

# $L_p$ -THEORY OF TYPE 1,1-OPERATORS

JON JOHNSEN

*Dedicated to Professor Hans Triebel on the Occasion of his Seventy Fifth Birthday*

ABSTRACT. This is a continuation of recent work on the general definition of pseudo-differential operators of type 1,1, in Hörmander's sense. Continuity in  $L_p$ -Sobolev spaces and Hölder–Zygmund spaces, and more generally in Besov and Lizorkin–Triebel spaces, is proved for positive smoothness; with extension to arbitrary smoothness for operators in the self-adjoint subclass. As a main tool the paradifferential decomposition is used for type 1,1-operators in combination with the Spectral Support Rule for pseudo-differential operators and pointwise estimates in terms of maximal functions of Peetre–Fefferman–Stein type.

## 1. INTRODUCTION

The understanding of pseudo-differential operators of type 1,1 and their applications developed crucially in the 1980's through works of Meyer [Mey81], Bony [Bon81], Bourdaud [Bou83, Bou88], Hörmander [Hör88, Hör89]; cf also the revised exposition in [Hör97, Ch. 9]. Their theory was taken up again more recently by the author, who showed that Lizorkin–Triebel spaces  $F_{p,q}^s$  are optimal for certain borderlines [Joh04, Joh05].

However, the first general definition of type 1,1-operators was given in 2008 by the author in [Joh08b] and used there in a discussion of unclosability, pseudo-locality, non-preservation of wavefront sets and the Spectral Support Rule. The present paper continues the work in [Joh08b] with a systematic approach to their  $L_p$ -theory.

Recall that by definition, the symbol  $a(x, \eta)$  of a type 1,1-operator of order  $d \in \mathbb{R}$  fulfils

$$|D_\eta^\alpha D_x^\beta a(x, \eta)| \leq C_{\alpha,\beta} (1 + |\eta|)^{d - |\alpha| + |\beta|} \quad \text{for } x, \eta \in \mathbb{R}^n. \quad (1.1)$$

The corresponding operator is for *Schwartz functions*, ie for  $u \in \mathcal{S}(\mathbb{R}^n)$ , given by

$$a(x, D)u = (2\pi)^{-n} \int e^{-ix \cdot \eta} a(x, \eta) \hat{u}(\eta) d\eta. \quad (1.2)$$

But a general definition for  $u \in \mathcal{S}' \setminus \mathcal{S}$  must take into account that in some cases they can only be defined on proper subspaces  $E \subset \mathcal{S}'$ .

---

2000 *Mathematics Subject Classification.* 35S05,47G30.

*Key words and phrases.* Exotic pseudo-differential operators, type 1,1, twisted diagonal condition, paradifferential decomposition, Spectral Support Rule, factorisation inequality, corona conditions.

Supported by the Danish Council for Independent Research, Natural Sciences (Grant No. 09-065927)

Appeared in *Mathematische Nachrichten*, **286** (2013), 712--729.

A rigorous definition of type 1, 1-operators was first given in [Joh08b]. Indeed, it was proposed to stipulate that  $u$  belongs to the domain  $D(a(x, D))$  and to set

$$a(x, D)u := \lim_{m \rightarrow \infty} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \psi(2^{-m} D_x) a(x, \eta) \psi(2^{-m} \eta) \hat{u}(\eta) d\eta \quad (1.3)$$

whenever this limit exists in  $\mathcal{D}'(\mathbb{R}^n)$  for all the  $\psi \in C_0^\infty(\mathbb{R}^n)$  with  $\psi = 1$  in a neighbourhood of the origin and does not depend on such  $\psi$ . (More precisely, one should replace the integral above by the action of  $\text{OP}(\psi(2^{-m} D_x) a(x, \eta) \psi(2^{-m} \eta))$  in  $\text{OP}(S^{-\infty})$  on  $u$ .)

This unconventional definition, by *vanishing frequency modulation*, was motivated by the applications of type 1, 1-operators in the theory of semi-linear elliptic boundary problems in the author's work [Joh08a].

In the present paper, the main question is to obtain boundedness

$$\|a(x, D)u\|_s \leq c_s \|u\|_{s+d}, \quad (1.4)$$

where the  $s$ -dependent norms can be those of the Sobolev spaces  $H_p^s$  (for a fixed  $p \in ]1, \infty[$ ), the Hölder–Zygmund spaces  $C_*^s$ , or even of the Besov spaces  $B_{p,q}^s$  or Lizorkin–Triebel spaces  $F_{p,q}^s$ . These  $L_p$ -results constitute an important justification of the definition in (1.3).

The proofs are based on Littlewood–Paley theory, where it has been most useful to adopt the pointwise estimates in the recent article [Joh11]. Indeed, this gives the *factorisation inequality*

$$|a(x, D)u(x)| \leq F_a(x) \cdot u^*(x) \quad (1.5)$$

in terms of the Peetre–Fefferman–Stein maximal function  $u^*(x) = \sup_{y \in \mathbb{R}^n} |u(x-y)| (1+R|y|)^{-N}$ . This was introduced in the theory of  $F_{p,q}^s$  spaces in 1975 by Peetre [Pee75], and soon adopted in the works of Triebel [Tri78, Tri83, Tri92] and others. The systematic use of  $u^*$  for control of pseudo-differential operators, cf. (1.5), was seemingly first proposed in [Joh11].

The symbol factor  $F_a(x)$  in (1.5) is easily controlled in terms of integrals reminiscent of the Mihlin–Hörmander multiplier theorem; cf. Theorem 3.1 below. This is useful for type 1, 1-operators, because the integrals themselves can be controlled for symbols in the self-adjoint subclass via their characterisation of Hörmander recalled in Theorem 2.4 below. In fact, in Section 5 this has led to estimates of such operators in spaces with  $0 < p \leq 1$ , which cannot be treated as duals of other spaces.

Notation is settled in Section 2 along with facts on operators of type 1, 1. Section 3 briefly recalls some facts on (1.5) from [Joh11]. Littlewood–Paley analysis of type 1, 1-operators is treated systematically in Section 4. Estimates in spaces over  $L_p$  are discussed in Section 5.

## 2. PRELIMINARIES ON TYPE 1, 1-OPERATORS

Notation and notions from distribution theory, such as the spaces  $C_0^\infty$ ,  $\mathcal{S}$ ,  $C^\infty$  of smooth functions and their duals  $\mathcal{D}'$ ,  $\mathcal{S}'$ ,  $\mathcal{E}'$  of distributions, and the Fourier transformation  $\mathcal{F}$ , will be as in Hörmander's book [Hör85], unless otherwise is mentioned. Eg  $\langle u, \varphi \rangle$  denotes the value of a distribution  $u$  on a test function  $\varphi$ . The space  $\mathcal{O}_M(\mathbb{R}^n)$  consists of the slowly increasing  $f \in C^\infty(\mathbb{R}^n)$ , ie the  $f$  that for each multiindex  $\alpha$  and some  $N > 0$  fulfils  $|D^\alpha f(x)| \leq c(1+|x|)^N$ .

As usual  $t_+ = \max(0, t)$  is the positive part and  $[t]$  denotes the greatest integer  $\leq t$ . In general,  $c$  will denote a real constant specific to the place of occurrence.

**2.1. The general definition of type 1,1-operators.** The reader may consult [Joh08b] for an overview of results on type 1,1-operators and a systematic treatment. The present paper is partly a continuation of [Joh04, Joh05, Joh08b], but it suffices to recall a few facts.

The operators are defined, as usual, on the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  by

$$a(x, D)u = \text{OP}(a)u(x) = (2\pi)^{-n} \int e^{ix \cdot \eta} a(x, \eta) \mathcal{F}u(\eta) d\eta, \quad u \in \mathcal{S}(\mathbb{R}^n). \quad (2.1)$$

Hereby the symbol  $a(x, \eta)$  is required to be in  $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ , of order  $d \in \mathbb{R}$  and type 1,1; ie for all multiindices  $\alpha, \beta \in \mathbb{N}_0^n$  it fulfils (1.1), or more precisely has finite seminorms

$$p_{\alpha, \beta}(a) := \sup_{x, \eta \in \mathbb{R}^n} (1 + |\eta|)^{-(d - |\alpha| + |\beta|)} |D_\eta^\alpha D_x^\beta a(x, \eta)| < \infty. \quad (2.2)$$

The Fréchet space of such symbols is denoted by  $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ , or just  $S_{1,1}^d$ .

For arbitrary  $u \in \mathcal{S}' \setminus \mathcal{S}$  it is quite delicate whether or not  $a(x, D)u$  is defined. To recall from [Joh08b] how type 1,1-operators can be defined in general, note that in terms of the partially Fourier transformed symbol

$$\hat{a}(\xi, \eta) = \mathcal{F}_{x \rightarrow \xi}(a(x, \eta)), \quad (2.3)$$

one can define a modified symbol  $\psi(2^{-m}D_x)a(x, \eta) = \mathcal{F}_{\xi \rightarrow x}^{-1}(\psi(2^{-m}\xi)\hat{a}(\xi, \eta))$ .

**Definition 2.1.** For a symbol  $a(x, \eta)$  in  $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$  and cut-off functions  $\psi \in C_0^\infty(\mathbb{R}^n)$  equal to 1 in a neighbourhood of the origin, let

$$a_\psi(x, D)u := \lim_{m \rightarrow \infty} \text{OP}(\psi(2^{-m}D_x)a(x, \eta)\psi(2^{-m}\eta))u. \quad (2.4)$$

If for each such  $\psi$  the limit  $a_\psi(x, D)u$  exists in  $\mathcal{D}'(\mathbb{R}^n)$  and moreover is independent of  $\psi$ , then  $u$  belongs to the domain  $D(a(x, D))$  by definition and

$$a(x, D)u = a_\psi(x, D)u. \quad (2.5)$$

Thus  $a(x, D)$  is a map  $\mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$  with dense domain.

Obviously the action on  $u$  is well defined for each  $m$  in (2.4) as the modified symbol is in  $S^{-\infty}$ . Since the removal of high frequencies in  $x$  and  $\eta$ , which is achieved from  $\psi(2^{-m}D_x)$  and  $\psi(2^{-m}\eta)$ , disappears for  $m \rightarrow \infty$ , this was called definition by vanishing frequency modulation in [Joh08b]; and accordingly  $\psi$  is said to be a modulation function.

While the calculus of type 1,1-operators is delicate in general, cf [Hör88, Hör89, Hör97], the following result is straightforward from the definition:

**Proposition 2.2.** *When  $a(x, \eta)$  is in  $S_{1,1}^{d_1}(\mathbb{R}^n \times \mathbb{R}^n)$  and  $b(\eta)$  belongs to  $S_{1,0}^{d_2}(\mathbb{R}^n \times \mathbb{R}^n)$ , then the symbol  $c(x, \eta) := a(x, \eta)b(\eta)$  is in  $S_{1,1}^{d_1+d_2}(\mathbb{R}^n \times \mathbb{R}^n)$  and*

$$c(x, D)u = a(x, D)b(D)u, \quad (2.6)$$

where  $D(c(x, D)) = D(a(x, D)b(D))$ ; that is, the two sides are simultaneously defined.

