POINTWISE ESTIMATES OF PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. As a new technique it is shown how general pseudo-differential operators can be estimated at arbitrary points in Euclidean space when acting on functions u with compact spectra. The estimate is a factorisation inequality, in which one factor is the Peetre–Fefferman–Stein maximal function of u, whilst the other is a symbol factor carrying the whole information on the symbol. The symbol factor is estimated in terms of the spectral radius of u, so that the framework is well suited for Littlewood–Paley analysis. It is also shown how it gives easy access to results on polynomial bounds and estimates in L_p, including a new result for type $1,1$-operators that they are always bounded on L_p-functions with compact spectra.

1. INTRODUCTION

The aim of this note is to show how one can estimate a pseudo-differential operator at an arbitrary point $x \in \mathbb{R}^n$. These pointwise estimates are applied to mapping properties and continuity results, in order to illustrate their efficacy.

The central theme is to show for a general symbol $a(x,\eta)$, with the associated operator $a(x,D)u(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\eta} a(x,\eta)\hat{u}(\eta)\,d\eta$, that for distributions with compact spectra, i.e. $u \in \mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^n)$,

$$|a(x,D)u(x)| \le c \cdot u^*(x) \quad \text{for every} \quad x \in \mathbb{R}^n. \tag{1}$$

Here u^* denotes the Peetre-Fefferman-Stein maximal function

$$u^*(x) = u^*(N,R;x) = \sup_y \frac{|u(x-y)|}{(1+|Ry|)^N} = \sup_y \frac{|u(y)|}{(1+R|x-y|)^N}, \tag{2}$$

where $N > 0$, $R > 0$ are parameters; R so large that $x \in \operatorname{supp}\hat{u}$ implies $|x| \le R$.

One obvious advantage of proving (1) in terms of (2) is the *immediate* L_p-estimate

$$\int |a(x,D)u(x)|^p\,dx \le c^p \int |u^*(x)|^p\,dx \le c^p C_p \|u\|_p^p, \qquad 1 \le p \le \infty, \tag{3}$$

where the last step is to invoke the *maximal inequality*

$$\int_{\mathbb{R}^n} |u^*(x)|^p\,dx \le C_p \int_{\mathbb{R}^n} |u(x)|^p\,dx, \qquad u \in L_p \cap \mathscr{F}^{-1}\mathscr{E}'. \tag{4}$$

This estimate of the non-linear map $u \mapsto u^*$ has for $Np > n$ been known since 1975 from a work of Peetre [22], who estimated $u^*(x)$ by the Hardy-Littlewood maximal function $Mu(x) = \sup_{\rho>0} c_n\rho^{-n}\int_{|y|<\rho}|u(x+y)|\,dy$ in order to invoke L_p-boundedness of the latter. A significantly simpler proof is given below.

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It is remarkable that little attention has been paid over the decades to pointwise estimates like (1) — in comparison Peetre's proof of (4) quickly got a central role in the theory of function spaces; cf. [27, 1.4.1]. However, to the author's knowledge, there has only been a similar attempt by Marschall, who in his thesis [16] suggested to estimate $a(x,D)u(x)$ in terms of Mu; this was followed up in a series of papers, e.g. [17, 18, 19], where the technique was used to derive boundedness under weak assumptions in spaces based on L_p (functions and symbols subject to Besov and Lizorkin–Triebel conditions).

In the present paper the point of view is quite different. First of all because u^* is rather easier to treat and work with than Mu. Secondly, the aim is to explain how pointwise estimates in terms of u^* will simplify well-known topics such as L_p-estimates and Littlewood–Paley analysis of $a(x,D)$.

So as a main result here, (1) is also shown to be straightforward to obtain; cf. Theorem 4.1–4.3 below. Indeed, the constant c in (1) is just an upper bound for the *symbol factor* $F_a(x)$, which is a continuous, bounded function carrying the entire information of the symbol in the *factorisation* inequality

$$|a(x,D)u(x)| \le F_a(x)u^*(x), \qquad u \in \mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^n). \tag{5}$$

As $F_a(x)$ only depends vaguely on u (cf. Section 4), this gives a somewhat surprising decoupling.

The inequality is well suited for Littlewood–Paley analysis of $a(x,D)$ as described in Section 5. The set-up there has recently been exploited by the author [14] in proofs of fundamental results for pseudo-differential operators of type $1,1$; this is briefly reviewed in Section 6, where also (3) is given as a new theorem for type $1,1$-operators.

2. THE PEETRE–FEFFERMAN–STEIN MAXIMAL FUNCTION

This section explores the definition of $u^*(x)$ in (2), in lack of a reference. It also gives a straightforward proof of the maximal inequality (4).

