.4), (4.5), (4.7) leads at once to bounds of pseudo-differential operators on $L_p \cap \mathcal{F}^{-1}\mathcal{E}'$,

$$\int |a(x, D)u(x)|^p dx \leq \|F_a\|_\infty^p \int u^*(x)^p dx \leq C_p \|F_a\|_\infty^p \int |u(x)|^p dx. \quad (4.8)$$

As $\|F_a\|_\infty = \sup |F_a(N, R; \cdot)|$ depends on R , this extends to all $u \in L_p$ only if $a(x, \eta)$ has further properties. But it is noteworthy that above boundedness holds whenever $0 < p \leq \infty$, so it was stated as a result in [Joh10a, Cor. 4.4], and in the type 1, 1-context in [Joh10a, Thm. 6.1]; cf Remark 6.4.3 below.

With a little more effort, mainly by renouncing on the compact spectrum of u , a transparent proof of the fact that $a(x, D)$ is a map $\mathcal{O}_M \rightarrow \mathcal{O}_M$ was also obtained in this way; cf [Joh10a, Cor. 4.3]. However, for type 1, 1-operators, this result requires another proof because it is not clear a priori that \mathcal{O}_M is contained in $D(a(x, D))$; cf Section 5.3 below.

These estimates of $a(x, D)$ are a bit paradoxical because the map $u \mapsto u^*$ is non-linear; but this is just a minor drawback as (4.7) was shown by elementary means in [Joh10a]. (The previous proofs of (4.7) in the literature invoke L_p -boundedness of the Hardy–Littlewood maximal function.)

REMARK 4.1.1. It deserves to be mentioned that somewhat different pointwise estimates were introduced by J. Marschall in his thesis [Mar85] and exploited in eg [Mar91, Mar95, Mar96]. For symbols $b(x, \eta)$ in $L_{1, \text{loc}}(\mathbb{R}^{2n}) \cap \mathcal{S}'(\mathbb{R}^{2n})$ with support in $\mathbb{R}^n \times \bar{B}(0, 2^k)$ and $\text{supp } \mathcal{F}u \subset \bar{B}(0, 2^k)$, $k \in \mathbb{N}$, Marschall’s inequality states that

$$|b(x, D)v(x)| \leq c \|b(x, 2^k \cdot)\|_{\dot{B}_{1,t}^{n/t}} M_t u(x), \quad 0 < t \leq 1. \quad (4.9)$$

Here $M_t u(x) = \sup_{r>0} (r^{-n} \int_{B(x,r)} |u(y)|^t dy)^{1/t}$ is the Hardy–Littlewood maximal function of u , when $t = 1$, while the norm of the homogeneous Besov space $\dot{B}_{1,t}^{n/t}$ falls on the dilated symbol $a(x, 2^k \cdot)$ parametrised by x . Under the natural condition that the right-hand side is in $L_{1, \text{loc}}(\mathbb{R}^n)$ it was proved in [Joh05], to which the reader is referred for details; some shortcomings in Marschall’s exposition in eg [Mar96] were pointed out in [Joh05, Rem. 4.2]. Cf also [Joh10a, Rem. 4.11] and [Joh10c, Rem. 7.3]. Marschall’s inequality is mentioned merely for the sake of completeness; it is not feasible for the general study of type 1, 1-operators.

In addition to the above observation that the symbol factor $F_a(x)$ is a bounded continuous function, basic properties of the Fourier transformation yield the following estimate, that is reminiscent of the Mihlin–Hörmander multiplier condition:

THEOREM 4.1.2 ([Joh10a, Thm. 4.5]). *Let the symbol factor $F_a(N, R; x)$ be given by (4.5) for parameters $R, N > 0$, with the auxiliary function taken as $\chi = \psi(R^{-1}\cdot)$ for $\psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a set with non-empty interior. Then it holds for all $x \in \mathbb{R}^n$ that*

$$0 \leq F_a(x) \leq c_{n,k} \sum_{|\alpha| \leq k} \left(\int_{R \operatorname{supp} \psi} |R^{|\alpha|} D_\eta^\alpha a(x, \eta)|^2 \frac{d\eta}{R^n} \right)^{1/2} \quad (4.10)$$

when k is the least integer satisfying $k > N + n/2$.

Although the above result has a straightforward proof, it nevertheless deserves to be presented as a theorem because it has a very central role. On the one hand, this will be clear later in the proof of Theorem 6.3.5, where it allows an exploitation of the profound condition on the twisted diagonal of L. Hörmander, which is phrased with similar integrals.

On the other hand, it is also most convenient for the more standard Littlewood–Paley analysis of pseudo-differential operators; but in this connection it applies through its corollaries given below.

First of all, more refined estimates in terms of symbol seminorms yield $\|F_a\|_\infty = \mathcal{O}(R^{d'})$ for $d' = \max(d, [N + n/2] + 1)$. However, the exponent can be much improved here in case the auxiliary function in the symbol factor is supported in a corona:

COROLLARY 4.1.3 ([Joh10a, Cor. 3.4]). *Let $a(x, \eta)$ be given in $S_{1,\delta}^d(\mathbb{R}^n \times \mathbb{R}^n)$ whilst N, R and ψ have the same meaning as in Theorem 4.1.2. When $R \geq 1$ and $k > N + n/2$, $k \in \mathbb{N}$, then there is a seminorm p on $S_{1,\delta}^d$ and some $c_k > 0$ independent of R such that*

$$0 \leq F_a(x) \leq c_k p(a) R^{\max(d,k)} \quad \text{for all } x \in \mathbb{R}^n. \quad (4.11)$$

Moreover, if $\operatorname{supp} \psi$ is contained in a corona

$$\{\eta \mid \theta_0 \leq |\eta| \leq \Theta_0\}, \quad (4.12)$$

and $\psi(\eta) = 1$ holds for $\theta_1 \leq |\eta| \leq \Theta_1$, whereby $0 \neq \theta_0 < \theta_1 < \Theta_1 < \Theta_0$, then

$$0 \leq F_a(x) \leq c'_k R^d p(a) \quad \text{for all } x \in \mathbb{R}^n, \quad (4.13)$$

with $c'_k = c_k \max(1, \theta_0^{d-k}, \theta_0^d)$.

The above asymptotics for $R \rightarrow \infty$ can be further reinforced when $a(x, \eta)$ has vanishing moments with respect to x , eg if $\hat{a}(\cdot, \eta)$ has a zero around $\xi = 0$. A simple result of this type is obtained by subjecting the symbol to a frequency modulation in its x -dependence, using a Fourier multiplier $\varphi(Q^{-1}D_x)$ that depends on a second spectral quantity Q :

COROLLARY 4.1.4 ([Joh10a, Cor. 4.9]). *When $a_Q(x, \eta) = \varphi(Q^{-1}D_x)a(x, \eta)$ for some $a \in S_{1,\delta}^d$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi = 0$ in a neighbourhood of $\xi = 0$, then there is a seminorm p on $S_{1,\delta}^d$ and constants c_M , depending only on M, n, N, ψ and φ , such that for $R \geq 1, M > 0, Q > 0$,*

$$0 \leq F_{a_Q}(N, R; x) \leq c_M p(a) Q^{-M} R^{\max(d+\delta M, [N+n/2]+1)}. \quad (4.14)$$

Here $d + \delta M$ can replace the maximum when the auxiliary function ψ in F_{a_Q} fulfils the corona condition in Corollary 4.1.3.

Not surprisingly, it is very convenient to have an adaptation of (4.6) to the frequency modulated symbols appearing in Definition 3.1.2. One such result is

PROPOSITION 4.1.5 ([Joh10c, Prop. 3.5]). *For $a(x, \eta)$ in $S_{1,\delta}^d(\mathbb{R}^n \times \mathbb{R}^n)$ and arbitrary $\Phi, \Psi \in C_0^\infty(\mathbb{R}^n)$, for which Ψ is constant in a neighbourhood of the origin and is supported by $\overline{B}(0, R)$ for $R \geq 1$, there is a constant $c > 0$ such that for all $k \in \mathbb{N}$, $N \geq \text{order}_{\mathcal{S}'}(\mathcal{F}v)$,*

$$|\text{OP}(\Phi(2^{-k}D_x)a(x, \eta)\Psi(2^{-k}\eta))v(x)| \leq c2^{k(N+d)_+}(1+|x|)^N. \quad (4.15)$$

Here the positive part $(N+d)_+ = \max(0, N+d)$ is redundant when $0 \notin \text{supp}\Psi$.

One of the points here is that the cutoff functions Φ, Ψ can be rather arbitrary, and that c is independent of k . The temperate order denoted $\text{order}_{\mathcal{S}'}$ in the proposition is for $u \in \mathcal{S}'$ introduced as the smallest integer N such that u fulfils the estimate

$$|\langle u, \psi \rangle| \leq c \sup\{(1+|x|)^N |D^\alpha \psi(x)| \mid x \in \mathbb{R}^n, |\alpha| \leq N\}, \quad \text{for } \psi \in \mathcal{S}. \quad (4.16)$$

Clearly one has $\text{order}_{\mathcal{S}'}(u) \geq \text{order}(u)$, but the notion plays only a minor technical role.

The inequalities (4.4), (4.7) are in fact relatively easy to show, but the passage to estimates in Sobolev spaces H_p^s requires Littlewood–Paley decompositions (which works well, cf (VII)). However, when treating these, the results of the next section are most convenient:

4.2. The spectral support rule

Seen as a temperate distribution, $a(x, D)u$ has a spectrum consisting of the frequencies belonging to $\text{supp}\mathcal{F}(a(x, D)u)$. Concerning this one has as a new result the *spectral support rule*, which in case $\text{supp}\hat{u} \Subset \mathbb{R}^n$ states that

$$\text{supp}\mathcal{F}a(x, D)u \subset \{\xi + \eta \mid (\cdot, \cdot) \in \text{supp}\hat{a}(\xi, \eta), \eta \in \text{supp}\hat{u}\}. \quad (4.17)$$

Cf the original statements in [Joh05, Joh08b] or [Joh10c, App. B] for more general versions.

It is instructive to note that (4.17) also can be written as

$$\text{supp}\mathcal{F}a(x, D)\mathcal{F}^{-1}\hat{u} \subset \text{supp}\mathcal{K} \circ \text{supp}\hat{u}, \quad (4.18)$$

where \mathcal{K} denotes the distribution kernel of $\mathcal{F}a(x, D)\mathcal{F}^{-1}$, ie of the conjugation of $a(x, D)$ by the Fourier transformation, that also appears on the left-hand side. Clearly this resembles the rule for $\text{supp}a(x, D)u$; cf (3.13). It is also related to the well-known formula for symbols $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$,

$$\mathcal{F}a(x, D)u(x) = (2\pi)^{-n} \int \hat{a}(\xi - \eta, \eta) d\eta, \quad u \in \mathcal{S}. \quad (4.19)$$

Indeed, an inspection shows that

$$\mathcal{K}(\xi, \eta) = (2\pi)^{-n} \hat{a}(\xi - \eta, \eta) = (2\pi)^{-n} \mathcal{F}_{(x,y) \rightarrow (\xi, \eta)} K(\xi, -\eta). \quad (4.20)$$

Therefore (4.17)–(4.18) are plausible, since this shows that \hat{a} essentially gives the full frequency content of the kernel K .

The result in (4.17) is a novelty already for classical $a(x, \eta)$. It holds trivially if $a(x, \eta)$ is an elementary symbol, which were introduced in 1978 by R. Coifman and Y. Meyer [CM78] specifically for the purpose of controlling the spectrum $\text{supp } \mathcal{F}a(x, D)u$ in Littlewood–Paley analysis of $a(x, D)u$. Indeed, elementary symbols are by definition given as a series of products

$$a(x, \eta) = \sum_{j=0}^{\infty} m_j(x) \Phi_j(\eta) \quad (4.21)$$

whereby (m_j) is a sequence in $L_{\infty}(\mathbb{R}^n)$ and $1 = \sum_{j=0}^{\infty} \Phi_j$ is a Littlewood–Paley partition of unity, that is Φ_j is in C^{∞} with support where $2^{j-1} \leq |\eta| \leq 2^{j+1}$ for $j \geq 1$. For such symbols in $S_{1,0}^d$ every $u \in \mathcal{F}^{-1} \mathcal{C}'(\mathbb{R}^n)$ gives a finite sum

$$a(x, D)u = \sum m_j(x) \Phi_j(D)u, \quad (4.22)$$

for which the support rule for convolutions immediately yields

$$\begin{aligned} \text{supp } \mathcal{F}(a(x, D)u) &= \text{supp } ((2\pi)^{-n} \sum \hat{m}_j * (\Phi_j \hat{u})) \\ &\subset \bigcup \{ \xi + \eta \mid \xi \in \text{supp } \hat{m}_j, \eta \in \text{supp } \Phi_j \cap \text{supp } \hat{u} \} \\ &\subset \{ \xi + \eta \mid (\xi, \eta) \in \text{supp } \hat{a}, \eta \in \text{supp } \hat{u} \}. \end{aligned} \quad (4.23)$$

This shows that the spectral support rule holds for elementary symbols.

However, it should be mentioned that there is an equally simple proof for *arbitrary* symbols $a \in S_{1,0}^d$: When $v \in C_0^{\infty}(\mathbb{R}^n)$ has support disjoint from $\text{supp } \mathcal{K} \circ \text{supp } \hat{u}$ and $\text{supp } \hat{u}$ is compact, then it is clear that $\text{dist}(\text{supp } \mathcal{K}, \text{supp}(v \otimes \hat{u})) > 0$. So by mollification, say $\hat{u}_{\varepsilon} = \varphi_{\varepsilon} * \hat{u}$ for some $\varphi \in C_0^{\infty}(\mathbb{R}^n)$ with $\hat{\varphi}(0) = 1$, $\varphi_{\varepsilon} = \varepsilon^{-n} \varphi(\cdot/\varepsilon)$, all sufficiently small $\varepsilon > 0$ give

$$\text{supp } \mathcal{K} \cap \text{supp } v \otimes \hat{u}_{\varepsilon} = \emptyset. \quad (4.24)$$

Therefore (4.18) follows at once, since

$$\langle \mathcal{F}a(x, D) \mathcal{F}^{-1} \hat{u}, v \rangle = \lim_{\varepsilon \rightarrow 0} \langle \mathcal{F}a(x, D) \mathcal{F}^{-1} \hat{u}_{\varepsilon}, v \rangle = \lim_{\varepsilon \rightarrow 0} \langle \mathcal{K}, v \otimes \hat{u}_{\varepsilon} \rangle = 0 \quad (4.25)$$

is obtained simply by using that $\mathcal{F}a(x, D) \mathcal{F}^{-1}$ is continuous in \mathcal{S}' and that $\hat{u}_{\varepsilon} \in C_0^{\infty}(\mathbb{R}^n)$. (This argument is taken from [Joh10c, App. B].)

The spectral support rule (4.17) was probably anticipated by some, but seemingly neither formulated nor proved. Indeed, in their works on L_p -estimates, J. Marschall [Mar91, Mar96] and T. Runst [Run85b] both tacitly avoided elementary symbols and as needed stated consequences of (4.17), albeit without adequate arguments; cf the remarks in [Joh05]. Anyhow, due to (4.17), the cumbersome reduction to elementary symbols is usually unnecessary.

Generalisations to the case in which $\text{supp } \hat{u}$ need not be compact (in which case one should take the closure of the right-hand sides of (4.17)–(4.18)) and to the case of type 1, 1-operators also exist, cf Section 5.6 below. However, the proofs for these cases were based on some subtle parts of distribution theory:

4.3. Stability of extended distributions under regular convergence

If $u, f \in \mathcal{D}'(\mathbb{R}^n)$ only “overlap” in a mild way, more precisely,

$$\text{supp } u \cap \text{supp } f \Subset \mathbb{R}^n \quad (4.26)$$

$$\text{sing supp } u \cap \text{sing supp } f = \emptyset, \quad (4.27)$$

and $\text{supp } u$ is compact, it is natural and classical (cf [Hör85, Sect. 3.1]) that fu is defined in $\mathcal{D}'(\mathbb{R}^n)$, whence $\langle u, f \rangle$ can be defined using $\text{supp } u \Subset \mathbb{R}^n$ as

$$\langle u, f \rangle = \langle fu, 1 \rangle. \quad (4.28)$$

It is easy to see that this well-known extension of the distribution u , or rather of the scalar product $\langle \cdot, \cdot \rangle$ is discontinuous in general. Eg $f = 0$ can be approached by $f_\nu = \exp(-\nu x^2)$ in $\mathcal{D}'(\mathbb{R})$, that for $u = \delta_0$ gives $fu = 0 \neq \delta_0 = \lim f_\nu u$, hence for the scalar product yields $\langle u, f \rangle = 0 \neq 1 = \lim \langle u, f_\nu \rangle$.

However, the extension does have an important property of stability:

THEOREM 4.3.1. *For the above extension it holds that*

$$\langle u, f_\nu \rangle \rightarrow \langle u, f \rangle \quad \text{for } \nu \rightarrow \infty, \quad (4.29)$$

provided $f_\nu \in C^\infty(\mathbb{R}^n)$ and $f_\nu \xrightarrow[\nu \rightarrow \infty]{} f$ both in $\mathcal{D}'(\mathbb{R}^n)$ and in $C^\infty(\mathbb{R}^n \setminus \text{sing supp } f)$.

The full set of results is collected in [Joh08b, Thm. 7.2]. Eg it is possible to have convergence of (f_ν) in the topology of C^∞ over a smaller open set if only this contains the singular support of u (which is unfulfilled for (f_ν) in the above example).

Sequences as in Theorem 4.3.1 have been used repeatedly for type 1,1-operators, so the following notion is introduced, inspired by a reference to $\mathbb{R}^n \setminus \text{sing supp } f$ as the regular set of f :

DEFINITION 4.3.2. A sequence $f_\nu \in C^\infty(\mathbb{R}^n)$ is said to converge *regularly* to the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ whenever $f = \lim_\nu f_\nu$ holds in $\mathcal{D}'(\mathbb{R}^n)$ as well as in $C^\infty(\mathbb{R}^n \setminus \text{sing supp } f)$, that is, if for $\nu \rightarrow \infty$,

$$\langle f_\nu - f, \varphi \rangle \rightarrow 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n) \quad (4.30)$$

$$\sup_{x \in K} |D^\alpha f_\nu(x) - D^\alpha f(x)| \rightarrow 0 \quad \text{for all } \alpha \in \mathbb{N}_0^n, \quad K \Subset \mathbb{R}^n \setminus \text{sing supp } f. \quad (4.31)$$

This definition was made (implicitly) in connection with [Joh08b, Thm. 7.2]. The result below shows that mollification automatically yields regular convergence, for which reason it was termed the Regular Convergence Lemma in [Joh08b, Lem. 6.1]:

LEMMA 4.3.3. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be given and take a sequence $\varepsilon_k \rightarrow 0^+$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\psi(\varepsilon_k D)u \rightarrow \psi(0) \cdot u \quad \text{for } k \rightarrow \infty \quad (4.32)$$

in the Fréchet space $C^\infty(\mathbb{R}^n \setminus \text{sing supp } u)$. If $\mathcal{F}^{-1}\psi \in C_0^\infty$ this extends to all $u \in \mathcal{D}'$, provided $\psi(\varepsilon_k D)u$ is replaced by $\mathcal{F}^{-1}(\psi(\varepsilon_k \cdot)) * u$.