*Proof.* That  $c(x, \eta)$  is in  $S_{1,1}^{d_1+d_2}$  can be verified in the usual way from symbolic estimates. For an arbitrary modulation function  $\psi$  it is obvious from (2.1) that for every  $u \in \mathcal{S}$ ,

$$\text{OP}(\psi(2^{-m}D_x)a(x, \eta)\psi(2^{-m}\eta))b(D)u = \text{OP}(\psi(2^{-m}D_x)a(x, \eta)\psi(2^{-m}\eta)b(\eta))u. \quad (2.7)$$

This extends to all  $u \in \mathcal{S}'$  since the symbols are in  $S^{-\infty}$  or  $S_{1,0}^{d_2}$ . Moreover, for  $m \rightarrow \infty$  the limit exists on both or none of the two sides for each  $u \in \mathcal{S}'$ , so in the notation of (2.4),

$$a_\psi(x, D)(b(D)u) = c_\psi(x, D)u. \quad (2.8)$$

Now  $u \in D(c(x, D))$  if and only if the right-hand side is independent of  $\psi$ , ie if the left-hand side is so, which is equivalent to  $b(D)u \in D(a(x, D))$ , ie to  $u \in D(a(x, D)b(D))$ .  $\square$

**Example 2.3.** A standard example of a symbol of type 1, 1 results by taking an auxiliary function  $A \in C_0^\infty(\mathbb{R}^n)$ , say with  $\text{supp} A \subset \{\eta \mid \frac{3}{4} \leq |\eta| \leq \frac{5}{4}\}$ , and  $\theta \in \mathbb{R}^n$  fixed:

$$a_\theta(x, \eta) = \sum_{j=0}^{\infty} 2^{jd} e^{-i2^j x \cdot \theta} A(2^{-j}\eta). \quad (2.9)$$

Clearly  $a_\theta \in S_{1,1}^d$  since the terms are disjointly supported.

Such symbols were used by Ching [Chi72] and Bourdaud [Bou88] for  $d = 0$ ,  $|\theta| = 1$  to show unboundedness on  $L_2$ . Refining this, Hörmander [Hör88] linked continuity from  $H^s$  with  $s > -r$  to the property that  $\theta$  is a zero of  $A$  of order  $r \in \mathbb{N}_0$ . Extension to  $d \in \mathbb{R}$  was given in [Joh08b].

Moreover, it was shown in [Joh08b, Lem. 3.2] that  $a_\theta(x, D)$  is unclosable in  $\mathcal{S}'$  when  $A$  is taken to have support in a small neighbourhood of  $\theta$ . Therefore Definition 2.1 cannot in general be replaced by a closure of the graph in  $\mathcal{S}' \times \mathcal{S}'$ .

As a general result, it was shown in [Joh08b, Sec. 4] that the subspace  $\mathcal{S}(\mathbb{R}^n) + \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n)$  always is contained in the domain of  $a(x, D)$  and that this is a map

$$a(x, D): \mathcal{S}(\mathbb{R}^n) + \mathcal{F}^{-1}\mathcal{E}'(\mathbb{R}^n) \rightarrow \mathcal{O}_M(\mathbb{R}^n). \quad (2.10)$$

In fact, if  $u = v + v'$  is an arbitrary splitting of  $u$  with  $v \in \mathcal{S}$  and  $v' \in \mathcal{F}^{-1}\mathcal{E}'$ , it was shown that

$$a(x, D)u = a(x, D)v + \text{OP}(a(1 \otimes \chi))v', \quad (2.11)$$

whereby  $a(1 \otimes \chi)(x, \eta) = a(x, \eta)\chi(\eta)$  and  $\chi \in C_0^\infty(\mathbb{R}^n)$  is chosen so that  $\chi = 1$  holds in a neighbourhood of  $\text{supp } \mathcal{F}v'$ , but otherwise arbitrarily. Here  $a(x, \eta)\chi(\eta)$  is in  $S^{-\infty} = \bigcap S_{1,1}^d$ .

In fact,  $\mathcal{O}_M(\mathbb{R}^n)$  is invariant under  $a(x, D)$ , and  $a(x, D): C^\infty \cap \mathcal{S}' \rightarrow C^\infty$ ; cf [Joh10, Thm. 2.7].

**2.2. Conditions along the twisted diagonal.** As the first explicit condition on the symbol of a type 1, 1-operator, Hörmander [Hör88] proved that (1.4) holds for the norms of  $H^s$  with arbitrary  $s \in \mathbb{R}$ ,  $u \in \mathcal{S}$ , whenever  $a \in S_{1,1}^d$  fulfils the *twisted diagonal condition*: for some  $B \geq 1$

$$\hat{a}(\xi, \eta) = 0 \quad \text{where} \quad B(1 + |\xi + \eta|) < |\eta|. \quad (2.12)$$

This means that the partially Fourier transformed symbol  $\hat{a}(\xi, \eta)$  vanishes in a conical neighbourhood of a non-compact part of the twisted diagonal

$$\mathcal{T} = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi + \eta = 0\}. \quad (2.13)$$

Localisations to conical neighbourhoods (of non-compact parts) of  $\mathcal{T}$  was also introduced by Hörmander in [Hör88, Hör89, Hör97], by passing to  $a_{\chi,\varepsilon}(x, \eta)$  defined by

$$\hat{a}_{\chi,\varepsilon}(\xi, \eta) = \hat{a}(\xi, \eta)\chi(\xi + \eta, \varepsilon\eta), \quad (2.14)$$

whereby  $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  is chosen so that  $\chi(t\xi, t\eta) = \chi(\xi, \eta)$  for  $t \geq 1$ ,  $|\eta| \geq 2$  and

$$\text{supp } \chi \subset \{(\xi, \eta) \mid 1 \leq |\eta|, |\xi| \leq |\eta|\} \quad (2.15)$$

$$\chi = 1 \quad \text{in} \quad \{(\xi, \eta) \mid 2 \leq |\eta|, 2|\xi| \leq |\eta|\}. \quad (2.16)$$

Using this, Hörmander analysed a milder condition than the strict vanishing in (2.12), namely that for some  $\sigma \in \mathbb{R}$ , it holds for all multiindices  $\alpha$  and  $0 < \varepsilon < 1$  that

$$N_{\chi,\varepsilon,\alpha}(a) := \sup_{R>0, x \in \mathbb{R}^n} R^{-d} \left( \int_{R \leq |\eta| \leq 2R} |R^{|\alpha|} D_\eta^\alpha a_{\chi,\varepsilon}(x, \eta)|^2 \frac{d\eta}{R^n} \right)^{1/2} \leq c_{\alpha,\sigma} \varepsilon^{\sigma+n/2-|\alpha|}. \quad (2.17)$$

This asymptotics for  $\varepsilon \rightarrow 0$  always holds for  $\sigma = 0$ , as was proved in [Hör97, Lem. 9.3.2].

For  $\sigma > 0$  the faster convergence to 0 in (2.17) was proved in [Hör89] to imply that  $a(x, D)$  is bounded on  $u \in \mathcal{S}(\mathbb{R}^n)$ ,

$$\|a(x, D)u\|_{H^s} \leq c_s \|u\|_{H^{s+d}} \quad \text{for } s > -\sigma. \quad (2.18)$$

The reader could consult [Hör97, Thm. 9.3.5] for this (and [Hör97, Thm. 9.3.7] for four pages of proof of necessity of  $s \geq -\sup \sigma$ , with supremum over all  $\sigma$  for which (2.17) holds).

If  $\hat{a}$  is so small along  $\mathcal{T}$  that (2.17) holds for all  $\sigma \in \mathbb{R}$ , consequently there is boundedness  $H^{s+d} \rightarrow H^s$  for all  $s \in \mathbb{R}$ . Eg this validity of (2.17) for all  $\sigma$  is implied by (2.12), for since

$$\text{supp } \hat{a}_{\chi,\varepsilon} \subset \{(\xi, \eta) \mid 1 + |\xi + \eta| \leq 2\varepsilon|\eta|\}, \quad (2.19)$$

it is clear that (2.12) gives  $a_{\chi,\varepsilon} \equiv 0$  whenever  $0 < 2\varepsilon < 1/B$ .

More generally (2.17) enters a characterisation of the  $a \in S_{1,1}^d$  for which the adjoint symbol

$$a^*(x, \eta) = e^{iD_x \cdot D_\eta} \bar{a}(x, \eta) \quad (2.20)$$

is again in  $S_{1,1}^d$ ; cf the below condition (i). Since adjoining is an involution, such symbols constitute the class

$$\tilde{S}_{1,1}^d := S_{1,1}^d \cap (S_{1,1}^d)^*. \quad (2.21)$$

**Theorem 2.4.** *For a symbol  $a(x, \eta)$  in  $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$  the following properties are equivalent:*

- (i)  $a(x, \eta)$  belongs to  $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ .
- (ii) For arbitrary  $N > 0$  and  $\alpha, \beta$  there is a constant  $C_{\alpha,\beta,N}$  such that

$$|D_\eta^\alpha D_x^\beta a_{\chi,\varepsilon}(x, \eta)| \leq C_{\alpha,\beta,N} \varepsilon^N (1 + |\eta|)^{d-|\alpha|+|\beta|} \quad \text{for } 0 < \varepsilon < 1. \quad (2.22)$$

- (iii) The seminorm  $N_{\chi,\varepsilon,\alpha}(a)$  fulfils (2.17) for all  $\sigma \in \mathbb{R}$ .

In the affirmative case  $a \in \tilde{S}_{1,1}^d$ , and there is an estimate

$$|D_\eta^\alpha D_x^\beta a^*(x, \eta)| \leq (C_{\alpha,\beta}(a) + C'_{\alpha,\beta,N})(1 + |\eta|)^{d-|\alpha|+|\beta|} \quad (2.23)$$

for a certain continuous seminorm  $C_{\alpha,\beta}$  on  $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$  and a finite sum  $C'_{\alpha,\beta,N}$  of constants fulfilling the inequalities in (ii).

It should be observed that  $a(x, \eta)$  fulfils (i) if and only if  $a^*(x, \eta)$  does so (neither (ii) nor (iii) make this obvious). But (ii) immediately gives the inclusion  $\tilde{S}_{1,1}^d \subset \tilde{S}_{1,1}^{d'}$  for  $d' > d$ . Condition (iii) is close in spirit to the Mihlin–Hörmander multiplier theorem and is useful for the estimates to follow in Section 5.

The theorem was undoubtedly known to Hörmander, who stated the equivalence of (i) and (ii) explicitly in [Hör88, Thm. 4.2] and [Hör97, Thm. 9.4.2], in the latter with brief remarks on (iii).

As a corollary to the proof of Theorem 2.4, for which the reader also may consult [Joh10], the vanishing frequency modulation gave the following main result in [Joh10, Thm. 4.6]:

**Theorem 2.5.** *If  $a(x, \eta)$  is in the class  $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ , characterised in Theorem 2.4, then*

$$a(x, D): \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n) \quad (2.24)$$

*is everywhere defined and continuous, and it equals the adjoint of  $\text{OP}(e^{iD_x \cdot D_\eta} \bar{a}(x, \eta))$ .*

### 3. POINTWISE ESTIMATES

A main technique in this paper will be to estimate  $|a(x, D)u(x)|$  at an arbitrary point of  $\mathbb{R}^n$ . The recent results on this by the author [Joh11] are recalled here for convenience of the reader.

**3.1. The factorisation inequality.** When  $\text{supp} \hat{u}$  is compact in  $\mathbb{R}^n$ , the action on  $u$  by  $a(x, D)$  can be separated from  $u$  at the cost of an estimate, which is the *factorisation inequality*

$$|a(x, D)u(x)| \leq F_a(N, R; x) u^*(N, R; x). \quad (3.1)$$

Here  $u^*$  denotes the maximal function of Peetre–Fefferman–Stein type, defined as

$$u^*(N, R; x) = \sup_{y \in \mathbb{R}^n} \frac{|u(x-y)|}{(1+R|y|)^N} = \sup_{y \in \mathbb{R}^n} \frac{|u(y)|}{(1+R|x-y|)^N} \quad (3.2)$$

when  $\text{supp} \hat{u} \subset \bar{B}(0, R)$ ; cf. (2.11). The parameter  $N$  may eg be chosen so that  $N \geq \text{order}(\hat{u})$ .