For the reader's sake a few easy facts are recalled first. To show that $u^*(x)$ is a 'slowly' varying function, note that

$$\frac{|u(x-z)|}{(1+|Rz|)^N} = \frac{|u(y-(z+y-x))|}{(1+R|z+y-x|)^N} \cdot \frac{(1+R|z+y-x|)^N}{(1+|Rz|)^N}, \tag{6}$$

so the inequality $1+|x+y| \le 1+|x|+|y|+|x||y| = (1+|x|)(1+|y|)$ gives

$$u^*(x) \le u^*(y)(1+R|x-y|)^N. \tag{7}$$

Therefore $u^*(x)$ is finite at every $x \in \mathbb{R}^n$ if it is so at one point y. So either $u^*(x) = \infty$ on the entire $\mathbb{R}^n$, or (7) implies that $u^*(x)$ is continuous on $\mathbb{R}^n$, i.e. $u^* \in C(\mathbb{R}^n)$.

Finiteness is for large N implied by the (often imposed) assumption that $u \in \mathscr{S}'(\mathbb{R}^n)$ should have its spectrum in the closed ball $\overline{B}(0,R)$ of radius R, ie

$$\operatorname{supp} \mathscr{F}u \subset \overline{B}(0,R). \tag{8}$$

Indeed, then $|u(x)| \le c_R(1+R|x|)^m$ by the Paley–Wiener–Schwartz theorem, when m is the order of $\hat{u}$. So $N \ge m$ gives $u^*(N,R;0) \le c_R$, hence $u^*(N,R;x) < \infty$ for all $x \in \mathbb{R}^n$.

In any case it is clear that $u \mapsto u^*$ is subadditive, i.e. $(u+v)^* \le u^* + v^*$, whence

$$|u^*(N,R;x) - v^*(N,R;x)| \le (u-v)^*(N,R;x). \tag{9}$$

Hence $u \mapsto u^*$ is Lipschitz continuous on $L_\infty(\mathbb{R}^n)$ with constant 1, as it is a shrinking map there, i.e. $\|u^*\|_\infty \le \|u\|_\infty$. With respect to the Hölder seminorm

$$|u|_\sigma := \sup_{x\neq y} |u(x)-u(y)|/|x-y|^\sigma, \qquad 0<\sigma<1, \tag{10}$$

it is also a shrinking map, for (9) gives that

$$|u^*(x+h)-u^*(x)| \le \sup_{\mathbb{R}^n} \frac{|u(x+h+\cdot)-u(x+\cdot)|}{(1+R|\cdot|)^N} \le |u|_\sigma |h|^\sigma. \tag{11}$$

Therefore $|u^*|_\sigma \le |u|_\sigma$ as claimed. In particular one has

Proposition 2.1. *The map $u\mapsto u^*$ is for all $N,R>0$ a shrinking map on the Hölder space $C^\sigma(\mathbb{R}^n)$, $0<\sigma<1$, defined by finiteness of the norm $|u|^*_\sigma = \sup|u| + |u|_\sigma$.*

A main case is when u is in $L_p(\mathbb{R}^n)$, $1\le p\le\infty$. For $p<\infty$ one has that $u^*\equiv\infty$ for u equal to $e^{|x|}$ times the characteristic function of $\bigcup_{k\in\mathbb{N}} B(ke_1, e^{-(k+1)2p})$. Such growth is impossible on the subspace of functions fulfilling the spectral condition (8), so this is imposed henceforth.

As an a priori analysis of this case, the Nikolskiĭ–Plancherel–Polya inequality implies $u\in L_p\cap L_\infty$, for it states that if $u\in L_p$ and (8) holds, then

$$\|u\|_r \le cR^{\frac{n}{p}-\frac{n}{r}}\|u\|_p \quad \text{for} \quad p<r\le\infty. \tag{12}$$

For its proof one can take an auxiliary function $\psi\in\mathscr{S}(\mathbb{R}^n)$ so that $\mathscr{F}\psi(\xi)=0$ for $|\xi|\ge 2$ and $\mathscr{F}\psi(\xi)=1$ around $B(0,1)$, for then $u=R^n\psi(R\cdot)*u$, and (12) follows from this identity at once by the Hausdorff–Young inequality $\|f*g\|_r\le\|f\|_p\|g\|_q$, where $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$; hereby $c=\|\psi\|_q$, that only depends on p, r and n.

To complete the picture, (12) extends as it stands to the range $0<p<r\le\infty$, provided u is given in $L_p\cap\mathscr{S}'(\mathbb{R}^n)$ with $\operatorname{supp}\mathscr{F}u\in\overline{B}(0,R)$; cf. [27, 1.4.1(ii)]. The direct treatment in [15] shows that one can take $c=\|\psi\|_\infty^{\frac{1}{p}-\frac{1}{r}}$ for $0<p\le 1$. (For $0<p<1$, the set $L_p\cap\mathscr{S}'$ itself consists of the $u\in L_1^{\rm loc}\cap\mathscr{S}'$ fulfilling $\int_{\mathbb{R}^n}|u|^p\,dx<\infty$, that *per se* requires stricter smallness than L_1 for $|x|\to\infty$ but gives a global condition on the singularities in the possibly non-compact region where $|u(x)|>1$.)