The last part of this lemma is easy to deduce, using a cutoff function equal to 1 on a neighbourhood of the given compact set, where the derivatives should converge uniformly. Only the \mathcal{S}' -part requires a more explicit proof.

However, despite the Regular Convergence Lemma's content, the broader notion of regular convergence is convenient because such sequences are invariant under eg linear coordinate changes, multiplication by cutoff functions and tensor products $f \mapsto f \otimes g$ when $g \in C^\infty$.

These remarks are useful in connection with Schwartz' kernel formula. Recall that for a continuous operator $A: \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$, its distribution kernel $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies, for all $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle Au, v \rangle = \langle K, v \otimes u \rangle. \quad (4.33)$$

First of all, this can be related to the vanishing frequency modulation adopted for type 1, 1-operators. Indeed, when $a(x, D)u = \lim_{m \rightarrow \infty} \text{OP}(\psi(2^{-m}D_x)a(x, \eta)\psi(2^{-m}\eta))u$ and the m^{th} term is written $A_m u$, then A_m has distribution kernel $K_m(x, y)$ given by a convolution conjugated by a change of coordinates (cf [Joh08b, Prop. 5.11]), namely

$$K_m(x, y) = 4^{mn} \mathcal{F}^{-1}(\psi \otimes \psi)(2^m \cdot) * (K \circ \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix})(x, x - y). \quad (4.34)$$

Because of the Regular Convergence Lemma, this C^∞ -function converges *regularly* to K for $m \rightarrow \infty$. Therefore $K_m \rightarrow K$ in $\mathcal{S}'(\mathbb{R}^n)$ as well as in $C^\infty(\mathbb{R}^n \setminus \{x = y\})$.

However, with a suitable cutoff function this gives convergence in the Schwartz space:

PROPOSITION 4.3.4 ([Joh08b, Prop. 6.3]). *If $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ has bounded derivatives of any order with $\text{supp } f$ bounded with respect to x and disjoint from the diagonal, then*

$$f(x, y)K_m(x, y) \xrightarrow{m \rightarrow \infty} f(x, y)K(x, y) \quad \text{in } \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n). \quad (4.35)$$

It is noteworthy that the proof of this plausible proposition relies on the mentioned less trivial part of the Regular Convergence Lemma, in which the function $\mathcal{F}^{-1}\psi$ there is in $\mathcal{S} \setminus C_0^\infty$. Certainly Proposition 4.3.4 sheds light on the limit in Definition 3.1.2, but it is also a convenient proof ingredient later.

Secondly, (4.33) is by (4.29) easily extended to the pairs $(u, v) \in \mathcal{S}'(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ fulfilling

$$\text{supp } K \cap \text{supp}(v \otimes u) \in \mathbb{R}^n \times \mathbb{R}^n, \quad (4.36)$$

$$\text{sing supp } K \cap \text{sing supp}(v \otimes u) = \emptyset. \quad (4.37)$$

THEOREM 4.3.5 ([Joh08b, Thm. 7.4]). *If $A: \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous and (4.36), (4.37) are fulfilled, then $\langle Au, v \rangle = \langle K, v \otimes u \rangle$ holds with extended action of the scalar product. This extends to the \mathcal{D}' -case.*

It is illuminating to give the short argument: the right-hand side of (4.33) is defined according to (4.36)–(4.37) and the extension of $\langle \cdot, \cdot \rangle$ in (4.28), so it only remains to verify the equality in (4.33) under the assumptions (4.36)–(4.37).

For this one can clearly take $\kappa, \chi \in C_0^\infty(\mathbb{R}^n)$ such that $\kappa = 1$ on $\text{supp } v$ and $\kappa(x)\chi(y) = 1$ on the compact set in (4.36). By letting $u_v \in C^\infty(\mathbb{R}^n)$ tend regularly to u , cf Lemma 4.3.3, the

convergence $v \otimes u_v \rightarrow v \otimes u$ is also regular, so one finds from Theorem 4.3.1,

$$\begin{aligned} \langle K, v \otimes u \rangle &= \langle (\kappa \otimes \chi)K, v \otimes u \rangle = \lim_{v \rightarrow \infty} \langle (\kappa \otimes \chi)K, v \otimes u_v \rangle \\ &= \lim_{v \rightarrow \infty} \langle K, (\kappa v) \otimes (\chi u_v) \rangle_{\mathcal{S}' \times \mathcal{S}} = \lim_{v \rightarrow \infty} \langle A(\chi u_v), \kappa v \rangle_{\mathcal{S}' \times \mathcal{S}} = \langle Au, v \rangle. \end{aligned} \quad (4.38)$$

This proves the theorem.

As consequences it should be pointed out that the support rule (3.13) follows at once from (4.33) for $A = a(x, D) \in \mathcal{S}_{1,0}^d(\mathbb{R}^n \times \mathbb{R}^n)$ by taking $v \in C_0^\infty(\mathbb{R}^n)$ with support disjoint from that of $\text{supp} K \circ \text{supp} u$. It is noteworthy that also the *spectral* support rule (3.14),(4.18) follows in this way for $A = \mathcal{F}a(x, D)\mathcal{F}^{-1}$, for this is also continuous on \mathcal{S}' for such $a(x, D)$.

Type 1, 1-operators requires some additional efforts due to the limit $m \rightarrow \infty$ in (3.4). The main line is the same as the above, that roughly speaking applies for each m ; for (3.13) the convergence in Proposition 4.3.4 was sufficient, cf [Joh08b, Sect. 7–8]. For the spectral support rule (4.17) the passage to the limit $m \rightarrow \infty$ required an extra assumption (\mathcal{S}' -convergence in (3.4)), but still the main ingredient was stability under regular convergence in the kernel formula.

4.3.1. Other extensions. Among the many possible extensions of $\langle \cdot, \cdot \rangle$, it is particularly relevant to recall the one related to the space \mathcal{D}'_Γ consisting of the $u \in \mathcal{D}'$ with $\text{WF}(u) \subset \Gamma$, whereby $\Gamma \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is a fixed closed, conical set (ie Γ is invariant under scaling by positive reals in the second entry). \mathcal{D}'_Γ is given a stronger topology than the relative by adding the seminorms

$$p_{\varphi, N, V}(u) = \sup_{\eta \in V} (1 + |\eta|)^N |\widehat{\varphi u}(\eta)|, \quad N = 1, 2, \dots, \quad (4.39)$$

where $\varphi \in C_0^\infty$ and the closed cone $V \subset \mathbb{R}^n$ run through those with $\Gamma \cap (\text{supp } \varphi \times V) = \emptyset$.

For cones Γ_1, Γ_2 such that $(x, -\eta) \notin \Gamma_2$ whenever $(x, \eta) \in \Gamma_1$, there is an extension of $\langle \cdot, \cdot \rangle$ to a bilinear map $\mathcal{D}'_{\Gamma_1} \times \mathcal{E}'_{\Gamma_2} \rightarrow \mathbb{C}$, which is sequentially continuous in each variable; this is eg explained in the notes of A. Grigis and J. Sjöstrand [GS94, Prop. 7.6]. Obviously the wavefront condition expressed via Γ_1, Γ_2 is weaker than disjointness of the singular supports.

On the other hand, any sequence $f_v \in C^\infty(\mathbb{R}^n)$ such that $f_v \rightarrow f$ in \mathcal{D}'_Γ automatically tends regularly to f in the sense of Definition 4.3.2, for the supremum over K there goes to 0 because it can be estimated by $p_{\varphi, |\alpha|+n+1, \mathbb{R}^n}(f_v - f)$ when $\varphi = 1$ on K and $\text{supp } \varphi \cap \text{sing supp } f = \emptyset$ (allowing $V = \mathbb{R}^n$), using that \mathcal{F} is bounded from L_1 to L_∞ .

The incompatibility of the two extensions becomes clearer by noting that $\langle u, f \rangle$ is defined whenever the product fu makes sense in \mathcal{E}' ; cf (4.28). Eg one may use the product $\pi(f, u)$ defined formally by regarding f as a (non-smooth) symbol independent of η (cf Remark 1.1 in [Joh08b], or the author's paper [Joh95] devoted to $\pi(f, u)$). It is well known that $\pi(\delta_0, H) = \delta_0/2$, when $H = 1_{\mathbb{R}_+}$ is the Heaviside function, so from this one finds $\langle \delta_0, H \rangle = \langle \frac{1}{2}\delta_0, 1 \rangle = \frac{1}{2}$. (As $\text{WF}(\delta_0) = \{0\} \times \mathbb{R}$, wavefront sets are not useful here.)

However, as the point of the regular convergence is to simplify (and to emphasise the essential), this notion should be well motivated.

REMARK 4.3.6. Both parts of the Regular Convergence Lemma could have been known since the 1950's in view of its content, of course. The same could be said about the stability in Theorem 4.3.1 and the resulting kernel convergence in Proposition 4.3.4 as well as the extended

kernel formula in Theorem 4.3.5. But it has not been possible to track any evidence of this, neither written nor as folklore.

CHAPTER 5

Review of qualitative results

This chapter gives a detailed account of the results summarised in items (I)–(VI) in Section 3.2. The review follows the order there.

For convenience $a(x, \eta)$ denotes an arbitrary symbol in $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$.

5.1. Consistency among extensions

The definition by vanishing frequency modulation has the merit of giving back most, if not all, of the previous extension of type 1, 1-operators. This is reviewed in the subsections below.

5.1.1. Extension to functions with compact spectrum. First of all there was in [Joh04, Joh05] a mild extension of $a(x, D)$ to $\mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n)$. The extension is rather elementary, but is easy to explain with a point of view from [Joh08b]: the defining integral may be seen as a scalar product for $u \in \mathcal{F}^{-1}C_0^\infty(\mathbb{R}^n)$

$$a(x, D)(x)u = \left\langle \hat{u}, a(x, \cdot) \frac{e^{i\langle x, \cdot \rangle}}{(2\pi)^n} \right\rangle_{\mathcal{E}' \times C_0^\infty}. \quad (5.1)$$

On the right-hand side one is free to insert any $\hat{u} \in \mathcal{E}'$, which is consistent with (1.1) because $\mathcal{S} \cap \mathcal{F}^{-1}\mathcal{E}' = \mathcal{F}^{-1}C_0^\infty$.

More precisely this gives an extension to a map $\tilde{a}(x, D): \mathcal{S} + \mathcal{F}^{-1}\mathcal{E}' \rightarrow C^\infty$ given by

$$\tilde{a}(x, D)u = a(x, D)v + \text{OP}(a(x, \eta)\chi(\eta))v' \quad (5.2)$$

when $u = v + v'$ for some $v \in \mathcal{S}$ and $v' \in \mathcal{F}^{-1}\mathcal{E}'$ whilst $\chi \in C^\infty$ is an arbitrary function equalling 1 on neighbourhood of $\text{supp } \hat{u}$. Indeed, χ can be inserted already in (5.1), and since the resulting symbol $a(x, \eta)\chi(\eta)$ is in $S^{-\infty}$ the formula for $\tilde{a}(x, D)$ makes sense. The value of $\tilde{a}(x, D)u$ is also independent of how v, v' are chosen, as can be seen using linearity and (5.1).

As examples of the above extension, type 1, 1-operators are always defined on polynomials $\sum_{|\alpha| \leq m} a_\alpha x^\alpha$, plane waves $e^{ix \cdot z}$ and also on the less trivial function $\frac{\sin x_1}{x_1} \dots \frac{\sin x_n}{x_n}$, since this is equal to $\pi^n \mathcal{F}^{-1}1_{[-1,1]^n}$.

It was verified in [Joh08b, Cor. 4.7] that this extension is contained in the operator defined by vanishing frequency modulation. However, this also results from the next section.

5.1.2. Extension to slowly growing functions. Following an early remark by G. Bounded [Bou88b] (who treated singular integral operators) one can obtain that every type 1, 1 symbol $a(x, \eta)$ gives rise to a map

$$\tilde{A}: \mathcal{O}_M(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n). \quad (5.3)$$

Hereby \mathcal{O}_M stands for the space of C^∞ -functions that together with all their derivatives have polynomial growth at infinity.

Indeed, for each $f \in \mathcal{O}_M$ one may take $\tilde{A}f$ as the distribution that for each $\varphi \in C_0^\infty(\mathbb{R}^n)$, and $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on a neighbourhood of φ , is given by

$$\langle \tilde{A}f, \varphi \rangle = \langle a(x, D)(\chi f), \varphi \rangle + \iint K(x, y)(1 - \chi(y))f(y)\varphi(x) dy dx. \quad (5.4)$$

Here the distribution kernel $K(x, y)$ decays rapidly for fixed x and $|y| \rightarrow \infty$, so that the integral makes sense. The right-hand side gives the same value for any other such cutoff function $\tilde{\chi}$, for an analogous integral defined from $\chi - \tilde{\chi}$ has the opposite sign of $\langle a(x, D)((\tilde{\chi} - \chi)f), \varphi \rangle$. In view of this independence, and since the absolute value is less than a constant times $\sup|\varphi|$, $\tilde{A}f$ defines a distribution in $\mathcal{D}'(\mathbb{R}^n)$.

It can also be seen that $\tilde{A}f$ is smooth and slowly increasing, and with some effort that \tilde{A} is in fact a restriction of $a(x, D)$ defined by vanishing frequency modulation:

PROPOSITION 5.1.1. *Each $a(x, D)$ in $\text{OP}(S_{1,1}^d)$ restricts to a map $\mathcal{O}_M(\mathbb{R}^n) \rightarrow \mathcal{O}_M(\mathbb{R}^n)$.*

This result contains the previous extension to $\mathcal{S} + \mathcal{F}^{-1}\mathcal{E}'$ in (5.2), and it is rather more precise. Of course it looks like being a completion, but this is not obvious as neither the topology on $\mathcal{O}_M(\mathbb{R}^n)$ nor continuity is involved in the statement.

The proposition is given without details here, as it is superseded by an extension to $C^\infty \cap \mathcal{S}'$, which is derived from a closer inspection of \tilde{A} . However, this result follows in Theorem 5.3.1 below, because it is rather more important in itself.

REMARK 5.1.2. In a remark preceding the proof of the $T1$ -theorem of G. David and J.-L. Journé [DJ84], it was explained that just a few properties of the distribution kernel of a continuous map $T: C_0^\infty(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ implies that $T(1)$ is well defined modulo constants. In particular this was applied to $T \in \text{OP}(S_{1,1}^0)$, but in that case their extension equals the above, hence by Proposition 5.1.1 also gives the same result as Definition 3.1.2.

5.1.3. Extension by continuity. In [Hör88, Hör89], L. Hörmander characterised the $s \in \mathbb{R}$ for which a given $a(x, D) \in \text{OP}(S_{1,1}^d)$ extends by continuity to a bounded operator $H^{s+d} \rightarrow H^s$; the only possible exception was a certain limit point s_0 that was not treated, cf [Hör97]. More precisely (paraphrasing his results) he obtained a largest interval $]s_0, \infty[\ni s$ together with constants C_s such that

$$\|a(x, D)u\|_{H^s} \leq C_s \|u\|_{H^{s+d}} \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^n). \quad (5.5)$$

Here $s_0 \leq 0$ always holds. Conversely existence of such a C_s was shown to imply $s \geq s_0$. More precisely, $s_0 = -\sup \sigma$ when σ runs through the values for which the symbol fulfils (1.17).

R. Torres [Tor90] worked with the full scale of Lizorkin–Triebel spaces $F_{p,q}^s$. His methods relied on the framework of atoms and molecules of M. Frazier and B. Jawerth [FJ85, FJ90], but he also estimated $a(x, D)u$ for $u \in \mathcal{S}(\mathbb{R}^n)$. This gave extensions by continuity to maps

$$A: F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n) \quad (5.6)$$

for $s > \max(0, \frac{n}{p} - n, \frac{n}{q} - n)$ and $0 < p < \infty$, $0 < q \leq \infty$, and more generally for all s so large that for all multiindices γ it holds true that

$$0 \leq |\gamma| < \max(0, \frac{n}{p} - n, \frac{n}{q} - n) - s \implies \text{supp } \mathcal{F}(a(x, D)^* x^\gamma) \in \mathcal{E}'(\mathbb{R}^n). \quad (5.7)$$

Of course the uniqueness of these extensions yield that they coincide with Definition 3.1.2 whenever the same continuity properties of $a(x, D)$ can be proved by other means. This has to a large extent been done with paradifferential decompositions, as reviewed in the section below.

Indeed, in the H_p^s context with $s > 0$ and $1 < p < \infty$ this was done in [Joh08b, Thm. 9.2], which covers the above result of L. Hörmander for $s > 0$, and also for $s \in \mathbb{R}$ in case $a(x, D)$ fulfils the twisted diagonal condition (1.16); and done in [Joh10c, Cor. 7.6] for $s \in \mathbb{R}$ when (1.17) holds for every $\sigma \in \mathbb{R}$. These results were in fact just special cases of the $F_{p,q}^s$ results in [Joh05, Joh10c], so for the same s also the extensions of R. Torres are restrictions of the operator defined by vanishing frequency modulation; cf Theorems 5.1.3 and 6.5.2 below.

However, it should be emphasized that full coherence has not yet been obtained. For $s \leq 0$ the above extensions by continuity still require treatment when $a(x, \eta)$ only satisfies (1.17) for a specific σ , for continuity has then been shown with the present methods for $s > -\sigma + [n/2] + 2$; cf Remark 7.11 in [Joh10c].

5.1.4. Extensions through paradifferential decompositions. As is well known, paradifferential decomposition of $a(x, D)$ yields three contributions to the limit (3.6),

$$a_\psi(x, D) = a_\psi^{(1)}(x, D) + a_\psi^{(2)}(x, D) + a_\psi^{(3)}(x, D). \quad (5.8)$$

The details of this decomposition will be given later in Section 6.1, for here it suffices to explain the philosophy behind it:

- $a_\psi^{(1)}(x, D)u$ has a regularity that depends on u alone (usually);
- the last term $a_\psi^{(3)}(x, D)u$ has a regularity determined by that of the symbol (usually);
- in between there is $a_\psi^{(2)}(x, D)u$ that may or may not be defined — depending on the fine interplay of u and $a(x, \eta)$. This term is the most regular of the three (usually).

In addition to this compelling description, it should be noted that the usefulness lies in the particular form the terms have (cf Section 6.1); they consist of three infinite series. More precisely, each of these can in its turn be treated by methods of harmonic analysis, which are quite simple owing to the above decomposition of the singularities in $a_\psi(x, D)u$.

In the type 1, 1 context, this decomposition goes back to Y. Meyer [Mey81a, Mey81b] and G. Bourdaud [Bou83, Bou88a], but has also been used by numerous authors in several fields.