The symbol factor  $F_a(x)$  only depends on  $u$  in a vague way, viz. through  $N$  and  $R$ :

$$F_a(N, R; x) = \int_{\mathbb{R}^n} (1+R|y|)^N |\mathcal{F}_{\eta \rightarrow y}^{-1}(a(x, \eta) \chi(\eta))| dy, \quad (3.3)$$

where the auxiliary function  $\chi \in C_0^\infty(\mathbb{R}^n)$  should equal 1 on a neighbourhood of  $\text{supp} \hat{u}$ . However,  $\chi$  is left out from the notation in  $F_a(x)$ , as this would be redundant by the results below in Theorem 3.1.

The estimate (3.1) is useful as both factors are easily controlled. Eg  $u^*(x)$  is polynomially bounded, for  $|u(y)| \leq c(1+|y|)^N \leq c(1+R|y-x|)^N(1+|x|)^N$  holds according to the Paley–Wiener–Schwartz Theorem if  $N \geq \text{order}(\hat{u})$ ,  $R \geq 1$ , and by (3.2) this implies

$$u^*(N, R; x) \leq c(1+|x|)^N, \quad x \in \mathbb{R}^n. \quad (3.4)$$

The non-linear map  $u \mapsto u^*$  is also bounded with respect to the  $L_p$ -norm on the subspace  $L_p \cap \mathcal{F}^{-1} \mathcal{E}'$ . This can be shown in an elementary way; cf [Joh11, Thm. 2.6].

Secondly, for the symbol factor one has  $F_a \in C(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$  with estimates highly reminiscent of the Mihlin–Hörmander conditions for Fourier multipliers:

**Theorem 3.1.** *Assume the symbol  $a(x, \eta)$  is in  $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$  and let  $F_a(N, R; x)$  be given by (3.3) for parameters  $R, N > 0$ , with the auxiliary function taken as  $\chi = \psi(R^{-1}\cdot)$  for  $\psi \in C_0^\infty(\mathbb{R}^n)$  equalling 1 in a set with non-empty interior. Then it holds for all  $x \in \mathbb{R}^n$  that*

$$0 \leq F_a(x) \leq c_{n,N} \sum_{|\alpha| \leq [N + \frac{n}{2}] + 1} \left( \int_{R \text{supp } \psi} |R^{|\alpha|} D_\eta^\alpha a(x, \eta)|^2 \frac{d\eta}{R^n} \right)^{1/2}. \quad (3.5)$$

For the elementary proof the reader is referred to Theorem 4.1 and Section 6 in [Joh11].

*Remark 3.2.* A further analysis of  $F_a$ 's dependence on  $R$  was given in [Joh11, Sect. 4]. Eg when the cut-off function  $\psi$  in Theorem 3.1 vanishes around the origin, then  $F_a(x) = \mathcal{O}(R^d)$  for  $a \in S_{1,1}^d$ . Moreover, when this is applied to symbols of the form  $a_Q(x, \eta) = \varphi(Q^{-1}D_x)a(x, \eta)$ ,  $Q > 0$ , with  $\varphi = 0$  around the origin, there is the sharpening  $F_{a_Q}(x) = \mathcal{O}(Q^{-M}R^{d+M})$ .

#### 4. LITTLEWOOD–PALEY ANALYSIS

For type 1,1-operators, Littlewood–Paley analysis will most conveniently depart from the limit in (2.4) with an arbitrary modulation function  $\psi$ . As  $\psi$  is a test function, this gives in the usual way a Littlewood–Paley decomposition  $1 = \psi(\eta) + \sum_{j=1}^\infty \varphi(2^{-j}\eta)$  by setting  $\varphi = \psi - \psi(2\cdot)$ . Note here that if  $\psi \equiv 1$  for  $|\eta| \leq r$  while  $\psi \equiv 0$  for  $|\eta| \geq R$ , one can fix an integer  $h \geq 2$  so that  $2R < r2^h$ . Then

$$\varphi(2^{-j}\eta) \neq 0 \implies r2^{j-1} \leq |\eta| \leq R2^j. \quad (4.1)$$

Inserting twice into (2.4) that  $\psi(2^{-m}\eta) = \psi(\eta) + \varphi(2^{-1}\eta) + \dots + \varphi(2^{-m}\eta)$ , the paradifferential splitting from the 1980's is recovered: if  $a(x, \eta)$  is in  $S_{1,1}^d$ , and  $u \in \mathcal{S}'(\mathbb{R}^n)$ , then

$$a_\psi(x, D)u = a_\psi^{(1)}(x, D)u + a_\psi^{(2)}(x, D)u + a_\psi^{(3)}(x, D)u, \quad (4.2)$$

whenever the three series below all converge in  $\mathcal{D}'$  (cf. Remark 4.2),

$$a_\psi^{(1)}(x, D)u = \sum_{k=h}^\infty \sum_{j \leq k-h} a_j(x, D)u_k = \sum_{k=h}^\infty a^{k-h}(x, D)u_k \quad (4.3)$$

$$a_\psi^{(2)}(x, D)u = \sum_{k=0}^\infty (a_{k-h+1}(x, D)u_k + \dots + a_{k-1}(x, D)u_k + a_k(x, D)u_k + a_k(x, D)u_{k-1} + \dots + a_k(x, D)u_{k-h+1}) \quad (4.4)$$

$$a_\psi^{(3)}(x, D)u = \sum_{j=h}^\infty \sum_{k \leq j-h} a_j(x, D)u_k = \sum_{j=h}^\infty a_j(x, D)u^{j-h}. \quad (4.5)$$

Here  $u_k = \varphi(2^{-k}D)u$  while  $a_k(x, \eta) = \varphi(2^{-k}D_x)a(x, \eta)$ ; by convention  $\varphi$  is replaced by  $\psi$  for  $k = 0$  and  $u_k \equiv 0 \equiv a_k$  for  $k < 0$ . In addition superscripts are used for the convenient short-hands  $u^{k-h} = \psi(2^{h-k}D)u$  and  $a^{k-h}(x, D) = \sum_{j \leq k-h} a_j(x, D) = \text{OP}(\psi(2^{h-k}D_x)a(x, \eta))$ . Using

this, there is a brief version of (4.4),

$$a_{\psi}^{(2)}(x, D)u = \sum_{k=0}^{\infty} ((a^k - a^{k-h})(x, D)u_k + a_k(x, D)(u^{k-1} - u^{k-h})). \quad (4.6)$$

Occasionally the subscripts  $\psi$  are omitted, as done already in the summands in (4.3)–(4.5).

The main point here is that the series have the following inclusions for the spectra of the summands in (4.3), (4.5) and (4.6), with  $R_h = \frac{r}{2} - R2^{-h} > 0$ :

$$\text{supp } \mathcal{F}(a^{k-h}(x, D)u_k) \subset \{ \xi \mid R_h 2^k \leq |\xi| \leq \frac{5R}{4} 2^k \}, \quad (4.7)$$

$$\text{supp } \mathcal{F}(a_k(x, D)u^{k-h}) \subset \{ \xi \mid R_h 2^k \leq |\xi| \leq \frac{5R}{4} 2^k \}, \quad (4.8)$$

$$\text{supp } \mathcal{F}(a_k(x, D)(u^{k-1} - u^{k-h})) \cup \text{supp } \mathcal{F}((a^k - a^{k-h})(x, D)u_k) \subset \bar{B}(0, 2R2^k) \quad (4.9)$$

Such spectral corona and ball properties have been known since the 1980's (e.g. [Yam86, (5.3)]) although they were verified then only for elementary symbols  $a(x, \eta)$ , in the sense of Coifman and Meyer [CM78]. However, this restriction is redundant because of the *Spectral Support Rule*, which for  $u \in \mathcal{F}^{-1} \mathcal{E}'(\mathbb{R}^n)$  states that

$$\text{supp } \mathcal{F}(a(x, D)u) \subset \{ \xi + \eta \mid (\xi, \eta) \in \text{supp } \mathcal{F}_{x \rightarrow \xi} a, \eta \in \text{supp } \mathcal{F} u \}, \quad (4.10)$$

A short proof of this can be found in [Joh10, App. B] (cf. also [Joh05, Joh08b] for the full version). Since (4.7)–(4.9) follow easily from (4.10), cf. [Joh05, Joh10], details are omitted.

Recently the pointwise estimates in Remark 3.2 were utilised for the following result. It was deduced in [Joh11, Thm. 5.1], with extension to type 1, 1 in Section 6 there.

**Theorem 4.1.** *For each  $a(x, \eta)$  in  $S_{1,1}^d$  the decomposition (4.2) is valid with the terms in (4.3)–(4.5) having spectral relations (4.7), (4.8), (4.9) and pointwise estimates, cf Section 3,*

$$|a^{k-h}(x, D)u_k(x)| \leq p(a)(R2^k)^d u_k^*(N, R2^k; x), \quad (4.11)$$

$$|(a^k - a^{k-h})(x, D)u_k(x)| \leq p(a)(R2^k)^d u_k^*(N, R2^k; x), \quad (4.12)$$

$$|a^k(x, D)(u^{k-1}(x) - u^{k-h}(x))| \leq p(a)(R2^k)^d \sum_{l=1}^{h-1} 2^{-ld} u_{k-l}^*(N, R2^{k-l}; x), \quad (4.13)$$

$$|a_j(x, D)u^{j-h}(x)| \leq c_M 2^{-jM} p(a) \sum_{k=0}^j (R2^k)^{d+M} u_k^*(N, R2^k; x). \quad (4.14)$$

Hereby  $p(a)$  denotes a continuous seminorm on  $S_{1,1}^d$  and  $M \in \mathbb{N}$ .

It is well known that in (4.14) one may treat the sum over  $k$  by the elementary inequality

$$\sum_{j=0}^{\infty} 2^{sjq} \left( \sum_{k=0}^j |b_k| \right)^q \leq c \sum_{j=0}^{\infty} 2^{sjq} |b_j|^q, \quad (4.15)$$

valid for all  $b_j \in \mathbb{C}$  and  $0 < q \leq \infty$  provided  $s < 0$ ; cf [Yam86].

*Remark 4.2.* There is the addendum that the series (4.3), (4.5) always converge for  $u \in \mathcal{S}'$ ; so that  $u$  is in  $D(a(x, D))$  if and only if the  $a^{(2)}$ -series converges. Cf. [Joh10, Thm. 6.3].

*Remark 4.3.* If  $a(x, \eta)$  satisfies the twisted diagonal condition (2.12) for some  $B \geq 1$ , the supports in (4.9) are for large  $k$  both contained in the corona

$$\left\{ \xi \mid \frac{r}{2^{h+1}B} 2^k \leq |\xi| \leq 2R2^k \right\}. \quad (4.16)$$

Indeed, (4.10) yields that  $\text{supp } \mathcal{F}a_k(x, D)(u^{k-1} - u^{k-h})$  is contained in

$$\left\{ \xi + \eta \mid (\xi, \eta) \in \text{supp}(\varphi_k \otimes 1)\hat{a}, r2^{k-h} \leq |\eta| \leq R2^{k-1} \right\}. \quad (4.17)$$

Therefore any  $\zeta = \xi + \eta$  in the support fulfils  $|\zeta| \leq R2^k + R2^{k-1} = (3R/2)2^k$ . But (2.12) implies that  $B(1 + |\xi + \eta|) \geq |\eta|$  on  $\text{supp } \mathcal{F}_{x \rightarrow \xi} a$  so that, for all  $k \geq h + 1 + \log_2(B/r)$ ,

$$|\zeta| \geq \frac{1}{B}|\eta| - 1 \geq \frac{1}{B}r2^{k-h} - 1 \geq \left(\frac{r}{2^h B} - 2^{-k}\right)2^k \geq \frac{r}{2^{h+1}B}2^k. \quad (4.18)$$

The term  $(a^k - a^{k-h})(x, D)u_k$  is analogous, but causes  $3R/2$  to be replaced by  $2R$ .