Quite simply, (5.8) induces an extension of $a(x, D)$ to those $u \in \mathcal{S}'(\mathbb{R}^n)$ for which all the three mentioned series converge in $\mathcal{D}'(\mathbb{R}^n)$. (This was taken as the definition of type 1, 1-operators in [Joh04, Joh05], but was superseded by Definition 3.1.2 in [Joh08b].) When combining the results of this analysis, one finds eg estimates of the form

$$\|a_\psi(x, D)u\|_{H_p^s} \leq \sum_{j=1,2,3} \|a_\psi^{(j)}(x, D)u\|_{H_p^s} \leq C\|u\|_{H_p^{s+d}}. \quad (5.9)$$

Here it is an important point that the inequality will be shown directly for all $u \in H_p^s$ (without extension by continuity). For this purpose the pointwise estimates and the spectral support rule reviewed in Sections 4.1–4.2 are particularly useful.

Moreover, there is here no dependence on the modulation function ψ , for \mathcal{S} is a dense subset on which $a_\psi(x, D) = a(x, D)$. This way the next result was obtained in [Joh05]:

THEOREM 5.1.3. *Let $a(x, \eta)$ be a symbol in $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$. Then for every $s > 0$, $1 < p < \infty$ the type 1, 1-operator $a(x, D)$ has $H_p^{s+d}(\mathbb{R}^n)$ in its domain and it is a continuous linear map*

$$a(x, D): H_p^{s+d}(\mathbb{R}^n) \rightarrow H_p^s(\mathbb{R}^n). \quad (5.10)$$

This property extends to all $s \in \mathbb{R}$ when $a(x, \eta)$ fulfils the twisted diagonal condition (1.16).

There is also a version for the general $F_{p,q}^s$ spaces, as reviewed in Theorem 6.5.2 below.

It is most noteworthy, though, that this result deals directly with the operator $a(x, D)$ defined by vanishing frequency modulation. Phrased with a few words this is because the modulation function, by dilation, gives rise to a Littlewood–Paley partition of unity that can be inserted twice in $a_\psi(x, D)u$ — whereafter bilinearity leads directly to the decomposition (5.8). Section 6.1 below gives the details of this.

5.2. Maximality of the definition by vanishing frequency modulation

It follows from standard results that $a(x, D)$ gives back the usual operator, written $\text{OP}(a)u$, if the definition is applied to some $a(x, \eta) \in S_{1,0}^\infty$; cf [Joh08b, Prop. 5.4]. In addition it is also consistent with the previous extensions, as elucidated in Section 5.1.

But even so the approximants in (3.5) might seem rather arbitrary. That this is not the case is evident from the following characterisation, which justifies the title of this section:

THEOREM 5.2.1 ([Joh08b, Thm. 5.9]). *The map $a \mapsto a(x, D)$ given by Definition 3.1.2 is one among the operator assignments $a \mapsto \widetilde{\text{OP}}(a)$ mapping each $a \in S_{1,1}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ into a linear map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ such that:*

- (i) $\widetilde{\text{OP}}(\cdot)$ is compatible with OP on $S^{-\infty}$, that is, $\widetilde{\text{OP}}(b)$ is defined for all $u \in \mathcal{S}'(\mathbb{R}^n)$ whenever $b(x, \eta) \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\widetilde{\text{OP}}(b)u = \text{OP}(b)u$;
- (ii) each $\widetilde{\text{OP}}(b)$ is stable under vanishing frequency modulation, or explicitly

$$\widetilde{\text{OP}}(b)u = \lim_{m \rightarrow \infty} \widetilde{\text{OP}}(\psi(2^{-m}D_x)b(x, \eta)\psi(2^{-m}\eta))u \quad (5.11)$$

for every modulation function ψ , $u \in D(\widetilde{\text{OP}}(b))$ for a fixed $b \in S_{1,1}^\infty$.

Whenever $\widetilde{\text{OP}}$ is such a map, then $\widetilde{\text{OP}}(a) \subset a(x, D)$ holds in the sense of operator theory for every $a \in S_{1,1}^\infty$.

It should be noted that “from...to” indicates that the operator $\widetilde{\text{OP}}(a)$ is defined just on a subspace of $\mathcal{S}'(\mathbb{R}^n)$, in analogy with the corresponding formulation in the theory of unbounded

operators in Hilbert space. In addition the domain of $\widetilde{\text{OP}}(a)$ is necessarily dense, for as a consequence of (i)–(ii) in the theorem, it follows from (1.5) that $\widetilde{\text{OP}}(a)$ extends $a(x, D)$ in (1.1), that is, $\widetilde{\text{OP}}(a)u = a(x, D)u$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

In [Joh08b, Sect. 4–5] it is also confirmed that Definition 3.1.2 is *strongly compatible* with eg $S_{1,0}^\infty$; ie, despite the limit procedure it gives the result

$$a(x, D)u = \text{OP}(a(x, \eta)\chi(\eta))u \quad (5.12)$$

whenever $u \in \mathcal{S}'(\mathbb{R}^n)$ and $a(x, \eta)\chi(\eta)$ is in $S_{1,0}^\infty$ for some C^∞ -function χ equalling 1 on a neighbourhood of $\text{supp} \hat{u}$. In particular (5.12) holds whenever $u \in \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n)$, as was already seen in Section 5.1, for $a(x, \eta)\chi(\eta)$ belongs to $S^{-\infty}$ when $\chi \in C_0^\infty(\mathbb{R}^n)$ is as above.

These and other questions of extension and compatibility are dealt with at length in [Joh08b, Sect. 4–5].

REMARK 5.2.2. It is of historic interest to relate Definition 3.1.2 to that used by E. M. Stein around 1972 in the extension to Hölder–Zygmund spaces $C^s(\mathbb{R}^n)$ with $s > 0$. For convenience the basis for this will be the exposition in Proposition 3 of [Ste93, VII, §1.3].

The indications will be brief, however, borrowing the Littlewood–Paley decomposition $1 = \sum_{j=0}^\infty \varphi(2^{-j}\eta)$ constructed from an arbitrary modulation function ψ in Section 6.1.1 below. The integer h there is such that $\tilde{\varphi}_j := \varphi(2^{-j+h+1}\cdot) + \dots + \varphi(2^{-j-h-1}\cdot)$ is identical equal to 1 on $\text{supp} \varphi(2^{-j}\cdot)$. Using this, E. M. Stein introduced for $a \in \text{OP}(S_{1,1}^0)$,

$$\widetilde{\text{OP}}(a)u = \sum_{j=0}^\infty \text{OP}(a(x, \eta)\varphi(2^{-j}\eta))\tilde{\varphi}_j(D)u \quad \text{for } u \in C^s(\mathbb{R}^n). \quad (5.13)$$

With arguments from harmonic analysis it was shown in [Ste93] that the series converges in L_∞ and that the induced map is bounded on $C^s(\mathbb{R}^n)$ for $s > 0$. (Independence of ψ was tacitly by-passed in [Ste93].)

With hindsight (5.13) applies as a definition of $\widetilde{\text{OP}}(a)u$ for the $u \in \mathcal{S}'$ for which the right-hand side converges in \mathcal{D}' . Of course the $\tilde{\varphi}_j(D)$ are redundant, so the above amounts to

$$\widetilde{\text{OP}}(a)u = \lim_{k \rightarrow \infty} \text{OP}(a(x, \eta)\psi(2^{-k}\eta))u. \quad (5.14)$$

First of all this shows that Stein's extension implicitly relied on vanishing *partial* frequency modulation, since the symbol is not modified by $\psi(2^{-m}D_x)$ in (5.14).

Secondly, whether $\widetilde{\text{OP}}(a)$ is a restriction of $a(x, D)$ is by means of Theorem 5.2.1 easily reduced to whether $\widetilde{\text{OP}}(a)$ is stable under vanishing frequency modulation, ie to show that

$$\widetilde{\text{OP}}(a)u = \lim_{m \rightarrow \infty} \widetilde{\text{OP}}(\Psi(2^{-m}D_x)a(x, \eta)\Psi(2^{-m}\eta))u \quad (5.15)$$

for every modulation function Ψ , $u \in D(\widetilde{\text{OP}}(a))$; which by definition of $\widetilde{\text{OP}}$ is equivalent to

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \text{OP}(\Psi(2^{-m}D_x)a(x, \eta)\Psi(2^{-m}\eta)\psi(2^{-k}\eta))u \\ = \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \text{OP}(\Psi(2^{-m}D_x)a(x, \eta)\Psi(2^{-m}\eta)\psi(2^{-k}\eta))u. \end{aligned} \quad (5.16)$$

So the question is reduced to that of commuting the two limits above. This is mentioned just to explicate the difficulties in the present question, and indeed in the theory as a whole.

5.3. The maximal smooth space

It turns out that every type 1, 1-operator $a(x, D)$, despite the pathologies it may display, always is defined on the largest possible space of smooth functions, which is $C^\infty \cap \mathcal{S}'$, of course.

The proof of this fact departs from the extension \tilde{A} of G . Bourdaud recalled in Section 5.1.2. To free the discussion from the slow growth in \mathcal{O}_M , one may restate the definition of $\tilde{A}f$ in terms of the tensor product $1 \otimes f$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ acting on $(\varphi \otimes (1 - \chi))K \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, ie

$$\langle \tilde{A}f, \varphi \rangle = \langle a(x, D)(\chi f), \varphi \rangle + \langle 1 \otimes f, (\varphi \otimes (1 - \chi))K \rangle, \quad (5.17)$$

The advantage here is that both terms obviously make sense as long as f is smooth and temperate, ie for every $f \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$. Moreover, the arguments in Section 5.1.2 can then essentially be repeated, which shows that every $f \in C^\infty \cap \mathcal{S}'$ is mapped by \tilde{A} to a well defined distribution; cf [Joh10c, Sect. 2.1.2].

Invoking Definition 3.1.2, this gives that $a(x, D)$ always is a map defined on the *maximal set* of smooth functions $C^\infty \cap \mathcal{S}'$:

THEOREM 5.3.1 ([Joh10c, Thm. 2.7]). *Every $a(x, D) \in \text{OP}(S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n))$ restricts to a map*

$$a(x, D): C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n), \quad (5.18)$$

which maps the subspace $\mathcal{O}_M(\mathbb{R}^n)$ into itself. The restriction is given by (5.17).

PROOF. Let for brevity $A_m = \text{OP}(\psi(2^{-m}D_x)a(x, \eta)\psi(2^{-m}\eta))$ with distribution kernel K_m , so that $a(x, D)u = \lim_m A_m u$ when $u \in D(a(x, D))$. With $f \in C^\infty \cap \mathcal{S}'$ and φ, χ as above, this is the case for $u = \chi f \in C_0^\infty$.

Since the support of $\varphi \otimes (1 - \chi)$ is disjoint from the diagonal and bounded in the x -direction, it was shown by means of the Regular Convergence Lemma in Proposition 4.3.4 that in the topology of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$

$$\varphi(x)(1 - \chi(y))K_m(x, y) \xrightarrow{m \rightarrow \infty} \varphi(x)(1 - \chi(y))K(x, y). \quad (5.19)$$

Exploiting these facts in (5.17) yields that

$$\langle \tilde{A}f, \varphi \rangle = \lim_m \langle A_m(\chi f), \varphi \rangle + \lim_m \iint K_m(x, y)(1 - \chi(y))f(y)\varphi(x) dy dx. \quad (5.20)$$

Here the integral equals $\langle A_m(f - \chi f), \varphi \rangle$ by the kernel relation, for $A_m \in S^{-\infty}$ and f may as an element of \mathcal{S}' be approached from C_0^∞ . So (5.20) yields

$$\langle \tilde{A}f, \varphi \rangle = \lim_m \langle A_m(\chi f), \varphi \rangle + \lim_m \langle A_m(f - \chi f), \varphi \rangle = \lim_m \langle A_m f, \varphi \rangle. \quad (5.21)$$

Thus $A_m f \rightarrow \tilde{A}f$, which is independent of ψ . Hence $\tilde{A} \subset a(x, D)$ as desired.

Moreover, $\tilde{A}f$ is smooth because $a(x, D)(f\chi) \in \mathcal{S}$ while the other contribution in (5.17) also acts like a C^∞ -function: for a suitable $\tilde{\varphi} \in C_0^\infty$ chosen to be 1 around $\text{supp } \varphi$ the second term equals

$$\int \langle f, (\tilde{\varphi}(x)(1 - \chi))K(x, \cdot) \rangle \varphi(x) dx, \quad (5.22)$$

where $x \mapsto \langle f, \tilde{\varphi}(x)(1 - \chi(\cdot))K(x, \cdot) \rangle$ is C^∞ as seen in the verification that $g \otimes f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ for $f, g \in \mathcal{S}'(\mathbb{R}^n)$. Therefore $\tilde{A}f$ is locally smooth, so $\tilde{A}f \in C^\infty(\mathbb{R}^n)$ follows.

When in addition $f \in \mathcal{O}_M$, then $(1 + |x|)^{-2N}D^\alpha \tilde{A}f$ is bounded for sufficiently large N , for when $r = \text{dist}(\text{supp } \varphi, \text{supp}(1 - \chi))$ one finds in the second contribution to (5.4) that

$$\begin{aligned} (1 + |y|)^{2N} |D_x^\alpha K(x, y)| &\leq (1 + |x|)^{2N} \max(1, 1/r)^{2N} (r + |x - y|)^{2N} |D_x^\alpha K(x, y)| \\ &\leq c(1 + |x|)^{2N} \sup_{x \in \mathbb{R}^n} \int |D_x^\alpha (2\Delta_\eta)^N a(x, \eta)| d\eta, \end{aligned} \quad (5.23)$$

where the supremum is finite for $2N > d + |\alpha| + n$ whilst $(1 + |y|)^{-2N}f(y)$ is in L_1 for large N . Hence $\tilde{A}f \in \mathcal{O}_M$ as claimed. \square

In view of the theorem, the difficulties for type 1, 1-operators are unrelated to growth at infinity for smooth functions. The space $C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ clearly contains functions of non-slow growth, eg

$$f(x) = e^{x_1 + \dots + x_n} \cos(e^{x_1 + \dots + x_n}). \quad (5.24)$$

The codomain C^∞ in Theorem 5.3.1 is of course not contained in \mathcal{S}' , but this is consistent with the use of \mathcal{D}' in Definition 3.1.2.

5.4. The pseudo-local property of type 1, 1-operators

For a classical pseudo-differential operator, say with symbol a in $S_{1,0}^\infty$, it is a well known fact that $a(x, D)$ has the so-called pseudo-local property. This means that it cannot create singularities, ie

$$\text{sing supp } a(x, D)u \subset \text{sing supp } u \quad \text{for all } u \in D(a(x, D)). \quad (5.25)$$

In this connection the domain $D(a(x, D))$ is simply $\mathcal{S}'(\mathbb{R}^n)$.

For a type 1, 1-operator $a(x, D)$ the above formulation also applies, although it is a delicate task to determine $D(a(x, D))$ exactly. But Definition 3.1.2 characterises $D(a(x, D))$ as the set of distributions u for which the limit there exists; and this suffices for the proof of

THEOREM 5.4.1 ([Joh08b, Thm. 6.4]). *Every operator $a(x, D)$ in $\text{OP}(S_{1,1}^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ has the pseudo-local property (5.25).*

This was partly anticipated already in 1978 by C. Parenti and L. Rodino [PR78]. More precisely, they formulated it as a result for the case where $a(x, D)$ is defined on the full distribution space (ie $\mathcal{E}'(\mathbb{R}^n)$ in their context of locally estimated symbols), but as justification they only observed that the distribution kernel $K(x, y)$ is C^∞ for $x \neq y$, tacitly leaving it to the reader to invoke the rest of the standard proof.

However, this is first of all not straightforward as the usual rules of pseudo-differential calculus are not available for type 1, 1-operators (a consequence of Ching's counter-example [Chi72]), so the question is rather more complicated than the impression [PR78] gives. But secondly, the remedy has to be sought, it seems, in the part they suppressed — namely, the substitute for the rules of calculus can be found precisely in the very definition of type 1, 1-operators, where the vanishing frequency modulation yields a useful regularisation.

This became clear with the proof in [Joh08b] of the above theorem. Indeed, in the three decades since [PR78], a proof of Theorem 5.4.1 has not been known. This is of course with

good reason, for both sides of (5.25) are empty for $u \in \mathcal{S}$, so one has to treat $u \in \mathcal{S}' \setminus \mathcal{S}$ directly. Obviously this requires a precise definition of $a(x, D)$ as well as further ideas to handle the possible discontinuity in \mathcal{S}' of $a(x, D)$.

To explain this, recall that the crux of the standard proof is to show that $C^\infty(\mathbb{R}^n)$ contains a certain localised term, namely $\psi(x)a(x, D)(\chi_1 u)(x)$ in the notation of [Joh08b, (6.18)]. Hereby $\psi \in C_0^\infty$ and $\chi_1 \in C^\infty$ have disjoint supports, so the distribution kernel $\tilde{K}(x, y) = \psi(x)K(x, y)\chi_1(y)$ of the composite map $u \mapsto \psi a(x, D)(\chi_1 u)$ belongs to $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, whence it suffices as usual to establish that

$$\langle \psi a(x, D)(\chi_1 u), \varphi \rangle = \langle \varphi \otimes u, \tilde{K} \rangle \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n). \quad (5.26)$$

In fact, the right-hand side equals $\int \langle u, \tilde{K}(x, \cdot) \rangle \varphi(x) dx$ by the definition of the tensor product, so that $\psi a(x, D)(\chi_1 u) = \langle u, \tilde{K}(x, \cdot) \rangle$, where the last expression is C^∞ .

However, even though both sides of (5.26) make sense as they stand, it is, because of the lack of continuity of $a(x, D)$, not a trivial task to show from scratch that they are equal. As indicated above, the details of the verification do not follow well-trodden paths:

The proof strategy in [Joh08b, Thm. 6.4] was to utilise the regularisation that one is given gratis from Definition 3.1.2. This departs from Proposition 4.3.4 that gives, because $\psi \otimes \chi_1$ has support disjoint from the diagonal,

$$\psi(x)K_m(x, y)\chi_1(y) \xrightarrow{m \rightarrow \infty} \psi(x)K(x, y)\chi_1(y) = \tilde{K}(x, y) \quad \text{in } \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n). \quad (5.27)$$

Using this on the right-hand side of (5.26), and approaching u by a sequence u_l from $C_0^\infty(\mathbb{R}^n)$, the formula follows from the usual continuity properties:

$$\begin{aligned} \langle \varphi \otimes u, \tilde{K} \rangle &= \lim_m \langle \varphi \otimes u, (\psi \otimes \chi_1)K_m \rangle_{\mathcal{S}' \otimes \mathcal{S}} \\ &= \lim_m \lim_l \langle \varphi \otimes u_l, (\psi \otimes \chi_1)K_m \rangle_{\mathcal{S}' \otimes \mathcal{S}} \\ &= \lim_m \langle a^m(x, D)(\chi_1 u)^m, \psi \varphi \rangle = \langle \psi a(x, D)(\chi_1 u), \varphi \rangle. \end{aligned} \quad (5.28)$$

In this calculation the order of the limits is essential, of course. But it might be instructive to note that in case $a(x, D)$ is continuous on \mathcal{S}' , then $u_l \rightarrow u$ and frequency modulation on the left-hand side of (5.26) gives $\lim_l \lim_m \langle K_m, u_l \otimes \varphi \rangle$, with limits in reverse order.

The full proof of the pseudo-local property in [Joh08b, Thm. 6.4] is not much longer. However, the above Schwartz space convergence of the distribution kernels required some preparation as noted around Proposition 4.3.4.