## 5. $L_p$ -ESTIMATES

**5.1. Function spaces.** To proceed from  $H^s$ -results, it would of course be natural to consider Sobolev spaces  $H_p^s$  and Hölder–Zygmund spaces  $C_*^s$  (cf. [Hör97, Def. 8.6.4]), but these are special cases of the Besov spaces  $B_{p,q}^s$  and Lizorkin–Triebel spaces  $F_{p,q}^s$ . In fact,

$$H_p^s = F_{p,2}^s \quad \text{for } 1 < p < \infty, \quad (5.1)$$

$$C_*^s = B_{\infty,\infty}^s \quad \text{for } s \in \mathbb{R}. \quad (5.2)$$

Because of the Littlewood–Paley analysis that will follow, it requires almost no extra effort in the estimates to cover the full  $B_{p,q}^s$  and  $F_{p,q}^s$  scales.

To invoke the  $B_{p,q}^s$  and  $F_{p,q}^s$  scales is natural in the context, for it was shown in [Joh04, Joh05] that every type 1, 1-operator  $a(x, D)$  of order  $d \in \mathbb{R}$  is a bounded map

$$a(x, D): F_{p,1}^d(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n) \quad \text{for } 1 \leq p < \infty. \quad (5.3)$$

Because  $B_{p,1}^d \subset F_{p,1}^d$  is a strict inclusion for  $p > 1$ , this sharpened the borderline analysis of Bourdaud [Bou88]; (5.3) was moreover proved to be optimal within the  $B_{p,q}^s$ - and  $F_{p,q}^s$ -scales.

To recall the definition of  $B_{p,q}^s$  and  $F_{p,q}^s$ , let a Littlewood–Paley partition of unity  $1 = \sum_{j=0}^{\infty} \Phi_j$  be chosen as in Section 4 with  $\Phi_j = \Phi(2^{-j}\cdot)$  for  $\Phi = \Psi - \Psi(2\cdot)$ , though  $\Phi_0 = \Psi$ , whereby  $\Psi \in C_0^\infty(\mathbb{R}^n)$  equal to 1 around the origin is fixed. Usually it has been required that  $\text{supp } \Phi$  should be contained in the corona with  $\frac{1}{2} \leq |\xi| \leq 2$ ; but this restriction is avoided here in order that  $\Psi$  can be taken equal to an arbitrary modulation function entering  $a(x, D)$ . That this is possible can be seen by adopting the approach in eg [Yam86, JS08]:

When  $\Psi$  is fixed as above, then the spaces are defined for  $s \in \mathbb{R}$  and  $p, q \in ]0, \infty]$  as follows, when  $\|\cdot\|_p$  denotes the (quasi-)norm of the Lebesgue space  $L_p(\mathbb{R}^n)$  for  $0 < p \leq \infty$  and  $\|\cdot\|_{\ell_q}$  stands for that of the sequence space  $\ell_q(\mathbb{N}_0)$ ,

$$B_{p,q}^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \left\| \left\{ 2^{sj} \|\Phi_j(D)u(\cdot)\|_p \right\}_{j=0}^{\infty} \right\|_{\ell_q} < \infty \right\}, \quad (5.4)$$

$$F_{p,q}^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \left\| \left\{ 2^{sj} \Phi_j(D)u \right\}_{j=0}^{\infty} \right\|_{\ell_q(\cdot)} \| \cdot \|_p < \infty \right\}. \quad (5.5)$$

Throughout it will be understood that  $p < \infty$  when Lizorkin–Triebel spaces  $F_{p,q}^s$  are considered.

In the definition the finite expressions are norms for  $p, q \geq 1$  (quasi-norms if  $p < 1$  or  $q < 1$ ). In general  $u \mapsto \|u\|^\lambda$  is subadditive for  $\lambda \leq \min(1, p, q)$ , so  $\|f - g\|^\lambda$  is a metric.

This implies continuous embeddings  $\mathcal{S} \hookrightarrow B_{p,q}^s \hookrightarrow \mathcal{S}'$  and  $\mathcal{S} \hookrightarrow F_{p,q}^s \hookrightarrow \mathcal{S}'$  in the usual way, thence completeness (cf [JS07, Tri83]). There are simple embeddings  $F_{p,q}^s \hookrightarrow F_{p,r}^{s'}$  for  $s' < s$  and arbitrary  $q, r$ , or for  $s' = s$  when  $r \geq q$ . Similarly for  $B_{p,q}^s$ .

**Example 5.1.** In the  $F_{p,q}^s$ -scale,  $f(t) = \sum_{j=0}^{\infty} 2^{-jd} e^{i2^j t}$  belongs locally to  $F_{p,\infty}^d(\mathbb{R})$ ; cf [Joh08b, Rem. 3.7]. This is for  $0 < d \leq 1$  a variant of Weierstrass' nowhere differentiable function.

Homogeneous distributions were characterised in the  $B_{p,q}^s$ -scale in Prop. 2.8 of [Joh08a]: when  $u \in \mathcal{D}'(\mathbb{R}^n)$  is  $C^\infty$  on  $\mathbb{R}^n \setminus \{0\}$  and homogeneous of degree  $a \in \mathbb{C}$  there (cf [Hör85, Def 3.2.2]), then (at  $x = 0$ )  $u$  is locally in  $B_{p,\infty}^{\frac{n}{p} + \operatorname{Re} a}(\mathbb{R}^n)$  for  $0 < p \leq \infty$ . If  $-n < \operatorname{Re} a < 0$  and  $p \in ] -\frac{n}{\operatorname{Re} a}, \infty]$  then  $u \in B_{p,\infty}^{\frac{n}{p} + \operatorname{Re} a}(\mathbb{R}^n)$ ; this holds also for  $p = \infty$  if  $\operatorname{Re} a = 0$ . These conclusions are optimal for  $s$  and  $q$ , unless  $u$  is a homogeneous polynomial (the only case in which  $u \in C^\infty(\mathbb{R}^n)$ ). Eg  $\delta_0 \in B_{p,\infty}^{\frac{n}{p}}$  while a quotient of two homogeneous polynomials of the same degree, say  $P(x)/Q(x)$  is locally in  $B_{p,\infty}^{\frac{n}{p}}$  for  $0 < p \leq \infty$ .

Invoking a multiplier result, one finds a dyadic ball and corona criterion:

**Lemma 5.2.** *Let  $s > \max(0, \frac{n}{p} - n)$  for  $0 < p < \infty$  and  $0 < q \leq \infty$  and suppose  $u_j \in \mathcal{S}'(\mathbb{R}^n)$  fulfil that, for some  $A > 0$ ,*

$$\operatorname{supp} \mathcal{F} u_j \subset B(0, A2^j), \quad F(q) := \left\| \left( \sum_{j=0}^{\infty} 2^{sjq} |u_j(\cdot)|^q \right)^{\frac{1}{q}} \right\|_p < \infty. \quad (5.6)$$

*Then  $\sum_{j=0}^{\infty} u_j$  converges in  $\mathcal{S}'(\mathbb{R}^n)$  to some  $u \in F_{p,r}^s(\mathbb{R}^n)$  for  $r \geq q$ ,  $r > \frac{n}{n+s}$ , and  $\|u\|_{F_{p,r}^s} \leq cF(r)$  for some  $c > 0$  depending on  $n, s, p$  and  $r$ .*

*When moreover  $\operatorname{supp} \mathcal{F} u_j \subset \{ \xi \mid A^{-1}2^j \leq |\xi| \leq A2^j \}$  for all  $j \geq J$ , for some  $J \geq 1$ , then the conclusions are valid for all  $s \in \mathbb{R}$  and  $r = q$ .*

This is an isotropic version of [JS08, Lem. 3.19-20], where the proof is applicable for arbitrary Littlewood–Paley partitions, though with other constants if  $\Psi$  is such that  $R > 2$ . Alternatively the reader may refer to the below Proposition 5.7, where the proof also covers the sufficiency of (5.6) and as a special case gives the last part of Lemma 5.2 as well.

From Lemma 5.2 it follows that  $F_{p,q}^s$  is independent of the particular Littlewood–Paley decomposition, and that different choices lead to equivalent quasi-norms.

The functions  $u_k = \Phi(2^{-k}D)u$  will play a central role below because their maximal functions  $u_k^*$ , cf Section 3, are controlled in terms of the Lizorkin–Triebel norm  $\|u\|_{F_{p,q}^s}$  as follows: for  $0 < t < \infty$  there is an estimate, cf [Yam86, Thm. 2.10], in terms of the modified Hardy–Littlewood maximal function given by  $M_t u_k(x) = \sup_{r>0} (r^{-n} \int_{|x-y| \leq r} |u(y)|^t dy)^{1/t}$ ,

$$u_k^*(N, R2^k; x) \leq u_k^*\left(\frac{n}{t}, R2^k; x\right) \leq cM_t u_k(x), \quad N \geq n/t. \quad (5.7)$$

So for  $t < \min(p, q)$  the Fefferman-Stein inequality (cf [Yam86, Thm. 2.2]) yields a basic inequality valid for the  $u_k^* = u_k^*(N, R2^k, \cdot)$  and any  $s \in \mathbb{R}$ ,

$$\int_{\mathbb{R}^n} \|2^{sk} u_k^*(\cdot)\|_{\ell_q}^p dx \leq c \int_{\mathbb{R}^n} \|2^{sk} M_t u_k(\cdot)\|_{\ell_q}^p dx \leq c' \int_{\mathbb{R}^n} \|2^{sk} u_k(\cdot)\|_{\ell_q}^p dx = c' \|u\|_{F_{p,q}^s}^p. \quad (5.8)$$

As general references to the theory of these function spaces, the reader is referred to the books [RS96, Tri83, Tri92]; the paper [Yam86] gives a concise (anisotropic) presentation.

*Remark 5.3.* As an alternative to the techniques in Section 3, there is an estimate for symbols  $b(x, \eta)$  in  $L_{1,\text{loc}}(\mathbb{R}^{2n}) \cap \mathcal{S}'(\mathbb{R}^{2n})$  with support in  $\mathbb{R}^n \times \overline{B}(0, 2^k)$  and  $\text{supp } \mathcal{F}u \subset \overline{B}(0, 2^k)$ ,  $k \in \mathbb{N}$ :

$$|b(x, D)v(x)| \leq c \|b(x, 2^k \cdot)\|_{\dot{B}_{1,t}^{n/t}} M_t u(x), \quad 0 < t \leq 1. \quad (5.9)$$

This is Marschall's inequality, it goes back to [Mar85, p.37] and was exploited in eg [Mar91]; in the above form it was proved in [Joh05] under the condition that the right-hand side is in  $L_{1,\text{loc}}(\mathbb{R}^n)$  (cf also [JS08]). While  $M_t u$  is as in (5.7), the norm  $\|b(x, 2^k \cdot)\|_{\dot{B}_{1,t}^{n/t}}$  of the symbol in the homogenous Besov space is of special interest here. It is defined in terms of a partition of unity  $1 = \sum_{j=-\infty}^{\infty} \Phi(2^{-j}\eta)$ , with  $\Phi$  as in (5.4), and (5.4) read with  $\ell_q$  over  $\mathbb{Z}$  gives the norm. This yields the well-known dyadic scaling property that

$$\|b(x, 2^k \cdot)\|_{\dot{B}_{1,t}^{n/t}} = 2^{k(\frac{n}{t}-n)} \|b(x, \cdot)\|_{\dot{B}_{1,t}^{n/t}}. \quad (5.10)$$

**5.2. Basic estimates in  $L_p$ .** For general type 1, 1-operators  $a(x, D)$  one has the next result. This appeared in [Joh05, Cor. 6.2], albeit with a rather sketchy explanation. Therefore a full proof is given here, now explicitly based on Definition 2.1 and the pointwise techniques in Section 3:

**Theorem 5.4.** *Every  $a(x, D)$  in  $\text{OP}(S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n))$  is continuous, for all  $s > \max(0, \frac{n}{p} - n)$ ,  $0 < p, q \leq \infty$ ,*

$$a(x, D): F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,r}^s(\mathbb{R}^n), \quad p < \infty, r \geq q, r > n/(n+s), \quad (5.11)$$

$$a(x, D): B_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n). \quad (5.12)$$

Here the twisted diagonal condition (2.12) implies (5.11) and (5.12) for all  $s \in \mathbb{R}$  and  $r = q$ .

*Proof.* Let  $\psi$  denote an arbitrary modulation function, and recall the notation from Section 4, in particular (4.2) and  $R, r$  and  $h$ . It is exploited below that  $\|u\|_{F_{p,q}^s}$  can be calculated in terms of the Littlewood–Paley partition associated with  $\psi$ .

For  $a^{(1)}(x, D)u = \sum_{k=h}^{\infty} a^{k-h}(x, D)u_k$  and  $u \in F_{p,q}^s$ , application of the norms of  $\ell_q$  and  $L_p$  to the pointwise estimate in (4.11) gives (if  $q < \infty$  for simplicity's sake)

$$\int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} 2^{skq} |a^{k-h}(x, D)u_k(x)|^q \right)^{\frac{p}{q}} dx \leq c_2 p(a)^p \left\| \left( \sum_{k=0}^{\infty} 2^{(s+d)kq} u_k^*(x)^q \right)^{\frac{1}{q}} \right\|_p^p. \quad (5.13)$$

Taking  $N > n/\min(p, q)$  in  $u_k^*$ , it is seen from (5.8) that one has the bound in Lemma 5.2 for all  $s \in \mathbb{R}$ , whilst the corona condition there holds by Theorem 4.1, so the lemma gives

$$\|a^{(1)}(x, D)u\|_{F_{p,q}^s} \leq c \left( \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} 2^{skq} |a^{k-h}(x, D)u_k(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \leq c' \|u\|_{F_{p,q}^{s+d}}. \quad (5.14)$$

In the contribution  $a^{(3)}(x, D)u = \sum_{j=h}^{\infty} a_j(x, D)u^{j-h}$  one may apply (4.15). For  $M > s$  this gives

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{sjq} |a_j(x, D)u^{j-h}(x)|^q &\leq \sum_{j=0}^{\infty} 2^{(s-M)jq} \left( \sum_{k=0}^j c_{MP}(a) (R2^k)^{d+M} u_k^*(N, R2^k; x) \right)^q \\ &\leq cp(a)^q \sum_{j=0}^{\infty} 2^{(s+d)jq} u_j^*(N, R2^j; x)^q. \end{aligned} \quad (5.15)$$

Proceeding by integration one arrives at

$$\left( \int_{\mathbb{R}^n} \left( \sum_{j=0}^{\infty} 2^{sjq} |a_j(x, D)u^{j-h}(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \leq c_3 p(a) \left\| \left( \sum_{j=0}^{\infty} 2^{(s+d)jq} u_j^*(x)^q \right)^{\frac{1}{q}} \right\|_p. \quad (5.16)$$

Hence the same application of Lemma 5.2 as for (5.14) now gives  $\|a^{(3)}(x, D)u\|_{F_{p,q}^s} \leq c \|u\|_{F_{p,q}^{s+d}}$ .

In estimates of  $a^{(2)}(x, D)u$  the terms can be treated similarly, now departing from (4.12) and (4.13). Thus one finds

$$\begin{aligned} \left( \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} 2^{skq} |(a^k - a^{k-h})(x, D)u_k(x) + a_k(x, D)(u^{k-1} - u^{k-h})|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\ \leq c'_2 p(a) \left\| \left( \sum_{k=0}^{\infty} 2^{(s+d)kq} u_k^*(x)^q \right)^{\frac{1}{q}} \right\|_p. \end{aligned} \quad (5.17)$$

In case (2.12) holds, Remark 4.3 shows that Lemma 5.2 is applicable once more, so the argument for (5.14) gives  $\|a^{(2)}(x, D)u\|_{F_{p,q}^s} \leq c \|u\|_{F_{p,q}^{s+d}}$ . So for all  $s \in \mathbb{R}$ ,

$$\|a_{\psi}(x, D)u\|_{F_{p,q}^s} \leq \sum_{j=1,2,3} \|a^{(j)}(x, D)u\|_{F_{p,q}^s} \leq cp(a) \|u\|_{F_{p,q}^{s+d}}. \quad (5.18)$$

Without (2.12) the spectra are by (4.9) just contained in balls, but the condition  $s > \max(0, \frac{n}{p} - n)$  and those on  $r$  imply that  $\|a^{(2)}(x, D)u\|_{F_{p,r}^s} \leq c \|u\|_{F_{p,q}^{s+d}}$ ; cf Lemma 5.2. This gives (5.18) with  $q$  replaced by  $r$  on the left-hand side.

Thus  $a_{\psi}(x, D): F_{p,q}^{s+d} \rightarrow F_{p,r}^s$  is continuous and coincides with  $a(x, D)$  on  $\mathcal{S}$ . Since  $\mathcal{S}$  is dense in  $F_{p,q}^s$  for  $q < \infty$  (and  $F_{p,\infty}^s \hookrightarrow F_{p,1}^{s'}$  for  $s' < s$ ), there is no dependence on  $\psi$ . Hence  $u \in D(a(x, D))$  and (5.18) holds for  $a(x, D)u$ . This proves (5.11) in all cases.

The Besov case is analogous; one can interchange the order of  $L_p$  and  $\ell_q$  and refer to the maximal inequality for scalar functions: Lemma 5.2 carries over to  $B_{p,q}^s$  in a natural way for  $0 < p \leq \infty$  with  $r = q$  in all cases; this is well known, cf [Yam86, Joh05, JS08].  $\square$

One may also obtain (5.12) by real interpolation of (5.11), cf [Tri83, 2.4.2], when  $0 < p < \infty$ .

The borderline analysis in (5.3) is a little simpler than the above, as completeness of  $L_p$  may replace the use of Lemma 5.2. In fact, the proof in [Joh04, Joh05] applies to Definition 2.1 with the addendum that the right-hand side of (4.2) does not depend on  $\psi$  for  $u \in F_{p,1}^d$ , because  $\mathcal{S}$  is dense there.

By duality, Theorem 5.4 extends to operators that merely fulfil the twisted diagonal condition of arbitrary real order.

**Theorem 5.5.** *Let  $a(x, \eta)$  belong to the self-adjoint subclass  $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ , characterised in Theorem 2.4. Then  $a(x, D)$  is a bounded map for all  $s \in \mathbb{R}$ ,*

$$a(x, D): F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,q}^s(\mathbb{R}^n), \quad 1 < p < \infty, 1 < q \leq \infty, \quad (5.19)$$

$$a(x, D): B_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow B_{p,q}^s(\mathbb{R}^n), \quad 1 < p \leq \infty, 1 < q \leq \infty. \quad (5.20)$$

*Proof.* When  $p' + p = p'p$  and  $q' + q = q'q$ , then  $F_{p,q}^s$  is the dual of  $F_{p',q'}^{-s}$  since  $1 < p' < \infty$  and  $1 \leq q' < \infty$ ; cf [Tri83, 2.11], the case  $q' = 1$  is covered by eg [FJ90, Rem. 5.14]. The adjoint symbol  $a^*(x, \eta)$  is in  $S_{1,1}^d$  by assumption, and  $p' \geq 1$  and  $q' \geq 1$ , so Theorem 5.4 gives that

$$a^*(x, D): F_{p',q'}^{-s-d}(\mathbb{R}^n) \rightarrow F_{p',q'}^{-s-d}(\mathbb{R}^n) \quad (5.21)$$

is continuous whenever  $-s - d > \max(0, \frac{n}{p'} - n) = 0$ , ie for  $s < -d$ . The adjoint  $a^*(x, D)^*$  is therefore bounded  $F_{p,q}^{s+d} \rightarrow F_{p,q}^s$ , and it is a restriction of  $a(x, D)$  in view of Theorem 2.5. When  $s > 0$  then (5.19) also holds by Theorem 5.4.

If  $d \geq 0$  the gap with  $s \in [-d, 0]$  can be closed since  $a(x, D) = b(x, D)\Lambda^t$  by Proposition 2.2 holds with  $\Lambda^t = \text{OP}((1 + |\eta|^2)^{t/2})$ ,  $t \in \mathbb{R}$  and  $b(x, \eta) = a(x, \eta)(1 + |\eta|^2)^{-t/2}$ . The latter is of type 1, 1 and order  $-1$  for  $t = d + 1$ , which by the just shown gives (5.19) for all  $s$ .

For the  $B_{p,q}^s$  scale similar arguments apply, also for  $p = \infty$ . □

Obviously Theorem 5.5 gives a natural generalisation of Hörmander's boundedness result mentioned after (2.18) to the  $L_p$ -setting. Specialisation of Theorems 5.4–5.5 to Sobolev and Hölder–Zygmund spaces, cf (5.1)–(5.2), gives

**Corollary 5.6.** *Every  $a(x, D) \in \text{OP}(S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n))$  is bounded*

$$a(x, D): H_p^{s+d}(\mathbb{R}^n) \rightarrow H_p^s(\mathbb{R}^n), \quad s > 0, 1 < p < \infty, \quad (5.22)$$

$$a(x, D): C_*^{s+d}(\mathbb{R}^n) \rightarrow C_*^s(\mathbb{R}^n), \quad s > 0. \quad (5.23)$$

*This is valid for all real  $s$  whenever  $a(x, \eta)$  belongs to the self-adjoint subclass  $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ .*

Previously *extensions* with similar properties were obtained for  $H_p^s$  by Meyer [Mey81] and for  $C_*^s$  by Stein (published in [Ste93]). By contrast, the corollary is valid for the operators in Definition 2.1.

**5.3. Direct estimates for the self-adjoint subclass.** To complement Theorem 5.5 with similar results valid for  $p, q$  in  $]0, 1]$  one can exploit the paradifferential decomposition (4.2) and the pointwise estimates used above.

However, in the results below there will be an arbitrarily small loss of smoothness. The reason is that the estimates of  $a_\psi^{(2)}(x, D)$  are based on a corona condition which is *non-symmetric* in the sense that the outer radii grow faster than the inner ones. That is, the last part of Lemma 5.2 will

now be extended to series  $\sum u_j$  fulfilling the more general condition, where  $0 < \theta \leq 1$  and  $A > 1$ ,

$$\begin{aligned} \text{supp } \mathcal{F}u_j &\subset \{ \xi \mid |\xi| \leq A2^j \} \quad \text{for all } j \geq 0, \\ \text{supp } \mathcal{F}u_j &\subset \{ \xi \mid \frac{1}{A}2^{\theta j} \leq |\xi| \leq A2^j \} \quad \text{for } j \geq J \geq 1. \end{aligned} \quad (5.24)$$

This situation is probably known to experts in function spaces, but in lack of a reference it is analysed here. The techniques should be standard, so the explanations will be brief.