5.5. Non-preservation of wavefront sets

In the counter-examples based on Ching's symbol $a_\theta(x, \eta)$ in (1.4), the role of the exponentials is to move all frequencies in the spectrum of the functions to a neighbourhood of the origin. Therefore it is perhaps not surprising that another variant of Ching's example will produce frequencies η that are moved to, say $-\eta$.

So, although $a_\theta(x, D)$ cannot create singularities, cf Section 5.4, at the singular points of $u(x)$ it may change all the high frequencies causing them.

This indicates that type 1,1-operators need not have the microlocal property. That is, the inclusion among wavefront sets

$$\text{WF}(a(x,D)u) \subset \text{WF}(u), \quad (5.29)$$

that always holds for pseudo-differential operators of type 1,0, is violated for certain symbols $a \in S_{1,1}^\infty$ and distributions u .

This was confirmed with explicit calculations in [Joh08b, Sec. 3.2], following C. Parenti and L. Rodino [PR78] who treated $d = 0$ and $n = 1$. The programme they suggested was carried out for all $d \in \mathbb{R}$, $n \in \mathbb{N}$ and arbitrary directions of θ . As a minor improvement, the wavefront sets was explicitly determined, and due to the fact that a certain v below has compact spectrum (rather than compact support as in [PR78]) and the uniformly estimated symbols, the proofs were also rather cleaner.

When $\theta \in \mathbb{R}^n$ is fixed with $|\theta| = 1$, one can introduce

$$w_\theta(x) = w(\theta, d; x) = \sum_{j=1}^{\infty} 2^{-jd} e^{i2^j \theta \cdot x} v(x) \quad (5.30)$$

for some $v \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp } \hat{v} \subset B(0, 1/20)$; or equivalently

$$\hat{w}_\theta(\eta) = \sum_{j=1}^{\infty} 2^{-jd} \hat{v}(\eta - 2^j \theta). \quad (5.31)$$

This distribution has the cone $\mathbb{R}^n \times (\mathbb{R}_+ \theta)$ as its wavefront set, as shown in [Joh08b, Prop. 3.3].

The counter-example arises by considering w_θ together with the symbol $a_{2\theta} \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ defined by (1.4) with auxiliary function A fulfilling in addition

$$A(\eta) = 1 \quad \text{for} \quad \frac{9}{10} \leq |\eta| \leq \frac{11}{10}. \quad (5.32)$$

PROPOSITION 5.5.1. *The distributions $w(\theta, d; x)$ are in $H^s(\mathbb{R}^n)$ precisely for $s < d$, and when $a_{2\theta}$ is chosen as in (1.4), (5.32) with $|\theta| = 1$, then*

$$a_{2\theta}(x, D)w(\theta, d; x) = w(-\theta, 0; x). \quad (5.33)$$

Moreover,

$$\text{WF}(w_\theta) = \mathbb{R}^n \times (\mathbb{R}_+ \theta), \quad (5.34)$$

$$\text{WF}(a_{2\theta}(x, D)w(\theta, d; x)) = \mathbb{R}^n \times (\mathbb{R}_+(-\theta)), \quad (5.35)$$

so the wavefront sets of w_θ and $a_{2\theta}(x, D)w_\theta$ are disjoint.

Since the above, as indicated, is a minor improvement of the result that has been known since [PR78], it should suffice here to refer to [Joh08b, Prop. 3.3] for details.

But a few remarks should be in order. As A vanishes around 2θ , it is easy to see that in this case every (ξ, η) in $\text{supp } \hat{a}_{2\theta}$ lies in the cone $|\eta| \leq 2|\xi + \eta|$ so that a fulfils (1.16) for $B = 2$. Hence $a_{2\theta}(x, D)$ has a large domain containing $\bigcup H^s$ and fulfils the twisted diagonal condition — but then neither of these properties can ensure the microlocal property of a type 1,1-operator, according to Proposition 5.5.1.

There is a clear reason why the counter-example w_θ in Proposition 5.5.1 is singular on all of \mathbb{R}^n : the function $w_\theta(x)$ equals $v(x)f(x \cdot \theta)$ where $v \in \mathcal{F}^{-1}C_0^\infty$ is analytic whilst $f(t) = \sum_{j=1}^\infty 2^{-jd}e^{i2^jt}$ is not just highly oscillating, but for $0 < d \leq 1$ equal to Weierstrass's continuous nowhere differentiable function, here in a complex version with its wavefront set along a half-ray. The link to this classical construction (that could have substantiated the argument for formula (19) in [PR78]) was first observed in [Joh08b, Rem. 3.5].

More remarks on the above f , including a short, explicit analysis of its regularity properties, can be found in Remarks 3.6–3.8 in [Joh08b]. In particular the nowhere differentiability was obtained with a short microlocalisation argument (further explored in [Joh10b]).

5.6. The support rule and its spectral version

This subject has already been explained in the context of type 1,0-operators in Section 4.2. It therefore suffices to comment on the modifications needed for type 1,1-operators.

First of all there is a satisfactory result for the extended kernel formula and the support rule, which for $u \in \mathcal{E}'(\mathbb{R}^n)$ is the well-known inclusion

$$\text{supp } Au \subset \text{supp } K \circ \text{supp } u. \quad (5.36)$$

Here $\text{supp } K \circ \text{supp } u$ stands for the set $\{x \mid \exists y \in \text{supp } u: (x, y) \in \text{supp } K\}$ as usual.

THEOREM 5.6.1. *When $a \in S_{1,1}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and K denotes its kernel, then $\langle a(x, D)u, v \rangle = \langle K, v \otimes u \rangle$ whenever $u \in D(a(x, D))$, $v \in C_0^\infty(\mathbb{R}^n)$ fulfil*

$$\text{supp } K \cap \text{supp } v \otimes u \in \mathbb{R}^n \times \mathbb{R}^n, \quad (5.37)$$

$$\text{sing supp } K \cap \text{sing supp } v \otimes u = \emptyset. \quad (5.38)$$

And for all $u \in D(a(x, D))$ the support rule holds, ie $\text{supp } a(x, D)u \subset \overline{\text{supp } K \circ \text{supp } u}$.

This was proved in [Joh08b, Thm. 8.1] by approaching u by a regularly converging sequence from C_0^∞ and applying Proposition 4.3.4.

The rule for spectra was amply described in Section 4.2, so the full statement for type 1,1-operators is just given here. Unfortunately it contains an undesirable assumption, which usually is redundant, cf the last part of The Spectral Support Rule:

THEOREM 5.6.2 ([Joh08b, Thm. 8.4]). *Let $a \in S_{1,1}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and let $u \in D(a(x, D))$ be such that $a(x, D)u$ is temperate and that, for some $\psi \in C_0^\infty(\mathbb{R}^n)$ equalling 1 around the origin, the convergence of Definition 3.1.2 holds in the topology of $\mathcal{S}'(\mathbb{R}^n)$, ie*

$$a(x, D)u = \lim_{m \rightarrow \infty} a^m(x, D)u^m \quad \text{in } \mathcal{S}'(\mathbb{R}^n). \quad (5.39)$$

Then (3.14) holds, that is with $\Xi = \text{supp } \mathcal{K} \circ \text{supp } \hat{u}$ one has

$$\text{supp } \mathcal{F}(a(x, D)u) \subset \overline{\Xi}, \quad (5.40)$$

$$\Xi = \{ \xi + \eta \mid (\xi, \eta) \in \text{supp } \hat{a}, \eta \in \text{supp } \hat{u} \}. \quad (5.41)$$

When $u \in \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n)$ then (5.39) holds automatically and Ξ is closed for such u .

In the theory of type 1, 1-operators it may seem unmotivated that the partially Fourier transformed symbol $\hat{a}(\xi, \eta)$ plays such a prominent role (cf the twisted diagonal condition). But the spectral support rule (4.17) gives an explanation as $\hat{a}(\xi, \eta)$ appears in Ξ too; thence $\hat{a}(\xi, \eta)$ should be a natural object for every pseudo-differential operator, as in Littlewood–Paley analysis control of $\text{supp } \mathcal{F}a(x, D)u$ is a central theme.

However, $\hat{a}(\xi, \eta)$ is particularly important for operators of type 1, 1, as the spectral support rule (5.41) clearly shows that the role of the twisted diagonal condition (1.16) is to ensure that $a(x, D)$ cannot change (large) frequencies in $\text{supp } \hat{u}$ to 0: (1.16) means that ξ cannot be close to $-\eta$ when $(\xi, \eta) \in \text{supp } \hat{a}$, which by (5.41) means that $\eta \in \text{supp } \hat{u}$ will be changed to $\xi + \eta \neq 0$.

The proof of Theorem 5.6.2 was given first in [Joh04, Joh05] in a special case, and the full result appeared in in [Joh08b, Thm. 8.3]. The main ingredient was to obtain an extended version of the kernel formula for the conjugated operator $\mathcal{F}a(x, D)\mathcal{F}^{-1}$. This is recalled from [Joh08b, Thm. 8.2] here.

THEOREM 5.6.3. *Let $a \in S_{1,1}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and let the distribution kernel of $\mathcal{F}a(x, D)\mathcal{F}^{-1}$ be denoted by $\mathcal{K}(\xi, \eta)$. When $u \in D(a(x, D))$ is such that, for some ψ as in Definition 3.1.2,*

$$a(x, D)u = \lim_{m \rightarrow \infty} a^m(x, D)u^m \quad \text{holds in } \mathcal{S}'(\mathbb{R}^n), \quad (5.42)$$

and when $\hat{v} \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\text{supp } \mathcal{K} \cap \text{supp } \hat{v} \otimes \hat{u} \in \mathbb{R}^n \times \mathbb{R}^n, \quad \text{sing supp } \mathcal{K} \cap \text{sing supp } \hat{v} \otimes \hat{u} = \emptyset, \quad (5.43)$$

then it holds

$$\langle \mathcal{F}a(x, D)\mathcal{F}^{-1}(\hat{u}), \hat{v} \rangle = \langle \mathcal{K}, \hat{v} \otimes \hat{u} \rangle, \quad (5.44)$$

with extended action of $\langle \cdot, \cdot \rangle$ on the right-hand side.

This was also obtained using regular convergence of test functions; cf [Joh08b, Thm. 8.2]. The convergence in the topology of \mathcal{S}' seems to be necessary for technical reasons in this proof. Anyhow, the assumption that $a(x, D)u$ be an element of \mathcal{S}' of course serves the purpose of making $\mathcal{F}a(x, D)u$ defined.

The remarks made in Section 4.2 also apply here: elementary symbols are redundant because of Theorem 5.6.2, but in the type 1, 1-context this simplification is particularly important, for else Definition 3.1.2 would contain a double limit procedure, with severe complications for the entire theory.

CHAPTER 6

Continuity results

Here the results summarised in (VII)–(XII) in Section 3.2 are described in detail.

6.1. Littlewood–Paley decompositions of type 1, 1-operators

Here it is described how the definition by vanishing frequency modulation leads at once to Littlewood–Paley analysis of $a(x, D)u$, and in particular to the paradifferential decomposition mentioned in Section 5.1.

6.1.1. Dyadic corona decompositions of symbols and operators. The basic step is to obtain a Littlewood–Paley decomposition from the modulation function ψ used in Definition 3.1.2.

As a preparation one may obviously, to each $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi = 1$ in a neighbourhood of 0, fix $R > r > 0$ satisfying

$$\psi(\xi) = 1 \quad \text{for } |\xi| \leq r; \quad \psi(\xi) = 0 \quad \text{for } |\xi| \geq R \geq 1. \quad (6.1)$$

It is also convenient to take an integer $h \geq 2$ so large that $2R < r2^h$.

As an auxiliary function one has $\varphi = \psi - \psi(2\cdot)$. Dilations of this function are supported in coronas, eg

$$\text{supp } \varphi(2^{-k}\cdot) \subset \{ \xi \mid r2^{k-1} \leq |\xi| \leq R2^k \}, \quad \text{for } k \geq 1, \quad (6.2)$$

and by calculating the telescopic sum

$$\psi(\xi) + \varphi(\xi/2) + \cdots + \varphi(\xi/2^m) = \psi(2^{-m}\xi), \quad (6.3)$$

it follows by letting $m \rightarrow \infty$ that one has the Littlewood–Paley partition of unity

$$1 = \psi(\xi) + \sum_{k=1}^{\infty} \varphi(2^{-k}\xi), \quad \text{for each } \xi \in \mathbb{R}^n. \quad (6.4)$$

Using this, $u(x)$ can now be (micro-)localised eg to frequencies $|\eta| \approx 2^j$ by setting

$$u_j = \varphi(2^{-j}D)u, \quad u^j = \psi(2^{-j}D)u. \quad (6.5)$$

Note that localisation to balls given by $|\eta| \leq R2^j$ are written with upper indices. For symbols $a(x, \eta)$ similar conventions apply to the first variable,

$$a_j(x, \eta) = \varphi(2^{-j}D_x)a(x, \eta) = \mathcal{F}_{\xi \rightarrow x}^{-1}(\varphi(2^{-j}\xi)\mathcal{F}_{x \rightarrow \xi}a(x, \eta)). \quad (6.6)$$

$$a^j(x, \eta) = \psi(2^{-j}D_x)a(x, \eta) = \mathcal{F}_{\xi \rightarrow x}^{-1}(\psi(2^{-j}\xi)\mathcal{F}_{x \rightarrow \xi}a(x, \eta)). \quad (6.7)$$

By convention $u^j = u_j$ and $a^j = a_j$ for $j = 0$; they are all taken to equal 0 for indices $j < 0$. (To avoid having several meanings of sub- and superscripts, dilations $\psi(2^{-j}\cdot)$ are written as

such, with the corresponding Fourier multiplier as $\psi(2^{-j}D)$, and similarly for φ . Note that as a consequence one has for operators that, eg, $a_j(x, D) = \text{OP}(\varphi(2^{-j}D_x)a(x, \eta))$.

Thus prepared with these classical dyadic corona decompositions, the point of this section is to insert the relation (6.3) twice in $a^m(x, D)u^m$, cf (3.9), and apply bilinearity. This gives

$$\begin{aligned} a^m(x, D)u^m &= \text{OP}((\psi(D_x) + \varphi(2^{-1}D_x) + \cdots + \varphi(2^{-m}D_x))a(x, \eta))(u_0 + u_1 + \cdots + u_m) \\ &= \sum_{j,k=0}^m a_j(x, D)u_k. \end{aligned} \quad (6.8)$$

Of course this sum may be split in three groups in which $j \leq k - h$, $|j - k| < h$ and $k \leq j - h$, respectively. Proceeding to the limit of vanishing frequency modulation, ie $m \rightarrow \infty$, one is therefore lead to consider the three infinite series

$$a_{\psi}^{(1)}(x, D)u = \sum_{k=h}^{\infty} \sum_{j \leq k-h} a_j(x, D)u_k = \sum_{k=h}^{\infty} a^{k-h}(x, D)u_k \quad (6.9)$$

$$\begin{aligned} a_{\psi}^{(2)}(x, D)u &= \sum_{k=0}^{\infty} (a_{k-h+1}(x, D)u_k + \cdots + a_{k-1}(x, D)u_k + a_k(x, D)u_k \\ &\quad + a_k(x, D)u_{k-1} + \cdots + a_k(x, D)u_{k-h+1}) \end{aligned} \quad (6.10)$$

$$a_{\psi}^{(3)}(x, D)u = \sum_{j=h}^{\infty} \sum_{k \leq j-h} a_j(x, D)u_k = \sum_{j=h}^{\infty} a_j(x, D)u^{j-h}. \quad (6.11)$$

More precisely one has

PROPOSITION 6.1.1. *If $a(x, \eta)$ is of type 1, 1 and ψ is an arbitrary modulation function, then the paradifferential decomposition*

$$a_{\psi}(x, D)u = a_{\psi}^{(1)}(x, D)u + a_{\psi}^{(2)}(x, D)u + a_{\psi}^{(3)}(x, D)u \quad (6.12)$$

holds for all $u \in \mathcal{S}'(\mathbb{R}^n)$ fulfilling that the three series converge in $\mathcal{D}'(\mathbb{R}^n)$.

One should note the shorthand $a^{k-h}(x, D)$ for $\sum_{j \leq k-h} a_j(x, D) = \text{OP}(\psi(2^{h-k}D_x)a(x, \eta))$ etc. In this way (6.10) also has a brief form, namely

$$a_{\psi}^{(2)}(x, D)u = \sum_{k=0}^{\infty} ((a^k - a^{k-h})(x, D)u_k + a_k(x, D)(u^{k-1} - u^{k-h})). \quad (6.13)$$

The importance of the decomposition in (6.9)–(6.11) lies in the fact that the summands have their spectra in balls and coronas. This has been anticipated since the 1980's, and verified for elementary symbols, eg in [Bou88a, Thm. 1]. In general it follows directly from the spectral support rule in Theorem 5.6.2:

PROPOSITION 6.1.2. *When $a \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, and r, R are chosen as in (6.1) for each modulation function ψ , then every $h \in \mathbb{N}$ such that $2R < r2^h$ gives*

$$\text{supp } \mathcal{F}(a^{k-h}(x, D)u_k) \subset \left\{ \xi \mid R_h 2^k \leq |\xi| \leq \frac{5R}{4} 2^k \right\} \quad (6.14)$$

$$\text{supp } \mathcal{F}(a_k(x, D)u^{k-h}) \subset \left\{ \xi \mid R_h 2^k \leq |\xi| \leq \frac{5R}{4} 2^k \right\}, \quad (6.15)$$

where $R_h = \frac{r}{2} - R2^{-h} > 0$.

PROOF. Since the type 1, 1-operator $a^{k-h}(x, D)$ is defined on $u_k \in \mathcal{F}^{-1}\mathcal{E}'$, the last part of Theorem 5.6.2 and (6.2) give

$$\text{supp } \mathcal{F}(a^{k-h}(x, D)u_k) \subset \{ \xi + \eta \mid (\xi, \eta) \in \text{supp}(\psi_{h-k} \otimes 1)\hat{a}, \quad r2^{k-1} \leq |\eta| \leq R2^k \}. \quad (6.16)$$

So by the triangle inequality every $\zeta = \xi + \eta$ in the support fulfils

$$r2^{k-1} - R2^{k-h} \leq |\zeta| \leq R2^{k-h} + R2^k \leq \frac{5}{4}R2^k, \quad (6.17)$$

as $h \geq 2$. This shows (6.14) and (6.15) follows analogously. \square

In contrast with this, the terms in $a^{(2)}(x, D)u$ only satisfy a dyadic ball condition, as was first observed for functions u in [Joh05]. But when the twisted diagonal condition (1.16) holds, the situation improves for large k :

PROPOSITION 6.1.3. *When $a \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$, $u \in \mathcal{S}'(\mathbb{R}^n)$, and r, R are chosen as in (6.1) for each auxiliary function ψ , then every $h \in \mathbb{N}$ such that $2R < r2^h$ gives*

$$\text{supp } \mathcal{F}(a_k(x, D)(u^{k-1} - u^{k-h}) + (a^k - a^{k-h})(x, D)u_k) \subset \{ \xi \mid |\xi| \leq 2R2^k \} \quad (6.18)$$

If $a(x, \eta)$ satisfies (1.16) for some $B \geq 1$, the support is eventually contained in the corona

$$\{ \xi \mid \frac{r}{2^{h+1}B}2^k \leq |\xi| \leq 2R2^k \}. \quad (6.19)$$

A proof of the general case above can be given with the same techniques as for Proposition 6.1.2; cf [Joh10c, Prop. 5.3].