The main point of (5.24) is that  $\sum u_j$  still converges for  $s \leq 0$ , albeit with a loss of smoothness; cf the cases below with  $s' < s$ . Actually the loss is proportional to  $(1 - \theta)/\theta$ , hence tends to  $\infty$  for  $\theta \rightarrow 0$ , which reflects that convergence in some cases fails for  $\theta = 0$  (take  $\hat{u}_j = \frac{1}{j}\psi \in C_0^\infty$ ,  $s = 0$ ,  $1 < q \leq \infty$ ).

**Proposition 5.7.** *Let  $s \in \mathbb{R}$ ,  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $J \in \mathbb{N}$  and  $0 < \theta \leq 1$  be given; with  $q > n/(n+s)$  if  $s > 0$ . For each sequence  $(u_j)_{j \in \mathbb{N}_0}$  in  $\mathcal{S}'(\mathbb{R}^n)$  fulfilling the corona condition (5.24) together with the bound (usual modification for  $q = \infty$ )*

$$F := \left\| \left( \sum_{j=0}^{\infty} |2^{sj} u_j(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L_p} < \infty, \quad (5.25)$$

the series  $\sum_{j=0}^{\infty} u_j$  converges in  $\mathcal{S}'(\mathbb{R}^n)$  to some  $u \in F_{p,q}^{s'}(\mathbb{R}^n)$  with

$$\|u\|_{F_{p,q}^{s'}} \leq cF, \quad (5.26)$$

whereby the constant  $c$  also depends on  $s'$ , which one can take as  $s' = s$  for  $\theta = 1$ , or in case  $0 < \theta < 1$ , take to fulfil

$$s' = s \quad \text{for } s > \max(0, \frac{n}{p} - n), \quad (5.27)$$

$$s' < s/\theta \quad \text{for } s \leq 0, p \geq 1, q \geq 1, \quad (5.28)$$

or in general

$$s' < s - \frac{1-\theta}{\theta} (\max(0, \frac{n}{p} - n) - s)_+. \quad (5.29)$$

(Here  $s' = s$  is possible by (5.27) if the positive part  $(\dots)_+$  has strictly negative argument.)

The conclusions carry over to  $B_{p,q}^{s'}$  for any  $q \in ]0, \infty]$  when  $B := (\sum_{j=0}^{\infty} 2^{sjq} \|u_j\|_p^q)^{\frac{1}{q}} < \infty$ .

*Remark 5.8.* The above restriction  $q > n/(n+s)$  for  $s > 0$  is not severe, for if (5.25) holds for a sum-exponent in  $]0, n/(n+s)]$ , then the constant  $F$  is also finite for any  $q > n/(n+s)$ , which yields the convergence and an estimate in a slightly larger space; cf the  $r$  in Lemma 5.2

*Proof.* Increasing  $A \geq 1$ , as we may, gives a reduction to the case  $J = 1$ :  $u = \sum u_j$  has the contributions  $0 + \dots + 0 + u_J + u_{J+1} + \dots$  and  $(u_0 + \dots + u_{J-1}) + 0 + \dots$ , where the former fulfils the conditions for  $J = 1$ ; the latter trivially converges, it fulfils (5.24) for  $J = 1$  if  $A$  is replaced by  $A2^J$  and (5.25) as  $\|u_0 + \dots + u_{J-1}\|_p \leq c_p 2^{|s|J} F < \infty$  with  $c_p = J^{\max(1, 1/p)}$ . Hence  $\|u\|_{F_{p,q}^{s'}} \leq C(c + c_p 2^{|s|J})F$  if  $C$  is the constant from the quasi-triangle inequality.

It is first assumed that  $u = \sum u_k$  converges in  $\mathcal{S}'$ . Then each term  $\Phi_j(D) \sum u_k$  in the expression for  $\|u\|_{F_{p,q}^{s'}}$  is defined; cf (5.5). Writing now  $\Phi_j(\eta)$  as  $\Phi(2^{-j}\eta)$  for clarity, one has

$$\Phi(2^{-j}D) \sum_{k \geq 0} u_k = \sum_{j-h \leq k \leq [j/\theta]+h} \Phi(2^{-j}D) u_k. \quad (5.30)$$

In fact, (5.24) gives an  $h \in \mathbb{N}$  such that  $\Phi(2^{-j}D) \mathcal{F} u_k = 0$  for all  $k \notin [j-h, \frac{j}{\theta} + h]$ .

To proceed it is convenient to use Marschall's inequality; cf Remark 5.3. This gives

$$|\Phi(2^{-j}D) u_k(x)| \leq c \|\Phi(R2^{v-j}\cdot)\|_{\dot{B}_{1,t}^{\frac{n}{t}}} M_t u_k(x), \quad \text{for } 0 < t \leq 1, \quad (5.31)$$

whereby  $v$  should be taken so large that  $B(0, R2^v)$  contains the supports of  $\Phi(2^{-j}\cdot)$  and  $\hat{u}_k$ ; also  $R \geq A$  can be arranged. Note that by Remark 5.3,

$$\|\Phi(R2^{v-j}\cdot)\|_{\dot{B}_{1,t}^{\frac{n}{t}}} = 2^{(v-j)(\frac{n}{t}-n)} \|\Phi(R\cdot)\|_{\dot{B}_{1,t}^{\frac{n}{t}}}. \quad (5.32)$$

This is applied in the following for some  $t \in ]0, 1]$  that also fulfils  $t < \min(p, q)$ , and the main point is to show that, with  $s'$  as in the statement, it holds in all cases that

$$\left( \sum_{j=0}^{\infty} 2^{s'jq} |\Phi(2^{-j}D) \sum_{k \geq 0} u_k(x)|^q \right)^{1/q} \leq c \left( \sum_{k=0}^{\infty} 2^{skq} M_t u_k(x)^q \right)^{1/q}. \quad (5.33)$$

The easiest case is for  $0 < q \leq 1$ . As  $\ell_q \hookrightarrow \ell_1$  for such  $q$ , one has

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{s'jq} |\Phi(2^{-j}D) \sum_{k \geq 0} u_k(x)|^q &\leq \sum_{j=0}^{\infty} \sum_{j-h \leq k \leq j/\theta+h} 2^{s'jq} |\Phi(2^{-j}D) u_k(x)|^q \\ &\leq c \sum_{k=0}^{\infty} \sum_{\theta k-h \leq j \leq k+h} 2^{s'jq} \|\Phi(R2^{v-j}\cdot)\|_{\dot{B}_{1,t}^{\frac{n}{t}}}^q M_t u_k(x)^q. \end{aligned} \quad (5.34)$$

Here  $v = j$  gives a constant for  $j \geq k$ , so the above is both for  $s' \geq 0$  estimated by

$$c \sum_{k=0}^{\infty} (h2^{s'kq} + \sum_{\theta k-h \leq j \leq k} 2^{s'jq + (\frac{n}{t}-n)(k-j)q}) M_t u_k(x)^q. \quad (5.35)$$

For  $\theta = 1$  the sum over  $j$  has a fixed number of terms, hence is  $\mathcal{O}(2^{skq})$  for  $s' = s$ ; cf (5.33).

In the case in (5.27) one may as  $q > n/(n+s)$  arrange that  $s' = s > \frac{n}{t} - n > \max(0, \frac{n}{p} - n, \frac{n}{q} - n)$  by taking  $t$  sufficiently close to  $\min(p, q)$ . Then the geometric series above is estimated by the last term, hence is  $\mathcal{O}(2^{skq})$ , as required in (5.33).

What remains of (5.29) are the cases in which  $s \leq \max(0, \frac{n}{p} - n)$ , that is

$$s' < s \leq \max(0, \frac{n}{p} - n, \frac{n}{q} - n) < \frac{n}{t} - n, \quad t \in ]0, \min(p, q)[. \quad (5.36)$$

By (5.29) a suitably small  $t > 0$  yields  $s = \theta s' + (1-\theta)(\frac{n}{t} - n)$ , and since  $s' - (\frac{n}{t} - n) < 0$  in the above sum an estimate by the first term gives  $\mathcal{O}(2^{(s'\theta + (1-\theta)(\frac{n}{t} - n))kq}) = \mathcal{O}(2^{skq})$ .

For  $1 < q < \infty$  the inequality (5.33) follows by use of Hölder's inequality in (5.30), for if  $q + q' = q'q$ , one can for  $s' < 0$  use  $2^{\theta s'(k-j)}$  as a summation factor to get

$$|\Phi(2^{-j}D) \sum_{k \geq 0} u_k(x)|^q \leq c \sum_{k=j-h}^{[j/\theta]+h} 2^{(k-j)s'\theta q} \|\Phi(R2^{v-j} \cdot)\|_{\dot{B}_{1,t}^{\frac{n}{t}}}^q M_t u_k(x)^q \left( \frac{2^{-(\frac{1}{\theta}-1)js'\theta q'}}{2^{-s'\theta q'} - 1} \right)^{\frac{q}{q'}}. \quad (5.37)$$

Therefore the above procedure yields an estimate of  $\sum_{j=0}^{\infty} 2^{s'jq} |\Phi(2^{-j}D) \sum_{k \geq 0} u_k(x)|^q$  by

$$\sum_{k=0}^{\infty} 2^{ks'\theta q} M_t u_k(x)^q \left( h + \sum_{\theta k-h \leq j < k} 2^{(k-j)(\frac{n}{t}-n)q} \right) \leq c \sum_{k=0}^{\infty} 2^{(s'\theta + (1-\theta)(\frac{n}{t}-n))kq} M_t u_k(x)^q, \quad (5.38)$$

which again gives (5.33) by using (5.29) to arrange  $s \geq s'\theta + (1-\theta)(\frac{n}{t}-n)$  for a  $t \in ]0, 1[$ . By making the last inequality strict for a slightly larger  $t$ , the argument is seen to extend to cases with  $0 \leq s' < s \leq \max(0, \frac{n}{p} - n)$  by using  $s' - (\frac{n}{t} - n) < 0$  instead of  $s'$  in Hölder's inequality. In fact, one gets  $\sum 2^{(s'\theta + (1-\theta)(\frac{n}{t}-n))kq} (h2^{h(\frac{n}{t}-n)} + (1+h+k(1-\theta))) M_t u_k(x)^q$ , which again is  $\mathcal{O}(2^{skq})$  as the term  $k(1-\theta)$  is harmless by the choice of  $t$  (or for  $\theta = 1$ ). Hence (5.33) holds.

In case  $s' = s > 0$ , cf (5.27), one may take  $s - \frac{n}{t} + n > 0$  (as for  $q \leq 1$ ) now with  $2^{(k-j)(s-\frac{n}{t}+n)/2}$  as a summation factor: then  $(\dots)^{q/q'} = \mathcal{O}(1)$ , so the factor in front of  $M_t u_k^q$  becomes

$$\sum_{\theta k-h \leq j \leq k+h} 2^{sjq + (k-j)(s-\frac{n}{t}+n)q/2 + (k-j)_+(\frac{n}{t}-n)q} = \mathcal{O}(2^{skq}). \quad (5.39)$$

For  $q = \infty$  a direct argument yields sup-norms weighted by  $2^{s'j}$  and  $2^{sk}$  in (5.33).