The decomposition in Proposition 6.1.1 is also useful because the terms of the series can be conveniently estimated at every $x \in \mathbb{R}^n$, using the pointwise approach of Section 4.1. For later reference, these estimates are collected in the next result.

PROPOSITION 6.1.4. *For every $a(x, \eta)$ in $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ there are pointwise estimates for $x \in \mathbb{R}^n$,*

$$|a^{k-h}(x, D)u_k(x)| \leq p(a)(R2^k)^d u_k^*(N, R2^k; x), \quad (6.20)$$

$$|(a^k - a^{k-h})(x, D)u_k(x)| \leq p(a)(R2^k)^d u_k^*(N, R2^k; x), \quad (6.21)$$

$$|a^k(x, D)(u^{k-1}(x) - u^{k-h}(x))| \leq p(a)(R2^k)^d \sum_{l=1}^{h-1} u_{k-l}^*(N, R2^{k-l}; x), \quad (6.22)$$

$$|a_j(x, D)u^{j-h}(x)| \leq c_M 2^{-jM} p(a) \sum_{k=0}^j (R2^k)^{d+M} u_k^*(N, R2^k; x). \quad (6.23)$$

Hereby $p(a)$ denotes a continuous seminorm on $S_{1,1}^d$ and $M \in \mathbb{N}$.

PROOF. From the factorisation inequality in Section 4.1 it follows that

$$|a^{k-h}(x, D)u_k(x)| \leq F_{a^{k-h}}(N, R2^k; x) u_k^*(N, R2^k; x) \leq c_1 \|\mathcal{F}^{-1}\psi\|_1 p(a)(R2^k)^d u_k^*(x). \quad (6.24)$$

Indeed, the estimate of $F_{a^{k-h}}$ builds on Corollary 4.1.3. Since \hat{u}_k is supported in a corona with outer radius $R2^k$, this states that

$$F_{a^{k-h}}(x) \leq cp(a^{k-h})(R2^k)^d. \quad (6.25)$$

Here it easy to see from a convolution estimate that $p(a^{k-h}) \leq c'p(a)$, since eg

$$|D_\eta^\alpha D_x^\beta a^k(x, \eta)| \leq \int |2^{n(k)} \check{\psi}(2^k y) D_\eta^\alpha D_x^\beta a(x-y, \eta)| dy \leq p_{\alpha, \beta}(a)(1 + |\eta|)^{d-|\alpha|+|\beta|} \int |\check{\psi}| dy. \quad (6.26)$$

Hence the first claim is obtained.

The estimate of $(a^k - a^{k-h})(x, D)u_k$ is highly similar. In fact the only change is in the integrals, where by the definition of the symbol $a^k - a^{k-h}$ the last integrand should be $|\mathcal{F}^{-1}(\psi - \psi(2^h \cdot))|$ instead of $|\mathcal{F}^{-1}\psi|$.

In the third line one has

$$a^k(x, D)(u^{k-1}(x) - u^{k-h}(x)) = a^k(x, D)u_{k-1}(x) + \dots + a^k(x, D)u_{k-h+1}(x). \quad (6.27)$$

Here each term is estimated as above, now with the factor $(R2^{k-l})^d$ as a result for $l = 1, \dots, h-1$; as this is less than $2^{(-ld)_+}$ times $(R2^k)^d$, the claim follows.

The last inequality results from the fact that, by (6.3) one has $u^{j-h} = u_0 + \dots + u_{j-h}$, whence a crude estimate yields

$$|a_j(x, D)u^{j-h}(x)| \leq \sum_{k=0}^{j-h} |a_j(x, D)u_k(x)| \leq \sum_{k=0}^j F_{a_j}(N, R2^k; x)u_k^*(N, R2^k; x). \quad (6.28)$$

Since $\hat{a}_j(\xi, \eta) = \varphi(2^{-j}\xi)\hat{a}(\xi, \eta)$ vanishes around the origin (there is nothing to show if $j < h$), Corollary 4.1.4 gives the estimate $F_{a_j}(x) \leq c_M p(a)2^{-jM}(R2^k)^{d+M}$ for $k \geq 1$, as for such k an auxiliary function supported in a corona may be used in F_{a_j} . The case $k = 0$ is similarly estimated if c_M is increased by a power of R ; cf Corollary 4.1.4. This completes the proof. \square

6.1.2. Calculation of symbols and remainder terms. In connection with the paradifferential splitting (6.9)–(6.11) there is an extra task for type 1, 1-operators, because the operator notation $a^{(j)}(x, D)$ requires a more explicit justification in this context.

Departing from the right hand sides of (6.9)–(6.11) one finds the following symbols,

$$a^{(1)}(x, \eta) = \sum_{k=h}^{\infty} a^{k-h}(x, \eta)\varphi(2^{-k}\eta) \quad (6.29)$$

$$a^{(2)}(x, \eta) = \sum_{k=0}^{\infty} ((a^k(x, \eta) - a^{k-h}(x, \eta))\varphi(2^{-k}\eta) + a_k(x, \eta)(\psi(2^{-(k-1)}\eta) - \psi(2^{-(k-h)}\eta))) \quad (6.30)$$

$$a^{(3)}(x, \eta) = \sum_{j=h}^{\infty} a_j(x, \eta)\psi(2^{-(j-h)}\eta). \quad (6.31)$$

Not surprisingly, these series converge in the Fréchet space $S_{1,1}^{d+1}(\mathbb{R}^n \times \mathbb{R}^n)$, for the sums are locally finite.

More intriguingly, an inspection shows that both $a^{(1)}(x, \eta)$ and $a^{(3)}(x, \eta)$ fulfil the twisted diagonal condition (1.16); cf [Joh10c, Prop. 5.5].

However, before this can be applied, it is clearly necessary to verify that the type 1, 1-operators corresponding to (6.29)–(6.31) are in fact given by the infinite series in (6.9)–(6.11). In particular it is a natural programme to show that the series for $a^{(j)}(x, D)u$, $j = 1, 2, 3$, converges precisely when u belongs to the domain of $a^{(j)}(x, D)$.

But in view of the definition by vanishing frequency modulation in (3.6), this will necessarily be lengthy because a second modulation function has to be introduced.

To indicate the details for $a^{(1)}(x, \eta)$, let $\psi, \Psi \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 around the origin, and let Ψ be used as the fixed auxiliary function entering the symbol $a_\Psi^{(j)}(x, \eta)$; and set $\Phi = \Psi - \Psi(2\cdot)$. The numbers r, R and h are then chosen in relation to Ψ as in (6.1); and one can take $\lambda < \Lambda$ playing similar roles for ψ . Moreover, ψ is used for the frequency modulation entering the definition of the operator $a_\Psi^{(1)}(x, D)$.

As shown in [Joh10c], this gives the following identity for $u \in \mathcal{S}'(\mathbb{R}^n)$, where prime indicates a finite sum and $\mu = \lceil \log_2(\lambda/R) \rceil$,

$$\begin{aligned} \text{OP}(\psi(2^{-m}D_x)a^{(1)}(x, \eta)\psi(2^{-m}\eta))u &= \sum_{k=h}^{m+\mu} a^{k-h}(x, D)u_k \\ &+ \sum'_{\mu < l < 1 + \log_2(\Lambda/r)} \text{OP}(\psi(2^{-m}D_x)\Psi(2^{h-l-m}D_x)a(x, \eta)\Phi(2^{-m-l}\eta)\psi(2^{-m}\eta))u. \end{aligned} \quad (6.32)$$

To complete the abovementioned programme, it remains to let $m \rightarrow \infty$ in (6.32), whereby the first term on the right-hand side converges to the limit in (6.9). But clearly it should also be shown that the remainder terms in the primed sum can be safely ignored.

This turns out to be possible for all $u \in \mathcal{S}'(\mathbb{R}^n)$, and as a result it is true that the type 1, 1-operator $a_\Psi^{(1)}(x, D)u$ simply equals the infinite series $\sum_{k=h}^\infty a^{k-h}(x, D)u_k$, with convergence for all u in the domain $D(a_\Psi^{(1)}(x, D)) = \mathcal{S}'(\mathbb{R}^n)$. However, the details of this will be indicated during the discussion of Theorem 6.2.3 below.

For the other operators $a_\Psi^{(2)}(x, D)$ and $a_\Psi^{(3)}(x, D)$ there are similar calculations, yielding remainder terms analogous to the primed sum above; cf [Joh10c, Sect. 5]. The outcome is equally satisfying for $a^{(3)}(x, D)$, which is also defined on $\mathcal{S}'(\mathbb{R}^n)$ and given by the infinite series. This extends to $a^{(2)}(x, D)$, at least if (1.16) holds (for weaker conditions, cf Theorem 6.3.5).

As indicated these issues will be taken up again in Section 6.2. A discussion of the applications of the paradifferential decomposition (6.12) in Proposition 6.1.1 was begun already in Section 5.1.4, and it continues in Section 6.2–6.5 below.

6.2. The twisted diagonal condition

For convenience it is recalled that boundedness $a(x, D): H^{s+d} \rightarrow H^s$ for all real s was proved by L. Hörmander [Hör88, Hör97] for every symbol in $S_{1,1}^d$ fulfilling the twisted diagonal condition, mentioned already in (1.16); namely for some $B \geq 1$,

$$\hat{a}(\xi, \eta) = 0 \quad \text{when} \quad B(|\xi + \eta| + 1) < |\eta|. \quad (6.33)$$

It is of course natural to conjecture that (6.33) also implies continuity $a(x, D): \mathcal{S}' \rightarrow \mathcal{S}'$. However, this question has neither been formulated nor treated before it was addressed in [Joh10c].

To prove the conjecture, one may argue by duality, which succeeds as follows. First a lemma on the adjoint symbols is recalled from [Hör88] and [Hör97, Lem. 9.4.1].

LEMMA 6.2.1. *When $a(x, \eta)$ is in $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ and for some $B \geq 1$ satisfies the twisted diagonal condition (6.33), then the adjoint $a(x, D)^* = b(x, D)$ has the symbol*

$$b(x, \eta) = e^{iD_x \cdot D_\eta} \overline{a(x, \eta)}, \quad (6.34)$$

which is in $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ in this case and $\hat{b}(\xi, \eta) = 0$ when $|\xi + \eta| > B(|\eta| + 1)$. Moreover,

$$|D_\eta^\alpha D_x^\beta b(x, \eta)| \leq C_{\alpha\beta}(a) B(1 + B^{d-|\alpha|+|\beta|})(1 + |\eta|)^{d-|\alpha|+|\beta|}, \quad (6.35)$$

for certain continuous seminorms $C_{\alpha\beta}$ on $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$, that do not depend on B .

The fact that $C_{\alpha,\beta}(a)$ is a continuous seminorm on $a(x, \eta)$ is added here (it is seen directly from Hörmander's proof). To emphasize its importance, note that if $a_k \rightarrow a$ in the topology of $S_{1,1}^d$ and they all fulfil (6.33) with the same $B \geq 1$, then insertion of $a_k - a$ in (6.35) yields that $b_k \rightarrow b$ in $S_{1,1}^d$.

In view of the lemma, it is clear that if $a(x, D)$ fulfils (6.33), it necessarily has continuous linear extension $\mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$, namely $b(x, D)^*$. This moreover coincides with Definition 3.1.2 of $a(x, D)$ by vanishing frequency modulation:

PROPOSITION 6.2.2 ([Joh08b, Prop. 4.2]). *When $a(x, \eta) \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ fulfils (6.33), then $a(x, D)$ is a continuous linear map $\mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ and it equals the adjoint of the mapping $b(x, D): \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$, when $b(x, \eta)$ is the adjoint symbol as in Lemma 6.2.1.*

PROOF. A simple convolution estimate, cf [Joh08b, Lem. 2.1], gives that in $S_{1,1}^{d+1}$,

$$\psi(2^{-m}D_x)a(x, \eta)\psi(2^{-m}\eta) \rightarrow a(x, \eta) \quad \text{for } m \rightarrow \infty. \quad (6.36)$$

As frequency modulation cannot increase supports, this sequence also fulfils (6.33) for the same B . So since the passage to adjoint symbols by (6.35) is continuous from the metric subspace of $S_{1,1}^d$ fulfilling (6.33) to $S_{1,1}^{d+1}$,

$$b_m(x, \eta) := e^{iD_x \cdot D_\eta} \overline{(\psi(2^{-m}D_x)a(x, \eta)\psi(2^{-m}\eta))} \xrightarrow{m \rightarrow \infty} e^{iD_x \cdot D_\eta} \overline{a(x, \eta)} =: b(x, \eta). \quad (6.37)$$

Moreover, since $b(x, D)$ as an operator on the Schwartz space depends continuously on the symbol, one has for $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} (b(x, D)^*u | \varphi) &= (u | b(x, D)\varphi) \\ &= (u | \lim_{m \rightarrow \infty} \text{OP}(b_m(x, \eta))\varphi) \\ &= \lim_{m \rightarrow \infty} (\text{OP}(\psi(2^{-m}D_x)a(x, \eta)\psi(2^{-m}\eta))u | \varphi). \end{aligned} \quad (6.38)$$

As the left-hand side is independent of ψ the limit in (3.6) is so, the definition of $a(x, D)$ gives that every $u \in \mathcal{S}'(\mathbb{R}^n)$ is in $D(a(x, D))$ and $a(x, D)u = b(x, D)^*u$ as claimed. \square

The mere extendability to \mathcal{S}' under the twisted diagonal condition (6.33) could have been observed already in [Hör88, Hör97]. The above result seems to be the first giving a sufficient condition for a type 1, 1-operator to be *defined* on the entire $\mathcal{S}'(\mathbb{R}^n)$.

Despite its success, the above is of limited value when it comes to continuity results, say in the H_p^s -scale. On the positive side it does show that the first part of Theorem 5.1.3 implies the last part there; but the former is not in itself related to duality.

For a direct proof of continuity it is convenient to invoke the paradifferential decomposition in Proposition 6.1.1, and the outcome of this is satisfying inasmuch as the series in (6.9)–(6.11) can be shown to converge for all temperate u , when $a(x, D)$ fulfils (6.33) as above. In fact, by summing up one arrives at a contraction of Theorems 6.3 and 6.5 in [Joh10c] as a main theorem of the analysis:

THEOREM 6.2.3. *When $a \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ fulfils the twisted diagonal condition (6.33), then the associated type 1, 1-operator $a(x, D)$ defined by vanishing frequency modulation is an everywhere defined continuous linear map*

$$a(x, D): \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n), \quad (6.39)$$

with its adjoint $a(x, D)^*$ also in $\text{OP}(S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n))$. For each modulation function ψ , the operator fulfils

$$a(x, D)u = a_\psi^{(1)}(x, D)u + a_\psi^{(2)}(x, D)u + a_\psi^{(3)}(x, D)u, \quad (6.40)$$

where the operators on the right-hand side all belong to $\text{OP}(S_{1,1}^d)$ and likewise fulfil (6.33) (hence have adjoints in $\text{OP}(S_{1,1}^d)$); they are given by the series in (6.9), (6.10), (6.11) that converge rapidly in $\mathcal{S}'(\mathbb{R}^n)$ for every $u \in \mathcal{S}'(\mathbb{R}^n)$.

The first statement is of course a repetition of Proposition 6.2.2. The rest of the proof relies on Proposition 6.1.2 and the second part of Proposition 6.1.3. Indeed, these show that the first assumptions in the following lemma is fulfilled, when applied in the basic case with $\theta_0 = 1 = \theta_1$ to the series in Proposition 6.1.1:

LEMMA 6.2.4 ([Joh10c, Lem. A.1]). *1° Let $(u_j)_{j \in \mathbb{N}_0}$ be a sequence in $\mathcal{S}'(\mathbb{R}^n)$ fulfilling that there exist $A > 1$ and $\theta_1 > \theta_0 > 0$ such that $\text{supp } \hat{u}_0 \subset \{ \xi \mid |\xi| \leq A \}$ while for $j \geq 1$*

$$\text{supp } \hat{u}_j \subset \{ \xi \mid \frac{1}{A} 2^{j\theta_0} \leq |\xi| \leq A 2^{j\theta_1} \}, \quad (6.41)$$

and that for suitable constants $C \geq 0$, $N \geq 0$,

$$|u_j(x)| \leq C 2^{jN\theta_1} (1 + |x|)^N \text{ for all } j \geq 0. \quad (6.42)$$

Then $\sum_{j=0}^{\infty} u_j$ converges rapidly in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution u , for which \hat{u} is of order N .

2° For every $u \in \mathcal{S}'(\mathbb{R}^n)$ both (6.41) and (6.42) are fulfilled for $\theta_0 = \theta_1 = 1$ by the functions $u_0 = \Phi_0(D)u$ and $u_j = \Phi(2^{-j}D)u$ when $\Phi_0, \Phi \in C_0^\infty(\mathbb{R}^n)$ and $0 \notin \text{supp } \Phi$. In particular this is the case for a Littlewood–Paley decomposition $1 = \Phi_0 + \sum_{j=1}^{\infty} \Phi(2^{-j}\xi)$.

The second assumption of this lemma, cf (6.42), is verified for the terms in (6.40) by using the pointwise estimates. Indeed, for the general term in $a_{\Psi}^{(1)}(x, D)u$ it is seen from Proposition 4.1.5 for $\Phi = \psi(2^{-h}\cdot)$ and $\Psi = \varphi$ that

$$|a^{k-h}(x, D)u_k(x)| = |\text{OP}(\psi(2^{k-h}D_x)a(x, \eta)\varphi(2^{-k}\eta))u(x)| \leq c2^{k(N+d_+)}(1+|x|)^{N+d_+}. \quad (6.43)$$

Similar estimates can be found for $a_{\Psi}^{(2)}(x, D)$ and $a_{\Psi}^{(3)}(x, D)$, cf [Joh10c, Prop. 6.1–2], hence the three series converge in \mathcal{S}' according to the above lemma.

Finally, to conclude that the three terms in (6.40) are type 1, 1-operators, one can apply Proposition 4.1.5 once more, this time with other choices of the two cut-off functions. For example, in the primed sum of remainders in (6.32), they should for each l be taken as $\psi \cdot \Psi(2^{h-l}\cdot)$ and $\Phi(2^{-l}\cdot)\psi$, respectively; then Lemma 6.2.4 yields that the remainders can be summed over $m \in \mathbb{N}$, hence that (each term in) the primed sum tends to 0 in \mathcal{S}' as $m \rightarrow \infty$. In this way Theorem 6.2.3 is proved.

REMARK 6.2.5. Theorem 6.2.3 and its proof generalises a result of R. R. Coifman and Y. Meyer [MC97, Ch. 15] in three ways. They stated Lemma 6.2.4 for $\theta_0 = \theta_1 = 1$ and derived a version of Theorem 6.2.3 for paramultiplication, though only with a treatment of the first and third term.

Without the twisted diagonal condition (6.33), the techniques behind Theorem 6.2.3 at least show that $a^{(1)}(x, D)u$ and $a^{(3)}(x, D)u$ are always defined. Thus one has

COROLLARY 6.2.6. *For $a(x, \eta)$ in $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the domain $D(a(x, D))$ if and only if the series for $a^{(2)}(x, D)u$ in (6.10) converges in $\mathcal{S}'(\mathbb{R}^n)$.*

As a follow-up to this corollary, it is natural to ask whether less strong assumptions on $a(x, D)$ along \mathcal{T} improves the picture. That is the topic of the next section.

6.3. The twisted diagonal condition of order σ

When the strict vanishing of $\hat{a}(\xi, \eta)$ in a conical neighbourhood of \mathcal{T} is replaced by vanishing to infinite order (in some sense) at \mathcal{T} , then one arrives at a characterisation of the self-adjoint subclass $\tilde{S}_{1,1}^d$. This result of L. Hörmander is recalled here and shown to imply continuity on \mathcal{S}' .

6.3.1. Localisation along the twisted diagonal. As a weakening of the twisted diagonal condition (6.33), L. Hörmander [Hör88, Hör89, Hör97] introduced certain localisations of the symbol to conical neighbourhoods of \mathcal{T} . Specifically this was achieved by passing from $a(x, \eta)$ to another symbol, that was denoted by $a_{\chi, \varepsilon}(x, \eta)$. This is defined by

$$\hat{a}_{\chi, \varepsilon}(\xi, \eta) = \hat{a}(\xi, \eta)\chi(\xi + \eta, \varepsilon\eta), \quad (6.44)$$

whereby $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is chosen so that

$$\chi(t\xi, t\eta) = \chi(\xi, \eta) \quad \text{for } t \geq 1, |\eta| \geq 2 \quad (6.45)$$

$$\text{supp } \chi \subset \{(\xi, \eta) \mid 1 \leq |\eta|, |\xi| \leq |\eta|\} \quad (6.46)$$

$$\chi = 1 \quad \text{in } \{(\xi, \eta) \mid 2 \leq |\eta|, 2|\xi| \leq |\eta|\}. \quad (6.47)$$

Here it should first of all be noted that by the choice of χ ,

$$\text{supp } \hat{a}_{\chi,\varepsilon} \subset \{(\xi, \eta) \mid 1 + |\xi + \eta| \leq 2\varepsilon|\eta|\}. \quad (6.48)$$

So when $a(x, \eta)$ fulfils the strict condition (6.33), then clearly $a_{\chi,\varepsilon} \equiv 0$ for $\varepsilon > 0$ so small that $B \leq 1/(2\varepsilon)$. Therefore milder conditions will result by imposing smallness requirements on $a_{\chi,\varepsilon}$ in the limit $\varepsilon \rightarrow 0$.

As a novelty in the analysis, L. Hörmander linked the above to well-known multiplier conditions (of Mihlin–Hörmander type) by introducing the condition that for some $\sigma \in \mathbb{R}$, it holds for all multiindices α and $0 < \varepsilon < 1$ that

$$\sup_{R>0, x \in \mathbb{R}^n} R^{-d} \left(\int_{R \leq |\eta| \leq 2R} |R^{|\alpha|} D_\eta^\alpha a_{\chi,\varepsilon}(x, \eta)|^2 \frac{d\eta}{R^n} \right)^{1/2} \leq c_{\alpha,\sigma} \varepsilon^{\sigma+n/2-|\alpha|}. \quad (6.49)$$

This is referred to as the twisted diagonal condition of order $\sigma \in \mathbb{R}$.

The above asymptotics for $\varepsilon \rightarrow 0$ always holds for $\sigma = 0$, regardless of $a \in S_{1,1}^d$, as was proved in [Hör89, Prop. 3.2], and given as a part of [Hör97, Lem. 9.3.2]:

LEMMA 6.3.1. *When $a \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 < \varepsilon \leq 1$, then $a_{\chi,\varepsilon} \in C^\infty$ and*

$$|D_\eta^\alpha D_x^\beta a_{\chi,\varepsilon}(x, \eta)| \leq C_{\alpha,\beta}(a) \varepsilon^{-|\varepsilon|} (1 + |\eta|)^{d-|\alpha|+|\beta|} \quad (6.50)$$

$$\left(\int_{R \leq |\eta| \leq 2R} |D_\eta^\alpha a_{\chi,\varepsilon}(x, \eta)|^2 d\eta \right)^{1/2} \leq C_\alpha R^d (\varepsilon R)^{n/2-|\alpha|}. \quad (6.51)$$

The map $a \mapsto a_{\chi,\varepsilon}$ is continuous in $S_{1,1}^d$.

The last remark on continuity was added in [Joh10c] because it plays a role in connection with the definition by vanishing frequency modulation. It is easily verified by deducing from the proof of [Hör97, Lem. 9.3.2] that the constant $C_{\alpha,\beta}(a)$ is a continuous seminorm in $S_{1,1}^d$.

In case $a(x, \eta)$ fulfils (6.49) for some $\sigma > 0$, the localised symbol $a_{\chi,\varepsilon}$ tends faster to 0, and this was proved in [Hör89, Hör97] to imply boundedness

$$a(x, D): H^{s+d}(\mathbb{R}^n) \rightarrow H^s(\mathbb{R}^n) \quad \text{for } s > -\sigma. \quad (6.52)$$

The reader could consult [Hör97, Thm. 9.3.5] for this (and [Hör97, Thm. 9.3.7] for four pages of proof of the sharpness of the condition $s > -\sigma$). Consequently, when $\hat{a}(\xi, \eta)$ satisfies (6.49) for all $\sigma \in \mathbb{R}$, then there is boundedness $H^{s+d} \rightarrow H^s$ for all $s \in \mathbb{R}$.

As accounted for below, $a(x, D): \mathcal{S}' \rightarrow \mathcal{S}'$ is furthermore everywhere defined and continuous when $a(x, \eta)$ fulfils (6.49) for every $\sigma \in \mathbb{R}$. However, this relies on L. Hörmander's characterisation of such symbols as those having adjoints of type 1, 1.

6.3.2. The self-adjoint subclass $\tilde{S}_{1,1}^d$. The next result characterises the $a \in S_{1,1}^d$ for which the adjoint symbol a^* is again in $S_{1,1}^d$; cf the below condition (i). Since passage to the adjoint is an involution, such symbols constitute the self-adjoint subclass

$$\tilde{S}_{1,1}^d := S_{1,1}^d \cap (S_{1,1}^d)^*. \quad (6.53)$$

THEOREM 6.3.2. *For every symbol $a(x, \eta)$ in $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ the following properties are equivalent:*

- (i) The adjoint symbol $a^*(x, \eta)$ also belongs to $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$.
(ii) For arbitrary $N > 0$ and α, β there is a constant $C_{\alpha,\beta,N}$ such that

$$|D_\eta^\alpha D_x^\beta a_{\chi,\varepsilon}(x, \eta)| \leq C_{\alpha,\beta,N} \varepsilon^N (1 + |\eta|)^{d-|\alpha|+|\beta|} \quad \text{for } 0 < \varepsilon < 1. \quad (6.54)$$

- (iii) For all $\sigma \in \mathbb{R}$ there is a constant $c_{\alpha,\sigma}$ such that (6.49) holds for $0 < \varepsilon < 1$, ie

$$\sup_{R>0, x \in \mathbb{R}^n} R^{|\alpha|-d} \left(\int_{R \leq |\eta| \leq 2R} |D_\eta^\alpha a_{\chi,\varepsilon}(x, \eta)|^2 \frac{d\eta}{R^n} \right)^{1/2} \leq c_{\alpha,\sigma} \varepsilon^{\sigma + \frac{n}{2} - |\alpha|}. \quad (6.55)$$

In the affirmative case $a \in \tilde{S}_{1,1}^d$, and there is an estimate

$$|D_\eta^\alpha D_x^\beta a^*(x, \eta)| \leq (C_{\alpha,\beta}(a) + C'_{\alpha,\beta,N})(1 + |\eta|)^{d-|\alpha|+|\beta|} \quad (6.56)$$

for a certain continuous seminorm $C_{\alpha,\beta}$ on $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ and a finite sum $C'_{\alpha,\beta,N}$ of constants fulfilling the inequalities in (ii).

It deserves to be mentioned that (i) holds for $a(x, \eta)$ if and only if it holds for $a^*(x, \eta)$ (neither (ii) nor (iii) make this obvious). But (ii) immediately gives that the inclusion $\tilde{S}_{1,1}^d \subset \tilde{S}_{1,1}^{d'}$ holds for $d' > d$, as one would expect. Condition (iii) involves L_2 -norms analogous to those in the Mihlin–Hörmander multiplier theorem and is useful for estimates.

As a small remark it is noted that (ii), (iii) both hold either for all χ satisfying (6.49) or for none, since (i) does not depend on χ . It moreover suffices to verify (ii) or (iii) for $0 < \varepsilon < \varepsilon_0$ for some convenient $\varepsilon_0 \in]0, 1[$, as can be seen from the second inequality in Lemma 6.3.1, since every power ε^p is bounded on the interval $[\varepsilon_0, 1]$.

The theorem is essentially due to L. Hörmander, who stated the equivalence of (i) and (ii) explicitly in [Hör88, Thm. 4.2] and [Hör97, Thm. 9.4.2], in the latter with brief remarks on (iii). Equivalence with continuous extensions $H^{s+d} \rightarrow H^s$ for all $s \in \mathbb{R}$ was also shown, whereas the estimate (6.56) was not mentioned.

However, a full proof of Theorem 6.3.2 was given in [Joh10c, Sec. 4.2], not only because a heavy burden of verification was left to the reader in [Hör97], but more importantly because some consequences for operators defined by vanishing frequency modulation were derived as corollaries to the proof.

It would lead too far here to give the details (that follow the line of thought in [Hör97, Thm. 9.4.2], and are available in [Joh10c, Sec. 4.2]); it should suffice to mention that the final estimate (6.56) in the theorem was derived from the proof that (ii) implies (i).

In its turn, this estimate was shown to imply that the self-adjoint class $\tilde{S}_{1,1}^d$, as envisaged, behaves nicely under full frequency modulation:

COROLLARY 6.3.3 ([Joh10c, Cor. 4.5]). *Whenever $a(x, \eta)$ belongs to $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ and ψ is a modulation function, then it holds for the adjoint symbols that*

$$\left(\psi(2^{-m} D_x) a(x, \eta) \psi(2^{-m} \eta) \right)^* \xrightarrow{m \rightarrow \infty} a(x, \eta)^* \quad (6.57)$$

in the topology of $S_{1,1}^{d+1}(\mathbb{R}^n \times \mathbb{R}^n)$.

Returning to the proof of Proposition 6.2.2, it is clear that the above establishes (6.37) under the weaker assumption that $a \in \tilde{S}_{1,1}^d$. So by repeating the rest of the proof there, one obtains the first main result on $\tilde{S}_{1,1}^d$:

THEOREM 6.3.4 ([Joh10c, Thm. 4.6]). *For every $a(x, \eta)$ of type 1, 1 belonging to the self-adjoint subclass $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$, that is characterised in Theorem 6.3.2, the operator*

$$a(x, D): \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n) \quad (6.58)$$

is everywhere defined and continuous, and it equals the adjoint of $\text{OP}(e^{iD_x \cdot D_\eta} \bar{a}(x, \eta))$.

6.3.3. Paradifferential decompositions for the self-adjoint subclass. The result below gives a generalisation of Theorem 6.2.3 to the operators $a(x, D)$ that merely fulfil the twisted diagonal condition (6.49) for every real σ instead of the strict condition (6.33).

THEOREM 6.3.5 ([Joh10c, Thm. 6.7]). *When $a(x, \eta)$ belongs to $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$, cf Theorem 6.3.2, then $a(x, D)$ is an everywhere defined continuous linear map*

$$a(x, D): \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n), \quad (6.59)$$

with its adjoint $a(x, D)^$ also in $\text{OP}(\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n))$. For each modulation function ψ , the operator fulfils*

$$a(x, D)u = a_\psi^{(1)}(x, D)u + a_\psi^{(2)}(x, D)u + a_\psi^{(3)}(x, D)u, \quad (6.60)$$

where the operators on the right-hand side all belong to $\text{OP}(\tilde{S}_{1,1}^d)$; they are given by the series in (6.9), (6.10), (6.11) that converge rapidly in $\mathcal{S}'(\mathbb{R}^n)$ for every $u \in \mathcal{S}'(\mathbb{R}^n)$.

The proof of this theorem is quite lengthy, and therefore only sketched here. First of all the continuity on \mathcal{S}' is obtained from Theorem 6.3.4, of course. The other statements involve a treatment of 9 infinite series, departing from the splitting obtained from (6.44) with $\varepsilon = 1$,

$$a(x, \eta) = (a(x, \eta) - a_{\chi,1}(x, \eta)) + a_{\chi,1}(x, \eta). \quad (6.61)$$

Here the difference $a - a_{\chi,1}$ fulfils the twisted diagonal condition (6.33) with $B = 1$, so this term is covered by Theorem 6.2.3 (that involves three series).

The remainder $a_{\chi,1}(x, \eta)$ is treated through the paradifferential decomposition (6.12). However, to simplify notation it is useful to note that $\hat{a}_{\chi,1}$ is supported by the set

$$\{ (\xi, \eta) \mid \max(1, |\xi + \eta|) \leq |\eta| \}. \quad (6.62)$$

It therefore suffices to prove the theorem for such symbols $a(x, \eta)$. Here $a_\psi^{(1)}(x, D)$ and $a_\psi^{(3)}(x, D)$ are again covered by Theorem 6.2.3; but the term $a_\psi^{(2)}(x, D)$ is first split into two using (6.30), where each term is subjected to Hörmander's localisation to \mathcal{T} (now applied for the second time).

More specifically, (6.30) gives rise to the series

$$\sum_{k=0}^{\infty} (a^k - a^{k-h})(x, D)u_k, \quad \sum_{k=1}^{\infty} a_k(x, D)(u^{k-1} - u^{k-h}); \quad (6.63)$$

they can be treated in much the same way, so only the latter will be discussed here. Set

$$v_k = u^{k-1} - u^{k-h} = \mathcal{F}^{-1}((\Phi(2^{1-k}\cdot) - \Phi(2^{h-k}\cdot))\hat{u}). \quad (6.64)$$

Using the cut-off function χ from (6.45)–(6.47) to set

$$\hat{a}_{k,\chi,\varepsilon}(\xi, \eta) = \hat{a}(\xi, \eta)\Phi(2^{-k}\xi)\chi(\xi + \eta, \varepsilon\eta), \quad (6.65)$$

the last of the above series is split into two sums by writing

$$a_k(x, D)v_k = a_{k,\chi,\varepsilon}(x, D)v_k + b_k(x, D)v_k. \quad (6.66)$$

Of course the remainder $b_k(x, D)v_k$ also depends on ε , which with $\varepsilon = 2^{-k\theta}$, say for $\theta = 1/2$ is taken to vary with the summation index (as also done in some proofs by L. Hörmander).

Convergence of $\sum b_k(x, D)v_k$ is obtained from Lemma 6.2.4, that this time applies with $\theta_0 = 1 - \theta = 1/2$ and $\theta_1 = 1$, as can be seen with a minor change of the argument for Proposition 6.1.2. The polynomial growth of its terms follows easily from similar control of a_k and of $a_{k,\chi,\varepsilon}$.

To control the $a_{k,\chi,\varepsilon}(x, D)v_k$ it is advantageous to adopt the factorisation inequality,

$$a_{k,\chi,\varepsilon}(x, D)v_k \leq F_{a_{k,\chi,\varepsilon}}(N, R2^k; x)v_k^*(N, R2^k; x), \quad (6.67)$$

where the symbol factor is controlled by certain L_2 -norms according to Theorem 4.1.2.

Indeed, a direct comparison of the L_2 -conditions in Theorem 4.1.2 with those in Theorem 6.3.2 reveal that they only differ in the domain of integration. But the integration area in Theorem 4.1.2 can be covered by a fixed finite number of the annuli appearing in the L_2 -conditions for $a \in \tilde{S}_{1,1}^d$ — and these furthermore yield control by powers of ε ; cf Theorem 6.3.2. It is therefore not surprising that the above gives an estimate of the form

$$|a_{k,\chi,\varepsilon}(x, D)v_k(x)| \leq cv_k^*(N, R2^k; x) \left(\sum_{|\alpha| \leq N + [n/2] + 1} c_{\alpha,\sigma} \varepsilon^{\sigma + n/2 - |\alpha|} \right) (R2^k)^d. \quad (6.68)$$

Here $N = \text{order } \hat{u}$, so for $\theta = 1/2$ the polynomial growth of v_k^* , cf (4.6), entails

$$|a_{k,\chi,2^{-k\theta}}(x, D)v_k(x)| \leq c(1 + |x|)^N 2^{-k(\sigma - 1 - 2d - 3N)/2}. \quad (6.69)$$

Now since $a \in \tilde{S}_{1,1}^d$ one can take σ such that $\sigma > 3N + 2d + 1$, whence $\sum_k \langle a_{k,\chi,\varepsilon}(x, D)v_k, \varphi \rangle$ converges rapidly for $\varphi \in \mathcal{S}$. Thereby all series in the theorem converge as claimed.

Finally, it only remains to note that the remainder terms appearing in connection with the operator $a_{\psi}^{(2)}(x, D)$ tend to 0 for $m \rightarrow \infty$, also under the weak assumption that $a \in \tilde{S}_{1,1}^d$. This can be seen with an argument analogous to the one for Theorem 6.2.3, by using the full generality of Proposition 4.1.5.

The paradifferential decomposition, that was analysed above, is applied in the L_p -estimates in the following sections.

6.4. Domains of type 1, 1-operators

For the possible domains of type 1, 1-operators, the scale $F_{p,q}^s(\mathbb{R}^n)$ of Lizorkin–Triebel spaces was recently shown to play a role, for it was proved in [Joh04, Joh05] that for all $p \in [1, \infty[$, every $a \in S_{1,1}^d$ gives a bounded linear map

$$F_{p,1}^d(\mathbb{R}^n) \xrightarrow{a(x,D)} L_p(\mathbb{R}^n). \quad (6.70)$$

The reader may refer to Section 2.1 for a review of the definition and basic properties of the $F_{p,q}^s$ spaces and of the related Besov spaces $B_{p,q}^s$.

The result in (6.70) is a substitute of boundedness $H_p^d \rightarrow L_p$, or of L_p -boundedness for $d = 0$ (neither of which can hold in general because of Ching’s counter-example [Chi72], recalled in Lemma 3.1.1). Indeed, $H_p^s = F_{p,2}^s$ for $1 < p < \infty$ whereas $F_{p,1}^s \subsetneq F_{p,q}^s$ for $q > 1$, so (6.70) means that $a(x, D)$ is bounded from a sufficiently small subspace of H_p^d .

Moreover, inside the $F_{p,q}^s$ and $B_{p,q}^s$ scales, (6.70) gives *maximal* domains for $a(x, D)$ in L_p , for it was noted in [Joh05, Lem. 2.3] that Ching’s operator is discontinuous $F_{p,q}^d \rightarrow \mathcal{D}'$ and $B_{p,q}^d \rightarrow \mathcal{D}'$ as soon as $q > 1$. This follows as in Lemma 3.1.1 above by simply calculating the norms of v_N in these spaces.