By the choice of  $t$ , the Fefferman–Stein inequality applies to (5.33), cf (5.8), whence

$$\left( \int_{\mathbb{R}^n} \left( \sum_{j=0}^{\infty} 2^{s'jq} |\Phi_j(D) \sum_{k \geq 0} u_k(x)|^q \right)^{p/q} dx \right)^{1/p} \leq c \left( \int \|2^{sk} u_k(\cdot)\|_{\ell_q}^p dx \right)^{1/p} = cF. \quad (5.40)$$

Convergence is trivial for the partial sums  $u^{(m)} = \sum_{j \leq m} u_j$ , hence for  $u^{(m+M)} - u^{(m)}$ . So (5.40) applies to  $(0, \dots, 0, u_{m+1}, \dots, u_{m+M}, 0, \dots)$ , which for  $q < \infty$  by majorisation for  $m \rightarrow \infty$  yields

$$\|u^{(m+M)} - u^{(m)}\|_{F_{p,q}^{s'}} \leq c \left( \int_{\mathbb{R}^n} \left( \sum_{k=m}^{\infty} 2^{skq} |u_k(x)|^q \right)^{p/q} dx \right)^{1/p} \searrow 0. \quad (5.41)$$

As  $F_{p,q}^{s'}$  is complete,  $\sum u_j$  converges to an element  $u(x)$  with norm  $\leq cF$  according to (5.40). For  $q = \infty$  there is convergence in the larger space  $F_{p,1}^{s'-1/\theta}$  since the constant  $F$  remains finite if  $s, \infty$  are replaced by  $s-1, 1$ ; and again  $\|u\|_{F_{p,q}^{s'}} \leq cF$  holds by (5.40).

For the Besov case the arguments are analogous. First of all the absolute value should be replaced by the norm of  $L_p$  in (5.34), that now pertains to  $0 < q \leq \min(1, p)$ . Hölder's inequality applies in this case if  $1/q + 1/q' = 1/\min(1, p)$ ; and (5.40) can be replaced by boundedness of  $M_t$  in  $L_p$  for  $t < p$ . Convergence is similarly shown.  $\square$

Thus prepared, one arrives at a general result for  $0 < p \leq 1$ .

**Theorem 5.9.** *If  $a(x, \eta)$  belongs to the self-adjoint subclass  $\tilde{S}_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$ , the operator  $a(x, D)$  is bounded for  $0 < p \leq 1$ ,  $0 < q \leq \infty$ ,*

$$a(x, D): F_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow F_{p,q}^{s'}(\mathbb{R}^n) \quad \text{for } s' < s \leq \frac{n}{p} - n, \quad (5.42)$$

$$a(x, D): B_{p,q}^{s+d}(\mathbb{R}^n) \rightarrow B_{p,q}^{s'}(\mathbb{R}^n) \quad \text{for } s' < s \leq \frac{n}{p} - n. \quad (5.43)$$

*Proof.* Using (2.15)–(2.16), the question is easily reduced to the case of symbols for which

$$\hat{a}(\xi, \eta) \neq 0 \implies \max(1, |\xi + \eta|) \leq |\eta|. \quad (5.44)$$

In fact  $a = a_{\chi,1} + (a - a_{\chi,1})$  where  $a_{\chi,1}$  has the above property, whilst Theorem 5.4 yields the boundedness for  $a - a_{\chi,1}$ , as this is easily seen to fulfil the twisted diagonal condition (2.12) for  $B = 1$ . (Note that  $a - a_{\chi,1} \in \tilde{S}_{1,1}^d$  is seen from Theorem 2.4, as in (2.19), so that also  $a_{\chi,1} \in \tilde{S}_{1,1}^d$ .)

First  $a^{(1)}(x, D)u$  and  $a^{(3)}(x, D)u$  are for all  $s \in \mathbb{R}$  covered by the proof of Theorem 5.4; cf (5.18). Thus it suffices to estimate the  $a^{(2)}$ -series in (4.6) for fixed  $s' < s \leq \frac{n}{p} - n$ ; a simple embedding of  $F_{p,q}^{s'}$  gives a reduction to the case  $q > n/(n+s)$  if  $s > 0$ ; cf also Remark 5.8.

To fix notation, the splitting (4.2) is considered for some modulation function  $\Psi$  for which the associated Littlewood–Paley decomposition  $1 = \sum \Phi_j$  is used in the definition of the norms on  $F_{p,q}^s$ , as described prior to (5.5). Subjecting the second term in (4.6) to Hörmander’s localisation to a neighbourhood of  $\mathcal{T}$ , cf (2.15)–(2.16), one arrives at

$$\hat{a}_{k,\chi,\varepsilon}(\xi, \eta) = \hat{a}(\xi, \eta)\Phi(2^{-k}\xi)\chi(\xi + \eta, \varepsilon\eta), \quad (5.45)$$

This leaves the remainder  $b_k(x, \eta) = a_k(x, \eta) - a_{k,\chi,\varepsilon}(x, \eta)$ , that applied to the difference  $v_k = u^{k-1} - u^{k-h} = \mathcal{F}^{-1}((\Phi(2^{1-k}\cdot) - \Phi(2^{h-k}\cdot))\hat{u})$  in (4.6) gives

$$a_k(x, D)v_k = a_{k,\chi,\varepsilon}(x, D)v_k + b_k(x, D)v_k. \quad (5.46)$$

To utilise the pointwise estimates, take  $\psi \in C_0^\infty(\mathbb{R}^n)$  equal to 1 around the corona given by  $\frac{r}{R}2^{-1-h} \leq |\eta| \leq 1$  and supported where  $\frac{r}{R}2^{-2-h} \leq |\eta| \leq 2$ . Using  $\psi(\eta/(R2^k))$  as the auxiliary function in the symbol factor, the factorisation inequality (3.1) and Theorem 3.1 give

$$\begin{aligned} |a_{k,\chi,\varepsilon}(x, D)v_k(x)| &\leq F_{a_{k,\chi,\varepsilon}}(N, R2^k; x)v_k^*(N, R2^k; x) \\ &\leq cv_k^*(x) \sum_{|\alpha|=0}^{[N+n/2]+1} \left( \int_{r2^{k-h-2} \leq |\eta| \leq R2^{k+1}} |(R2^k)^{|\alpha|-n/2} D_\eta^\alpha a_{k,\chi,\varepsilon}(x, \eta)|^2 d\eta \right)^{1/2}. \end{aligned} \quad (5.47)$$

Here the ratio of the limits is  $2R/(r2^{-h-2}) > 32$ , so the integration can be extended to  $L \geq 6$  dyadic coronas, with  $|\eta| \in [R2^{k+1-L}, R2^{k+1}]$ . This gives an estimate by  $c(R2^k)^d L^{1/2} N_{\chi,\varepsilon,\alpha}(a_k)$ . In addition, Minkowski’s inequality gives

$$N_{\chi,\varepsilon,\alpha}(a_k) \leq \sup_{\rho>0} \rho^{|\alpha|-d} \int_{\mathbb{R}^n} |2^{kn} \check{\Phi}(2^k y)| \left( \int_{\rho \leq |\eta| \leq 2\rho} |D_\eta^\alpha a_{\chi,\varepsilon}(x-y, \eta)|^2 \frac{d\eta}{\rho^n} \right)^{1/2} dy \leq cN_{\chi,\varepsilon,\alpha}(a). \quad (5.48)$$

So it follows from the above and (iii) in Theorem 2.4 that for all  $\sigma > 0$ ,

$$|a_{k,\chi,\varepsilon}(x,D)v_k(x)| \leq cv_k^*(N, R2^k; x)2^{k+d} \sum_{|\alpha| \leq [N+n/2]+1} c_{\alpha,\sigma} \varepsilon^{\sigma+n/2-|\alpha|}. \quad (5.49)$$

Now  $\theta \in ]0, 1[$  is taken so small that  $s' < s - \frac{\theta}{1-\theta}(\frac{n}{p} - n - s)$ , which is the last condition in Proposition 5.7 with  $1 - \theta$  instead of  $\theta$ . Then  $\varepsilon = 2^{-k\theta}$  in (5.49) clearly gives

$$2^{k(s+M)} |a_{k,\chi,\varepsilon}(x,D)v_k(x)| \leq cv_k^*(N, R2^k; x)2^{k(s+d)} 2^{-k\theta(\sigma-1-N-M/\theta)}. \quad (5.50)$$

Here one may first of all take  $N > n/\min(p, q)$  so that (5.8) applies. Secondly,  $\sigma$  can for any  $M$  (with  $\theta$  fixed as above) be chosen so that  $2^{-k\theta(\sigma-1-N-M/\theta)} \leq 1$ . This gives

$$\begin{aligned} \left( \int \|2^{k(s+M)} a_{k,\chi,\varepsilon}(x,D)v_k(\cdot)\|_{\ell_q}^p dx \right)^{\frac{1}{p}} &\leq c \left( \int \|2^{k(s+d)} v_k^*(N, R2^k; \cdot)\|_{\ell_q}^p dx \right)^{\frac{1}{p}} \\ &\leq c' \left( \int \|2^{k(s+d)} v_k(\cdot)\|_{\ell_q}^p dx \right)^{\frac{1}{p}} \leq c'' \|u\|_{F_{p,q}^{s+d}}. \end{aligned} \quad (5.51)$$

Here the last inequality follows from the (quasi-)triangle inequality in  $\ell_q$  and  $L_p$ .

Since  $a_{k,\chi,\varepsilon}(x,D)v_k$  according to (4.9) has its spectrum in  $\bar{B}(0, 2R2^k)$ , the above estimate allows application of Lemma 5.2, if  $M$  is so large that

$$M > 0, \quad M + s > 0, \quad M + s > \frac{n}{p} - n. \quad (5.52)$$

This gives convergence of  $\sum a_{k,\chi,2^{-k\theta}}(x,D)v_k$  to a function in  $F_{p,\infty}^{s+M}$  fulfilling

$$\left\| \sum_{k=1}^{\infty} a_{k,\chi,2^{-k\theta}}(x,D)v_k \right\|_{F_{p,\infty}^{s+M}} \leq c \|u\|_{F_{p,q}^{s+d}}. \quad (5.53)$$

On the left-hand side the embedding  $F_{p,\infty}^{s+M} \hookrightarrow F_{p,q}^s$  applies, of course.

For the remainder  $\sum_{k=1}^{\infty} b_k(x,D)v_k$ , cf (5.45) ff, note that (5.51) holds for  $M = 0$  with the same  $\sigma$ . If combined with a part of (5.17), a crude use of the (quasi-)triangle inequality gives

$$\int \|2^{ks} b_k(x,D)v_k(\cdot)\|_{\ell_q}^p dx \leq \int \|2^{ks} (a_k(x,D) - a_{k,\chi,2^{-k\theta}}(x,D))v_k(\cdot)\|_{\ell_q}^p dx \leq c \|u\|_{F_{p,q}^{s+d}}^p. \quad (5.54)$$

The series also fulfils a corona condition with inner radius  $2^{(1-\theta)k}$  for all large  $k$ , namely

$$\text{supp } \mathcal{F} b_k(x,D)v_k \subset \{ \zeta \mid (r2^{-h-2})2^{k(1-\theta)} \leq |\zeta| \leq R2^k \}. \quad (5.55)$$

Indeed,  $\hat{b}_k(x, \eta) = 0$  holds if  $\chi(\xi + \eta, 2^{-k\theta}\eta) = 1$ , so at least for  $2 \max(1, |\xi + \eta|) \leq 2^{-k\theta}|\eta|$ ; whence by (5.44),

$$\text{supp } \hat{b}_k \subset \{ (\xi, \eta) \mid 2^{-1-k\theta}|\eta| \leq \max(1, |\xi + \eta|) \leq |\eta| \}. \quad (5.56)$$

The Spectral Support Rule (4.10) shows that  $\zeta = \xi + \eta$  only belongs to  $\text{supp } \mathcal{F} b_k(x,D)v_k$  if

$$|\zeta| \leq |\eta| \leq R2^k \quad (5.57)$$

$$\max(1, |\zeta|) \geq 2^{-1-k\theta}|\eta| \geq r2^{k(1-\theta)-h-2}. \quad (5.58)$$

When  $2^{k(1-\theta)} > 2^{h+2}/r$  (so that the last right-hand side is  $> 1$ ) this shows (5.55). Hence Proposition 5.7 applies, and the choice of  $\theta$  gives

$$\left\| \sum_{k=1}^{\infty} b_k(x, D)v_k \right\|_{F_{p,q}^{s'}} \leq c \|u\|_{F_{p,q}^{s+d}}. \quad (5.59)$$

The other contribution  $\sum (a^k(x, D) - a^{k-h}(x, D))u_k$  in (4.6) is analogous, with a splitting of  $\tilde{a}_k = a^k - a^{k-h}$  into  $\tilde{a}_{k,\chi,\varepsilon} + \tilde{b}_k$  as in (5.45). In particular the inequality (5.49) can be carried over to  $\tilde{a}_{k,\chi,\varepsilon}(x, D)u_k$ , with just another constant because Minkowski's inequality now leads to an estimate in terms of  $\int |\Psi - \Psi(2^h \cdot)| dy$ . Consequently (5.50) carries over, and with (5.52) the same arguments as for (5.53), (5.59) give

$$\left\| \sum_{k=h}^{\infty} (a^k - a^{k-h})_{\chi,\varepsilon}(x, D)u_k \right\|_{F_{p,\infty}^{s+M}} + \left\| \sum_{k=h}^{\infty} \tilde{b}_k(x, D)u_k \right\|_{F_{p,q}^{s'}} \leq c \|u\|_{F_{p,q}^{s+d}}. \quad (5.60)$$

Altogether the estimates (5.53), (5.59), (5.60) show that

$$\left\| a_{\psi}^{(2)}(x, D)u \right\|_{F_{p,q}^{s'}} \leq c \|u\|_{F_{p,q}^{s+d}}. \quad (5.61)$$

Via the decomposition (4.2),  $a_{\psi}(x, D)$  is therefore a bounded linear map  $F_{p,q}^{s+d} \rightarrow F_{p,q}^{s'}$ . Since  $\mathcal{S}$  is dense for  $q < \infty$  (a case one can reduce to), there is no dependence on the modulation function  $\psi$ , so the type 1, 1-operator  $a(x, D)$  is defined and continuous on  $F_{p,q}^{s+d}$  as stated.

The arguments are similar for the Besov spaces: it suffices to interchange the order of the norms in  $\ell_q$  and  $L_p$ , and to use the estimate in (5.8) for each single  $k$ .  $\square$

The proof extends to cases with  $0 < p \leq \infty$  when  $s' < s \leq \max(0, \frac{n}{p} - n)$ , but this barely fails to reprove Theorem 5.5, so only  $p \leq 1$  is included in Theorem 5.9. Cf also Remark 5.10 below.

One particular interest of Theorem 5.9 is that  $F_{p,2}^0(\mathbb{R}^n)$  identifies with the so-called local Hardy space  $h_p(\mathbb{R}^n)$  for  $0 < p \leq 1$ ; cf [Tri83] and especially [Tri92, Ch. 1.4]. In this case Theorem 5.9 gives boundedness as a map  $a(x, D): h_p(\mathbb{R}^n) \rightarrow F_{p,2}^{s'}(\mathbb{R}^n)$  for every  $s' < 0$ , but this can probably be improved in view of recent results:

*Remark 5.10.* Extensions to  $h_p(\mathbb{R}^n)$  of operators in the self-adjoint subclass  $\text{OP}(\tilde{S}_{1,1}^0)$  were treated by Hounie and dos Santos Kapp [HdSK09], who used atomic estimates to carry over the  $L_2$ -boundedness of Hörmander [Hör89, Hör97] to  $h_p$ , ie to obtain estimates with  $s' = s = 0$ . However, they worked without a precise definition of type 1, 1-operators. Torres [Tor90] obtained extensions by continuity using the atomic decompositions in [FJ90], but for  $s < 0$  he relied on conditions on the adjoint  $a(x, D)^*$  rather than on the symbol  $a(x, \eta)$  itself. In the  $F_{p,q}^s$ -scales, general type 1, 1-operators were first estimated by Runst [Run85], though with insufficient control of the spectra as noted in [Joh05]; a remedy is provided by the Spectral Support Rule (4.10).

*Remark 5.11.* Together Theorems 5.4, 5.5 and 5.9 give a satisfactory  $L_p$ -theory of operators  $a(x, D)$  in the self-adjoint subclass  $\text{OP}(\tilde{S}_{1,1}^d)$ , inasmuch as for the domain  $D(a(x, D))$  they cover all possible  $s, p$ . Only a few of the codomains seem barely unoptimal, and these all concern cases with  $0 < q < 1$  or  $0 < p \leq 1$ ; cf the role of the parameter  $r$  in Theorem 5.4 and that of  $s'$  in Theorem 5.9.

*Remark 5.12.* As a corollary to Theorem 5.9, its proof (extended to  $p \geq 1$ ) gives that if  $a(x, D)$  fulfils the twisted diagonal condition of order  $\sigma > 0$ , i.e. (2.17) holds for a specific  $\sigma$ , then it is not difficult to see that

$$B_{p,q}^s \bigcup F_{p,q}^s \subset D(a(x, D)) \quad \text{for } s > -\sigma + [N + n/2] + 1 - n/2, \quad 1 \leq p \leq \infty. \quad (5.62)$$

Hereby  $N > n/p$  must hold (as  $q = \infty$  suffices now), so the condition has the form  $s > -\sigma + k$ , where  $k = [n/p] + 1$  in even dimensions, while in odd dimensions  $k$  should be the least number in  $\frac{1}{2} + \mathbb{N}_0$  such that  $k > n/p$ . While this does provide a result in the  $L_p$  set-up, it is hardly optimal; cf Hörmander's condition  $s > -\sigma$  for  $p = 2$ , recalled in (2.18).

## REFERENCES

- [Bon81] J.-M. Bony, *Calcul symbolique et propagations des singularités pour les équations aux dérivées partielles non linéaires*, Ann. scient. Éc. Norm. Sup. **14** (1981), 209–246.
- [Bou83] G. Bourdaud, *Sur les opérateurs pseudo-différentiels à coefficients peu réguliers*, Thèse, Univ. de Paris-Sud, 1983.
- [Bou88] ———, *Une algèbre maximale d'opérateurs pseudo-différentiels*, Comm. Partial Differential Equations **13** (1988), no. 9, 1059–1083.
- [Chi72] Chin Hung Ching, *Pseudo-differential operators with nonregular symbols*, J. Differential Equations **11** (1972), 436–447.
- [CM78] R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque, vol. 57, Société Mathématique de France, Paris, 1978.
- [FJ90] M. Frazier and B. Jawerth, *A discrete transform and decomposition of distribution spaces*, J. Func. Anal. **93** (1990), 34–170.
- [HdSK09] J. Hounie and R. A. dos Santos Kapp, *Pseudodifferential operators on local Hardy spaces*, J. Fourier Anal. Appl. **15** (2009), 153–178.
- [Hör85] L. Hörmander, *The analysis of linear partial differential operators*, Grundlehren der mathematischen Wissenschaften, Springer Verlag, Berlin, 1983, 1985.
- [Hör88] ———, *Pseudo-differential operators of type 1, 1*, Comm. Partial Differential Equations **13** (1988), no. 9, 1085–1111.
- [Hör89] ———, *Continuity of pseudo-differential operators of type 1, 1*, Comm. Partial Differential Equations **14** (1989), no. 2, 231–243.
- [Hör97] ———, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications, vol. 26, Springer Verlag, Berlin, 1997.
- [Joh04] J. Johnsen, *Domains of type 1, 1 operators: a case for Triebel–Lizorkin spaces*, C. R. Acad. Sci. Paris Sér. I Math. **339** (2004), no. 2, 115–118.
- [Joh05] ———, *Domains of pseudo-differential operators: a case for the Triebel–Lizorkin spaces*, J. Function Spaces Appl. **3** (2005), 263–286.
- [Joh08a] ———, *Parametrices and exact parilinearisation of semi-linear boundary problems*, Comm. Part. Diff. Eqs. **33** (2008), 1729–1787.
- [Joh08b] ———, *Type 1, 1-operators defined by vanishing frequency modulation*, New Developments in Pseudo-Differential Operators (L. Rodino and M. W. Wong, eds.), Operator Theory: Advances and Applications, vol. 189, Birkhäuser, 2008, pp. 201–246.
- [Joh10] ———, *Type 1, 1-operators on spaces of temperate distributions*, Tech. Report R-2010-13, Aalborg University, 2010, (Available at <http://vbn.aau.dk/files/38938995/R-2010-13.pdf>).
- [Joh11] ———, *Pointwise estimates of pseudo-differential operators*, J. Pseudo-Diff. Ops. Appl. **2** (2011), 377–398, DOI:10.1007/s11868-011-0029-2.
- [JS07] J. Johnsen and W. Sickel, *A direct proof of Sobolev embeddings for quasi-homogeneous Lizorkin–Triebel spaces with mixed norms*, J. Function Spaces Appl. **5** (2007), 183–198.

- [JS08] J. Johnsen and W. Sickel, *On the trace problem for Lizorkin–Triebel spaces with mixed norms*, Math. Nachr. **281** (2008), 1–28.
- [Mar85] J. Marschall, *Pseudo-differential operators with non-regular symbols*, Ph.D. thesis, Free University of Berlin, 1985.
- [Mar91] ———, *Weighted parabolic Triebel spaces of product type. Fourier multipliers and pseudo-differential operators*, Forum Math. **3** (1991), no. 5, 479–511.
- [Mey81] Y. Meyer, *Remarques sur un théorème de J.-M. Bony*, Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980), 1981, pp. 1–20.
- [Pee75] J. Peetre, *On spaces of Triebel–Lizorkin type*, Ark. Mat. (1975), 123–130.
- [RS96] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskiĭ operators and non-linear partial differential equations*, Nonlinear analysis and applications, vol. 3, de Gruyter, Berlin, 1996.
- [Run85] T. Runst, *Pseudodifferential operators of the “exotic” class  $L_{1,1}^0$  in spaces of Besov and Triebel–Lizorkin type*, Ann. Global Anal. Geom. **3** (1985), no. 1, 13–28.
- [Ste93] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993.
- [Tor90] R.H. Torres, *Continuity properties of pseudodifferential operators of type 1,1*, Comm. Partial Differential Equations **15** (1990), 1313–1328.
- [Tri78] H. Triebel, *Spaces of Besov–Hardy–Sobolev type*, Teubner–Texte zur Mathematik, vol. 15, Teubner Verlagsgesellschaft, Leipzig, 1978.
- [Tri83] ———, *Theory of function spaces*, Monographs in mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.
- [Tri92] ———, *Theory of function spaces II*, Monographs in mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.
- [Yam86] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **33** (1986), 131–174.

DEPARTMENT OF MATHEMATICAL SCIENCES, AALBORG UNIVERSITY, FREDRIK BAJERS VEJ 7G, DK-9220 AALBORG ØST, DENMARK

*E-mail address:* jjohnsen@math.aau.dk