In comparison G. Bourdaud [Bou83, Bou88a] showed the borderline result that every $a(x, D)$ in $\text{OP}(S_{1,1}^0)$ has a bounded extension

$$A: B_{p,1}^0 \rightarrow L_p \quad \text{for } 1 \leq p \leq \infty. \quad (6.71)$$

In view of the embedding $B_{p,1}^s \hookrightarrow F_{p,1}^s$ valid for all finite $p \geq 1$, the result in (6.70) is sharper already for $d = 0$ (unless $p = \infty$)—and for $1 \leq p < \infty$ the best possible within the scales $B_{p,q}^s$, $F_{p,q}^s$ for the full class $\text{OP}(S_{1,1}^d)$; cf the above. Thus the $F_{p,q}^s$ -spaces with $q = 1$ are indispensable for a sharp description of the borderline $s = 0$ for type 1, 1-operators; cf [Joh05, Rem. 1.1].

The above considerations can be summed up thus:

THEOREM 6.4.1 ([Joh05]). *Every $a \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$, $d \in \mathbb{R}$, yields a bounded operator*

$$a(x, D): F_{p,1}^d(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n) \quad \text{for } p \in [1, \infty[, \quad (6.72)$$

$$a(x, D): B_{\infty,1}^d(\mathbb{R}^n) \rightarrow L_\infty(\mathbb{R}^n). \quad (6.73)$$

The class $\text{OP}(S_{1,1}^d)$ contains operators $a(x, D): \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$, that are discontinuous when $\mathcal{S}(\mathbb{R}^n)$ is given the induced topology from any of the Triebel–Lizorkin spaces $F_{p,q}^d(\mathbb{R}^n)$ or Besov spaces $B_{p,q}^d(\mathbb{R}^n)$ with $p \in [1, \infty]$ and $q > 1$ (while \mathcal{D}' has the usual topology).

By the remarks prior to the statements, it remains to discuss the boundedness. Unlike [Joh05] that was based on Marschall’s inequality mentioned in Remark 4.1.1, the present exposition of the proof will be indicated with as much use of the pointwise estimates in Section 4.1 as possible.

However, it is still necessary to invoke the Hardy–Littlewood maximal function, which is given on locally integrable functions $f(x)$ by

$$Mf(x) = \sup_{R>0} \frac{1}{\text{meas}(B(y,R))} \int_{B(y,R)} |f(y)| dy. \quad (6.74)$$

This is needed for the following well-known inequality:

LEMMA 6.4.2. *When $0 < p < \infty$, $0 < q \leq \infty$ and $N > n/\min(p, q)$ are given, there is a constant c such that*

$$\int_{\mathbb{R}^n} \|u_k^*(N, R2^k; x)\|_{\ell_q}^p dx \leq c \int_{\mathbb{R}^n} \|u_k(x)\|_{\ell_q}^p dx \quad (6.75)$$

for every sequence $(u_k)_{k \in \mathbb{N}_0}$ in $L_p(\mathbb{R}^n)$ satisfying $\text{supp } \hat{u}_k \subset \bar{B}(0, R2^k)$, $k \geq 0$.

This inequality and its proof are due to J. Peetre. It might be illuminating to give an outline: taking $r \in]0, \min(p, q)[$ so that $N \geq n/r$ it can be shown (eg as in the paper of M. Yamazaki [Yam86a, Thm. 2.10]) that

$$u_k^*(N, R2^k; x) \leq u_k^*(\frac{n}{r}, R2^k; x) \leq c(Mu_k^r(x))^{1/r}; \quad (6.76)$$

so it suffices to estimate $\int \|(Mu_k^r(x))^{1/r}\|_{\ell_q}^p dx$, but since $r < \min(p, q)$ the fundamental inequality of C. Fefferman and E. M. Stein [FS72] (cf also [Yam86a, Thm. 2.2]) yields an estimate by the right-hand side of (6.75).

6.4.1. Proof of Theorem 6.4.1. To obtain the boundedness one may simply take a finite sum of the inequalities in Proposition 6.1.4 and integrate. Indeed, for (6.20) this gives according to Lemma 6.4.2 that

$$\int |\sum_k a^{k-h}(x, D)u_k(x)|^p dx \leq c \int (\sum_k (R2^k)^d |u_k^*(N, R2^k; x)|)^p dx \leq c' \int (\sum_k 2^{kd} |u_k(x)|)^p dx. \quad (6.77)$$

In particular this holds if the modulation function used in the splitting of Proposition 6.1.1 coincides with the function generating the Littlewood–Paley decomposition in the definition of $F_{p,q}^s$, which can be arranged by Remark 2.2.1. Then the integral on the right-hand side above is less than $\|u\|_{F_{p,1}^d}^p$, so that the terms $2^{kd} |u_k(x)|$ tend to 0 a.e. on \mathbb{R}^n for $k \rightarrow \infty$.

So by taking k in a set of the form $\{K+1, \dots, K+K'\}$, it follows by majorised convergence for $K \rightarrow \infty$ that $\sum a^{k-h}(x, D)u_k$ is a Cauchy series, hence converges in $L_p(\mathbb{R}^n)$. The above inequalities therefore hold verbatim for the sum over all $k \geq h$, that is, $\|\sum a^{k-h}(x, D)u_k\|_{L_p} \leq c\|u\|_{F_{p,1}^d}$. This means that $a^{(1)}(x, D)$ is bounded $F_{p,1}^d \rightarrow L_p$.

The boundedness of $a^{(2)}(x, D)$ is analogous, for in view of (6.13) the above procedure need only be applied to each of the inequalities (6.21) and (6.22). For the latter, the fixed sum over $l = 1, \dots, h-1$ is easily treated via the triangle inequality.

In case of $a^{(3)}(x, D)$ the approach needs a minor modification since the sum in (6.23) has its number of terms increasing with j . But this is easily handled by taking $M > 0$ there and using a well-known summation lemma (that goes back at least to [Yam86a], where it was used for similar purposes): for $s < 0$, $0 < q \leq \infty$ and $b_j \in \mathbb{C}$,

$$\sum_{j=0}^{\infty} 2^{sjq} \left(\sum_{k=0}^j |b_k| \right)^q \leq c \sum_{j=0}^{\infty} 2^{sjq} |b_j|^q. \quad (6.78)$$

Indeed, from this and Lemma 6.4.2 it follows that

$$\begin{aligned} \int |\sum_j a_j(x, D) u^{j-h}(x)|^p dx &\leq c \int (\sum_j 2^{-jM} (\sum_{k=0}^j (R2^k)^{d+M} u_k^*(N, R2^k; x)))^p dx \\ &\leq c' \int (\sum_j 2^{jd} |u_j(x)|)^p dx. \end{aligned} \quad (6.79)$$

Again this yields the convergence in L_p of the series, hence boundedness of $a^{(3)}(x, D)$. By Proposition 6.1.1 the operator $a_\psi(x, D)$ is therefore continuous $F_{p,1}^d \rightarrow L_p$, but since it coincides with $a(x, D)$ on the dense subset $\mathcal{S} \subset F_{p,1}^d$, it does not depend on ψ ; therefore $a(x, D): F_{p,1}^d \rightarrow L_p$ is everywhere defined and bounded.

The Besov case with $p = \infty$ is analogous, although the boundedness of Mf on L_∞ suffices (instead of Lemma 6.4.2) due to the fact that the norms of ℓ_q and L_p are interchanged for $B_{p,q}^s$. Thus $a_\psi(x, D): B_{\infty,1}^d \rightarrow L_\infty$ is bounded for each modulation function ψ . It is well known that $B_{p,1}^d$ has $\mathcal{F}^{-1}\mathcal{E}'$ as a dense subset, and since $a(x, D)$ by its extension to $\mathcal{F}^{-1}\mathcal{E}'$ coincides with each $a_\psi(x, D)$ there, again there is no dependence on ψ ; whence $a(x, D): B_{\infty,1}^d \rightarrow L_\infty$ is everywhere defined and bounded. This yields the proof.

REMARK 6.4.3. It should be mentioned that L_p -boundedness on $L_p \cap \mathcal{F}^{-1}\mathcal{E}'$ is much easier to establish, as indicated in (4.8). In view of Ching's counter-example, this result is somewhat striking, hence was formulated as Theorem 6.1 in [Joh10a].

6.5. General Continuity Results

In addition to the borderline case $s = 0$, that was treated above, it is natural to expect that the $F_{p,q}^s$ scale is invariant under $a(x, D)$ as soon as $s > 0$. More precisely, the programme is to show that if $a(x, \eta)$ is of order, or rather degree $d \in \mathbb{R}$, then $a(x, D)$ is bounded from $F_{p,q}^{s+d}$ to $F_{p,q}^s$.

This is true to a wide extent, and in the verification one may by and large use the procedure from Section 6.4. However, for $q \neq 2$ one cannot just appeal to the completeness of L_p , but the spectral properties in Proposition 6.1.2 and 6.1.3 make it possible to apply the following lemma.

LEMMA 6.5.1. *Let $s > \max(0, \frac{n}{p} - n)$ for $0 < p < \infty$ and $0 < q \leq \infty$ and suppose $u_j \in \mathcal{S}'(\mathbb{R}^n)$ such that, for some $A > 0$,*

$$\text{supp } \mathcal{F} u_j \subset B(0, A2^j), \quad F(q) := \left\| \left(\sum_{j=0}^{\infty} 2^{sjq} |u_j(\cdot)|^q \right)^{\frac{1}{q}} \right\|_p < \infty. \quad (6.80)$$

Then $\sum_{j=0}^{\infty} u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to some u in the space $F_{p,r}^s(\mathbb{R}^n)$ for $r \geq q$, $r > \frac{n}{n+s}$, fulfilling $\|u\|_{F_{p,r}^s} \leq cF(r)$ for some $c > 0$ depending on n, s, p and r .

When moreover $\text{supp } \mathcal{F} u_j \subset \{\xi \mid A^{-1}2^j \leq |\xi| \leq A2^j\}$ for $j \geq J$ for some $J \geq 1$, then the conclusions are valid for all $s \in \mathbb{R}$ and $r = q$.

This has been known for $r = q$ at least since the pseudo-differential L_p -estimates of M. Yamazaki [Yam86a, Yam86b], who proved it under the stronger assumption that

$$s > \max(0, \frac{n}{p} - n, \frac{n}{q} - n). \quad (6.81)$$

However, the above sharpening was derived as an addendum in [Joh05] by noting that $F(r) \leq F(q) < \infty$ for $r \geq q$. (The case $J > 1$ is also an addendum, which was used tacitly in [Yam86a, Yam86b].)

The following theorem is taken from [Joh05], where the methods were essentially the same as here, except that there was no explicit reference to the definition by vanishing frequency modulation (this definition was first crystallised in [Joh08b], inspired by the proof of the theorem in [Joh05]). However, the proof was only sketched in [Joh05], so full explanations were given in [Joh10c, Thm. 7.4].

Previous works on such L_p -results include those of G. Bourdaud [Bou83, Bou88a], T. Runst [Run85a] and R. Torres [Tor90]. Besides the remarks in Section 1.2 it should be noted here, that they worked under the assumption (6.81), which has the disadvantage of excluding arbitrarily large values of s (even for $p = 2$) in the limit $q \rightarrow 0^+$. As indicated this is unnecessary:

THEOREM 6.5.2 ([Joh05, Cor. 6.2],[Joh10c, Thm. 7.4]). *If $a \in S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ the corresponding operator $a(x, D)$ is a bounded map for $s > \max(0, \frac{n}{p} - n)$, $0 < p, q \leq \infty$,*

$$a(x, D): F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,r}^s(\mathbb{R}^n) \quad (p < \infty); \quad (6.82)$$

$$a(x, D): B_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n). \quad (6.83)$$

Hereby $r \geq q$ and $r > n/(n+s)$. If (6.33) holds, then (6.82) and (6.83) do so for all $s \in \mathbb{R}$ and $r = q$.

REMARK 6.5.3. It should be noted that in the Banach space case (ie when $p \geq 1$ and $q \geq 1$), one can always take $r = q$ in (6.82), since $q \geq 1 > n/(n+s)$ then.

PROOF. If ψ is an arbitrary modulation function, then Remark 2.2.1 shows that $\|u\|_{F_{p,q}^s}$ can be calculated in terms of the Littlewood–Paley partition associated with ψ ; cf Section 6.1.

For the treatment of $a^{(1)}(x, D)u = \sum_{k=h}^{\infty} a^{k-h}(x, D)u_k$ with $u \in F_{p,q}^s$ note that an application of the norms of ℓ_q and L_p on both sides of the pointwise estimate (6.20) gives, if $q < \infty$ for simplicity's sake,

$$\int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} 2^{skq} |a^{k-h}(x, D)u_k(x)|^q \right)^{\frac{p}{q}} dx \leq c_2 \left\| \left(\sum_{k=0}^{\infty} 2^{(s+d)kq} u_k^*(N, R2^k; x)^q \right)^{\frac{1}{q}} \right\|_{F_{p,q}^s}^p. \quad (6.84)$$

If $N > n/\min(p, q)$ here, (6.75) gives an estimate from above by $\|u\|_{F_{p,q}^{s+d}}$, so that one has the bound in Lemma 6.5.1 for all $s \in \mathbb{R}$, whilst the corona condition holds by Proposition 6.1.2. Thus the lemma gives

$$\|a^{(1)}(x, D)u\|_{F_{p,q}^s} \leq c \left(\int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} 2^{skq} |a^{k-h}(x, D)u_k(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \leq c' \|u\|_{F_{p,q}^{s+d}}. \quad (6.85)$$

In the contribution $a^{(3)}(x, D)u = \sum_{j=h}^{\infty} a_j(x, D)u^{j-h}$ one may apply (6.78) to the estimate (6.23) for $M > s$ to get that

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{sjq} |a_j(x, D)u^{j-h}(x)|^q &\leq \sum_{j=0}^{\infty} 2^{(s-M)jq} \left(\sum_{k=0}^j c_M (R2^k)^{d+M} u_k^*(N, R2^k; x) \right)^q \\ &\leq c \sum_{j=0}^{\infty} 2^{(s+d)jq} u_j^*(N, R2^j; x)^q, \end{aligned} \quad (6.86)$$

which by integration entails

$$\left(\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{sjq} |a_j(x, D)u^{j-h}(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \leq c_3 \left\| \left(\sum_{j=0}^{\infty} 2^{(s+d)jq} u_j^*(x)^q \right)^{\frac{1}{q}} \right\|_p. \quad (6.87)$$

Repeating the argument for (6.85) one arrives at $\|a^{(3)}(x, D)u\|_{F_{p,q}^s} \leq c\|u\|_{F_{p,q}^{s+d}}$.

In estimates of $a^{(2)}(x, D)u$ the various terms can be treated similarly, now departing from (6.21) and (6.22). This gives

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} 2^{skq} |(a^k - a^{k-h})(x, D)u_k(x) + a_k(x, D)(u^{k-1} - u^{k-h})|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\ \leq c'_2 \left\| \left(\sum_{k=0}^{\infty} 2^{(s+d)kq} u_k^*(x)^q \right)^{\frac{1}{q}} \right\|_p. \end{aligned} \quad (6.88)$$

If moreover the twisted diagonal condition (6.33) holds, the last part of Proposition 6.1.3 and (6.75) show that the argument for (6.85), mutatis mutandis, gives $\|a^{(2)}(x, D)u\|_{F_{p,q}^s} \leq c\|u\|_{F_{p,q}^{s+d}}$. Altogether one has then, for all $s \in \mathbb{R}$,

$$\|a_{\psi}(x, D)u\|_{F_{p,q}^s} \leq \sum_{j=1,2,3} \|a^{(j)}(x, D)u\|_{F_{p,q}^s} \leq cp(a)\|u\|_{F_{p,q}^{s+d}}. \quad (6.89)$$

Otherwise the spectra are by Proposition 6.1.3 only contained in balls, but the condition $s > \max(0, \frac{n}{p} - n)$ and those on r imply that $\|a^{(2)}(x, D)u\|_{F_{p,r}^s} \leq c\|u\|_{F_{p,q}^{s+d}}$; cf Lemma 6.5.1. This gives the above inequality with q replaced by r on the left-hand side.

Thus $a_{\psi}(x, D): F_{p,q}^{s+d} \rightarrow F_{p,r}^s$ is continuous, but since \mathcal{S} is dense in $F_{p,q}^{s+d}$ for $q < \infty$ (and $F_{p,\infty}^{s+d} \hookrightarrow F_{p,1}^{s'}$ for $s' < s+d$), there is no dependence on ψ . Hence $u \in D(a(x, D))$ and the above inequalities hold for $a(x, D)u$, which proves (6.82) in all cases.

The Besov case is analogous; one can interchange the order of L_p and ℓ_q and refer to the maximal inequality for scalar functions: Lemma 6.5.1 carries over to $B_{p,q}^s$ in a natural way for $0 < p \leq \infty$ with $r = q$ in all cases; this is well known, cf [Yam86a, Joh05, JS08]. \square

In the above proof, the Besov result (6.83) can also be obtained by real interpolation of (6.82), since the sum exponent is inherited from the interpolation method; cf [Tri83, 2.4.2]. However, this will not cover $p = \infty$ in (6.83), as this is excluded in (6.82).

By duality, Theorem 6.5.2 implies an extension to operators that fulfil the twisted diagonal condition of arbitrary real order. This is a main result.

THEOREM 6.5.4 ([Joh10c, Thm. 7.5]). *Let $a(x, \eta)$ belong to the self-adjoint subclass $\tilde{S}_{1,1}^d$, as characterised in Theorem 6.3.2. Then $a(x, D)$ is a bounded map for all $s \in \mathbb{R}$,*

$$a(x, D): F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n), \quad 1 < p < \infty, 1 < q \leq \infty, \quad (6.90)$$

$$a(x, D): B_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n), \quad 1 < p \leq \infty, 1 < q \leq \infty. \quad (6.91)$$

PROOF. When $p' + p = p'p$ and $q' + q = q'q$, then $F_{p,q}^s$ is the dual of $F_{p',q'}^{-s}$ since $1 < p' < \infty$ and $1 \leq q' < \infty$; cf [Tri83, 2.11]. The adjoint symbol $a^*(x, \eta)$ is in $S_{1,1}^d$ by assumption, so

$$a^*(x, D): F_{p',q'}^{-s}(\mathbb{R}^n) \rightarrow F_{p',q'}^{-s-d}(\mathbb{R}^n) \quad (6.92)$$

is continuous whenever $-s - d > \max(0, \frac{n}{p'} - n) = 0$, ie for $s < -d$; this follows from Theorem 6.5.2 since $p' \geq 1$ and $q' \geq 1$. The adjoint $a^*(x, D)^*$ is therefore bounded $F_{p,q}^{s+d} \rightarrow F_{p,q}^s$, and it equals $a(x, D)$ according to Theorem 6.3.4.

For $s > 0$ the property (6.90) holds by Theorem 6.5.2. If $d \geq 0$ the gap with $s \in [-d, 0]$ can be closed by a reduction to order -1 ; cf [Joh10c].

For the $B_{p,q}^s$ scale similar arguments apply, also for $p = \infty$. \square

For symbols $a(x, \eta)$ in $\tilde{S}_{1,1}^d$ the special case $p = 2 = q$ of the above corollary, ie continuity $a(x, D): H^{s+d} \rightarrow H^s$ for all $s \in \mathbb{R}$, was obtained by Hörmander as an immediate consequence of [Hör89, Thm. 4.1], but first formulated in [Hör97, Thm. 9.4.2].

In comparison Theorem 6.5.4 may appear as a rather wide generalisation to the L_p -setting. However, a specialisation of the two above results to Sobolev and Hölder–Zygmund spaces gives the following result directly from the standard identifications (2.12), (2.13).

COROLLARY 6.5.5 ([Joh10c, Cor. 7.6]). *Every $a(x, D) \in \text{OP}(S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n))$ is bounded*

$$a(x, D): H_p^{s+d}(\mathbb{R}^n) \rightarrow H_p^s(\mathbb{R}^n), \quad s > 0, 1 < p < \infty, \quad (6.93)$$

$$a(x, D): C_*^{s+d}(\mathbb{R}^n) \rightarrow C_*^s(\mathbb{R}^n), \quad s > 0. \quad (6.94)$$

This holds for all real s whenever $a(x, \eta)$ belongs to the self-adjoint subclass $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$.

The last extension to $p \neq 2$ of the results on $\tilde{S}_{1,1}^d$ in [Hör89, Hör97] is also new.

6.6. Direct estimates for the self-adjoint subclass

To complement Theorem 6.5.4 with similar results valid for p, q in $]0, 1]$, it is natural to exploit the paradifferential decomposition (6.12) and the pointwise estimates used above.

However, in the results to follow below there will be an arbitrarily small loss of smoothness. The reason is that the estimates of $a_{\psi}^{(2)}(x, D)$ will be based on a corona condition, which is now *asymmetric* in the sense that the outer radii grow faster than the inner ones. That is, the last part of Lemma 6.5.1 will now be extended to series $\sum u_j$ fulfilling the more general condition, where $0 < \theta \leq 1$ and $A > 1$,

$$\begin{aligned} \text{supp } \mathcal{F}u_0 &\subset \{ \xi \mid |\xi| \leq A2^j \}, \quad \text{for } j \geq 0, \\ \text{supp } \mathcal{F}u_j &\subset \{ \xi \mid \frac{1}{A}2^{\theta j} \leq |\xi| \leq A2^j \} \quad \text{for } j \geq J \geq 1. \end{aligned} \quad (6.95)$$

This situation is probably known to experts in function spaces, but in lack of a reference it was analysed with standard techniques from harmonic analysis in [**Joh10c**].

The main point of (6.95) is that $\sum u_j$ still converges for $s \leq 0$, albeit with a loss of smoothness that arises in the cases below with $s' < s$. Actually the loss is proportional to $(1 - \theta)/\theta$, hence tends to ∞ for $\theta \rightarrow 0$, which reflects that convergence in some cases fails for $\theta = 0$ (as can be easily seen for $\hat{u}_j = \frac{1}{j}\psi \in C_0^\infty$, $s = 0$, $1 < q \leq \infty$).

PROPOSITION 6.6.1 ([**Joh10c**, Prop. 7.7]). *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $J \in \mathbb{N}$ and $0 < \theta < 1$ be given; with $q > n/(n+s)$ if $s > 0$. For each sequence $(u_j)_{j \in \mathbb{N}_0}$ in $\mathcal{S}'(\mathbb{R}^n)$ fulfilling the corona condition (6.95) together with the bound (usual modification for $q = \infty$)*

$$F := \left\| \left(\sum_{j=0}^{\infty} |2^{sj} u_j(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L_p} < \infty, \quad (6.96)$$

the series $\sum_{j=0}^{\infty} u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to some $u \in F_{p,q}^{s'}(\mathbb{R}^n)$ with

$$\|u\|_{F_{p,q}^{s'}} \leq cF, \quad (6.97)$$

whereby the constant c also depends on s' , that one can take to fulfil

$$s' = s \quad \text{for } s > \max(0, \frac{n}{p} - n), \quad (6.98)$$

$$s' < s/\theta \quad \text{for } s \leq 0, p \geq 1, q \geq 1, \quad (6.99)$$

or in general

$$s' < s - \frac{1-\theta}{\theta} (\max(0, \frac{n}{p} - n) - s)_+. \quad (6.100)$$

(Note that $s' = s$ is possible if the positive part $(\dots)_+$ has strictly negative argument.)

The conclusions carry over to $B_{p,q}^{s'}$ for any $q \in]0, \infty]$ when $B := (\sum_{j=0}^{\infty} 2^{sjq} \|u_j\|_p^q)^{\frac{1}{q}} < \infty$.

REMARK 6.6.2. The restriction above that $q > n/(n+s)$ for $s > 0$ is not severe, for if (6.96) holds for a sum-exponent in $]0, n/(n+s)]$, then the constant F is also finite for any $q > n/(n+s)$, which yields convergence and an estimate in a slightly larger space (but for the same s and p).

The proof of the proposition is omitted here, since it is lengthy and only has little in common with the treatment of type 1, 1-operators.

However, thus prepared one can obtain the next result, which provides a general result for $0 < p \leq 1$.

THEOREM 6.6.3 ([**Joh10c**, Thm. 7.9]). *For every symbol $a(x, \eta)$ belonging to the self-adjoint subclass $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ the operator $a(x, D)$ is bounded for $0 < p \leq 1$ and $0 < q \leq \infty$,*

$$a(x, D): F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,q}^{s'}(\mathbb{R}^n), \quad \text{for } s' < s \leq \frac{n}{p} - n, \quad (6.101)$$

$$a(x, D): B_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow B_{p,q}^{s'}(\mathbb{R}^n) \quad \text{for } s' < s \leq \frac{n}{p} - n. \quad (6.102)$$

PROOF. The theorem follows by elaboration of the proof of Theorem 6.3.5. By applying the last part of Theorem 6.5.2 to the difference $a - a_{\chi,1}$, the question is reduced to the case of $a_{\chi,1}$. Again this will be covered by treating general $a \in \tilde{S}_{1,1}^d$ for which $\hat{a}(\xi, \eta) \neq 0$ only holds for $\max(1, |\xi + \eta|) \leq |\eta|$.

Under this assumption, $a^{(1)}(x, D)u$ and $a^{(3)}(x, D)u$ are covered by Theorem 6.5.2 for all $s \in \mathbb{R}$; cf (6.89). Thus it suffices to estimate the series in (6.63) for fixed $s' < s \leq 0$.

Taking in (6.68) the parameter $\varepsilon = 2^{-k\theta}$ for $\theta \in]0, 1[$ so small that s' fulfils the last condition in Proposition 6.6.1 with $1 - \theta$ instead of θ (cf remarks prior to (6.108)), clearly

$$2^{k(s+M)} |a_{k,\chi,\varepsilon}(x, D)v_k(x)| \leq cv_k^*(N, R2^k; x) 2^{k(s+d)} 2^{-k\theta(\sigma-1-N-M/\theta)}. \quad (6.103)$$

Here one may first of all take $N > n/\min(p, q)$ so that (6.75) applies. Secondly, since by assumption $a(x, \eta)$ fulfils the twisted diagonal condition (6.49) of any real order, σ can for any M (with θ fixed as above) be chosen so that $2^{-k\theta(\sigma-1-N-M/\theta)} \leq 1$. This gives

$$\begin{aligned} \left(\int \|2^{k(s+M)} a_{k,\chi,\varepsilon}(x, D)v_k(\cdot)\|_{\ell_q}^p dx \right)^{\frac{1}{p}} &\leq c \left(\int \|2^{k(s+d)} v_k^*(N, R2^k; \cdot)\|_{\ell_q}^p dx \right)^{\frac{1}{p}} \\ &\leq c' \left(\int \|2^{k(s+d)} v_k(\cdot)\|_{\ell_q}^p dx \right)^{\frac{1}{p}} \leq c'' \|u\|_{F_{p,q}^{s+d}}. \end{aligned} \quad (6.104)$$

Here the last inequality follows from the (quasi-)triangle inequality in ℓ_q and L_p .

Since the spectral support rule and Proposition 6.1.3 imply that $a_{k,\chi,\varepsilon}(x, D)v_k$ also has its spectrum in the ball $B(0, 2R2^k)$, the above estimate allows application of Lemma 6.5.1, if M is so large that

$$M > 0, \quad M + s > 0, \quad M + s > \frac{n}{p} - n. \quad (6.105)$$

This gives convergence of $\sum a_{k,\chi,2^{-k\theta}}(x, D)v_k$ to a function in $F_{p,\infty}^{s+M}$ fulfilling

$$\left\| \sum_{k=1}^{\infty} a_{k,\chi,2^{-k\theta}}(x, D)v_k \right\|_{F_{p,\infty}^{s+M}} \leq c \|u\|_{F_{p,q}^{s+d}}. \quad (6.106)$$

On the left-hand side the embedding $F_{p,\infty}^{s+M} \hookrightarrow F_{p,q}^s$ applies, of course.

For the remainder $\sum_{k=1}^{\infty} b_k(x, D)v_k$, cf (6.65) ff, note that (6.104) holds for $M = 0$ with the same σ . If combined with (6.88), it follows by the (quasi-)triangle inequality that

$$\int \|2^{ks} b_k(x, D)v_k(\cdot)\|_{\ell_q}^p dx \leq \int \|2^{ks} (a_k(x, D) - a_{k,\chi,2^{-k\theta}}(x, D))v_k(\cdot)\|_{\ell_q}^p dx \leq c \|u\|_{F_{p,q}^{s+d}}^p. \quad (6.107)$$

In addition the series can be shown (cf the proof of Theorem 6.3.5) to fulfil a corona condition with inner radius $2^{(1-\theta)k}$ for all sufficiently large k , so that Proposition 6.6.1 applies. By the choice of θ , this gives

$$\left\| \sum_{k=1}^{\infty} b_k(x, D)v_k \right\|_{F_{p,q}^{s'}} \leq c \|u\|_{F_{p,q}^{s+d}}. \quad (6.108)$$

In $a_{\psi}^{(2)}(x, D)u$ the other contribution $\sum (a^k(x, D) - a^{k-h}(x, D))u_k$, cf (6.63), can be treated similarly. This was also done in the proof of Theorem 6.3.5, where in particular (6.68) was

shown to hold for $(a^k - a^{k-h})_{\chi,\varepsilon}(x,D)u_k$, with just a change of the constant. Consequently (6.103) carries over, and with (6.105) the same arguments as for (6.106), (6.108) give

$$\left\| \sum_{k=h}^{\infty} (a^k - a^{k-h})_{\chi,\varepsilon}(x,D)u_k \right\|_{F_{p,\infty}^{s+M}} + \left\| \sum_{k=h}^{\infty} \tilde{b}_k(x,D)u_k \right\|_{F_{p,q}^{s'}} \leq c \|u\|_{F_{p,q}^{s+d}}. \quad (6.109)$$

Altogether the estimates (6.106), (6.108), (6.109) show that

$$\|a_{\psi}^{(2)}(x,D)u\|_{F_{p,q}^{s'}} \leq c \|u\|_{F_{p,q}^{s+d}}. \quad (6.110)$$

Via the paradifferential decomposition (6.12), the operator $a_{\psi}(x,D)$ is therefore a bounded linear map $F_{p,q}^{s+d} \rightarrow F_{p,q}^{s'}$. Since \mathcal{S} is dense for $q < \infty$ (a case one can reduce to), there is no dependence on the modulation function ψ , so the type 1, 1-operator $a(x,D)$ is defined and continuous on $F_{p,q}^{s+d}$ as stated.

The arguments are similar for the Besov spaces: it suffices to interchange the order of the norms in ℓ_q and L_p , and to use the estimate in (6.75) for each single k . \square

The proof above extends to cases with $0 < p \leq \infty$ when $s' < s \leq \max(0, \frac{n}{p} - n)$. But even though it like the proof of Theorem 6.3.5 uses a delicate splitting of $a(x,D)$ into 9 infinite series, it barely fails to reprove Theorem 6.5.4 due to the loss of smoothness. Therefore only $p \leq 1$ is included in Theorem 6.6.3.

When taken together, Theorems 6.5.2, 6.5.4 and 6.6.3 give a satisfactory L_p -theory of operators $a(x,D)$ in $\text{OP}(\tilde{S}_{1,1}^d)$, inasmuch as for the domain $D(a(x,D))$ they cover all possible s, p . Only a few of the codomains seem barely unoptimal, and these all concern cases with $0 < q < 1$ or $0 < p \leq 1$; cf the parameters r in Theorem 6.5.2 and s' in Theorem 6.6.3.

One particular interest of Theorem 6.6.3 concerns the well-known identification of $F_{p,2}^0(\mathbb{R}^n)$ with the so-called local Hardy space $h_p(\mathbb{R}^n)$ for $0 < p \leq 1$, which is described in [Tri83] and especially [Tri92, Ch. 1.4]. Here Theorem 6.6.3 gives boundedness $a(x,D): h_p(\mathbb{R}^n) \rightarrow F_{p,2}^{s'}(\mathbb{R}^n)$ for every $s' < 0$, but this can probably be improved in view of recent results:

REMARK 6.6.4. Extensions to $h_p(\mathbb{R}^n)$ of operators in the self-adjoint subclass $\text{OP}(\tilde{S}_{1,1}^0)$ were treated by J. Hounie and R. A. dos Santos Kapp [HdSK09], who used atomic estimates to carry over the L_2 -boundedness of Hörmander [Hör89, Hör97] to h_p , ie to obtain estimates with $s' = s = 0$. However, they worked without a precise definition of type 1, 1-operators.

REMARK 6.6.5. As an additional merit, the proof above gives that when $a(x,D)$ fulfils the twisted diagonal condition of a specific order $\sigma > 0$, then for $1 \leq p \leq \infty$

$$B_{p,q}^s \cup F_{p,q}^s \subset D(a(x,D)) \quad \text{for } s > -\sigma + [n/2] + 2. \quad (6.111)$$

While this does provide a result in the L_p set-up, it is hardly optimal in view of L. Hörmander's condition $s > -\sigma$ for $p = 2$, that was recalled in (6.52).

CHAPTER 7

Final remarks

In view of the satisfying results on type 1, 1-operators in Chapter 5 and the continuity results in $\mathcal{S}'(\mathbb{R}^n)$ and in the scales H_p^s , C_*^s , $F_{p,q}^s$ and $B_{p,q}^s$ presented in Chapter 6, their somewhat unusual definition by vanishing frequency modulation in Definition 3.1.2 should be well motivated.

As an open problem, it remains to characterise the type 1, 1-operators $a(x, D)$ that are everywhere defined and continuous on $\mathcal{S}'(\mathbb{R}^n)$. For this it was shown above to be sufficient that $a(x, \eta)$ is in $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$, and it could of course be conjectured that this is necessary as well.

Similarly, since the works of G. Bourdaud and L. Hörmander, cf [Bou83, Ch. IV], [Bou88a], [Hör88, Hör89] and also [Hör97], it has remained an open problem to determine the operator class

$$\mathbb{B}(L_2(\mathbb{R}^n)) \cap \text{OP}(S_{1,1}^0). \quad (7.1)$$

Indeed, it was shown by G. Bourdaud that this contains the self-adjoint subclass $\text{OP}(\tilde{S}_{1,1}^0)$, and this sufficient condition has led some authors to somewhat misleading statements, eg that lack of L_2 -boundedness for $\text{OP}(S_{1,1}^0)$ is “attributable to the lack of self adjointness”. But self-adjointness is not necessary, since already G. Bourdaud, by modification of Ching’s symbol (1.4), gave an example [Bou88a, p. 1069] of an operator $\sigma(x, D)$ in the subset $\mathbb{B}(L_2) \cap \text{OP}(S_{1,1}^0 \setminus \tilde{S}_{1,1}^0)$, for which $\sigma(x, D)^*$ is not of type 1, 1.

As a summary, it is noted that the work grew out of the analysis of semi-linear boundary value problems, which was reviewed in Section 1.3. In particular the need for a proof of pseudo-locality of type 1, 1-operators was identified, as was the question of how they can be defined at all. Subsequently Definition 3.1.2 by vanishing frequency modulation was crystallised from the borderline analysis in Section 6.4. And during investigation of the definition’s consequences, the general techniques of pointwise estimates, the spectral support rule and stability under regular convergence was developed. With these tools, the properties (I)–(XII) in Chapter 3 were derived for pseudo-differential operators of type 1, 1.

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Resumé

Denne afhandling vedrører den type af pseudodifferential operatorer, der er kendt i litteraturen som type 1, 1-operatorer. Disse har været kendt i moderne matematisk analyse især siden 1980, da det blev vist, at de spiller en væsentlig rolle for behandlingen af fuldt ikke-lineære partielle differentiaalligninger.

Bidragene i denne afhandling skal ses på baggrund af de fundamentale resultater, der blev opnået i 1988–89 af G. Bourdaud og L. Hörmander. Disse viste at type 1, 1-operatorer har en række egenskaber, der afviger væsentligt fra andre pseudodifferential operatorers.

I afhandlingens arbejder fremføres for første gang en præcis definition, af hvad en type 1, 1-operator er i almindelighed. Dette følges op af en redegørelse for, at definitionen indeholder flere af de tidligere udvidelser af begrebet.

Med udgangspunkt deri bevises en tredive år gammel formodning om, at type 1, 1-operatorer er pseudolokale; det vil sige, at de ikke kan skabe nye singulariter i de funktioner, de virker på. Desuden almindeliggøres et tidligere resultat om, at disse operatorer kan ændre eksisterende singulariteter; dette gøres ved at inddrage Weierstrass' intetsteds differentiable funktion.

Det udledes også, hvorledes type 1, 1-operatorer ændrer støtten og spektret af den funktion, der virkes på. Spørgsmålet, om hvilke funktioner en given operator *kan* virke på, er også diskuteret indgående. Som et nyt resultat er det vist, at enhver type 1, 1-operator kan anvendes på alle glatte funktioner, der er tempererede i L. Schwartz' forstand.

For generelle tempererede distributioner er det vist, at type 1, 1-operatorer kan virke på dem, hvis deres symboler efter delvis Fouriertransformering forsvinder i en kegleformet omegn af skævdiagonalen i det fulde frekvensrum. Mere generelt er dette bevist for de operatorer, der tilhører den selvadjungerede delklasse.

Ydermere er tilsvarende egenskaber blevet givet en omfattende behandling i flere skalaer af funktionsrum, så som Hölderrum og Sobolevrum samt de mere generelle Besov og Lizorkin–Triebelrum. Disse beskriver en række forskellige differentierbarheds- og integrabilitetsegenskaber.

Som en vigtig metode til opnåelse af disse resultater er der blevet indført en almen ramme for punktvis vurderinger af pseudodifferential operatorer. Et generelt resultat er den såkaldte faktoriseringsulighed, som udsiger at virkningen af en operator på en funktion, med kompakt spektrum for eksempel, altid er mindre end en vis symbolfaktor multipliceret med en maksimalfunktion af Peetre–Fefferman–Stein type. Via en analyse af symbolfaktorens asymptotiske egenskaber er uligheden vist at have et frugtbart samspil med de dyadiske dekompositioner, der indgår som en hovedingrediens i kontinuitetsanalysen af type 1, 1-operatorer.