By (12), pointwise estimates of $u^*(x)$ hold for L_p-functions with compact spectra:

Lemma 2.1. *For every $u\in L_p\cap\mathscr{S}'(\mathbb{R}^n)$, $0<p\le\infty$ with $\operatorname{supp}\mathscr{F}u\subset\overline{B}(0,R)$, it holds true on $\mathbb{R}^n$ that*

$$|u(x)|\le u^*(N,R;x)\le\|u\|_\infty<\infty \quad \textit{for every} \quad N>0. \tag{13}$$

Proof. With $r=\infty$ in (12) it follows that $\|u\|_\infty$ is finite; and it dominates $u^*(x)$ as stated, by the definition of u^* in (2). Taking $y=0$ there yields $|u(x)|\le u^*(x)$. □

Note that $\mathscr{F}L_p\subset\mathscr{D}'^k$ for the least integer $k>\frac{n}{2}-\frac{n}{p}$ if $p>2$, cf. [8, Sec. 7.9], so the Paley–Wiener–Schwartz theorem would give the poor condition $N\ge[\frac{n}{2}-\frac{n}{p}]+1$ for finiteness of u^*.

For convenience in the following, the auxiliary function f_N is introduced as

$$f_N(z)=(1+|z|)^{-N}. \tag{14}$$

Example 2.1. As is well known, u^* is useful (when finite) for pointwise control of convolutions, since e.g. the assumptions $\varphi\in\mathscr{S}$, $u\in\mathscr{F}^{-1}\mathscr{E}'$ clearly give

$$|\varphi*u(x)|\le\int(1+R|y|)^N|\varphi(y)|\frac{|u(x-y)|}{(1+R|y|)^N}\,dy\le cu^*(N,R;x). \tag{15}$$

Example 2.2. Conversely $u^*(x)$ may be controlled by convolving $|u|$ with the above function f_N; cf. (14). Hereby cases with $N > n$ are particularly simple as one has

$$u^*(N,R;x) \le C_N R^n f_N(R\cdot) * |u|(x). \tag{16}$$

Indeed, when $u \in L_p$, $1 \le p \le \infty$ with $\operatorname{supp} \mathscr{F} u \subset \overline{B}(0,R)$ the compact spectrum of u can be exploited by taking ψ as after (12) above, which gives $u = R^n \psi(R\cdot) * u$. Thence

$$\begin{aligned} |u(y)|(1+R|x-y|)^{-N} &\le \int \frac{R^n |\psi(R(y-z))u(z)|}{(1+R|x-y|)^N}\,dz \\ &\le \int (1+R|y-z|)^N |\psi(R(y-z))| \frac{R^n |u(z)|}{(1+R|x-z|)^N}\,dz \end{aligned} \tag{17}$$

by using $(1+R|x-y|)(1+R|y-z|) \ge (1+R|x-z|)$ in the denominator. This gives

$$u^*(x) \le C_N \int \frac{R^n |u(z)|}{(1+R|x-z|)^N}\,dz, \tag{18}$$

where $C_N := \sup(1+|v|)^N |\psi(v)| < \infty$ because $\psi \in \mathscr{S}$. This shows the claim in (16).

As an addendum to Example 2.2, a basic estimate gives in (16) that for $p \ge 1$

$$\|u^*\|_p \le C_N \|R^n f_N(R\cdot) * |u|\|_p \le C_N \int (1+|z|)^{-N}\,dz \cdot \|u\|_p. \tag{19}$$

So for $N > n$ this short remark proves a special case of the maximal inequality (4).

However, $N > n$ is far from an optimal assumption for (4). But a few changes give the improvement $N > n/p$; and also every $p \in]0,\infty]$ can be treated using the Nikolskiĭ–Plancherel–Polya inequality (12).

The idea is to utilise the powerful pointwise estimate in (16), where e.g. both sides can be integrated over $\mathbb{R}^n$ (unlike (13)). But first it is generalised thus:

Proposition 2.2. *If $u \in \mathscr{S}'$, $\operatorname{supp} \mathscr{F} u \subset \overline{B}(0,R)$ and $N, p \in]0,\infty[$ are arbitrary, then*

$$u^*(N,R;x) \le C_{n,N,p} \Big(\int R^n \frac{|u(x-z)|^p}{(1+R|z|)^{Np}}\,dz \Big)^{\frac{1}{p}} = C_{n,N,p} \big(R^n f_N^p(R\cdot) * |u|^p(x) \big)^{\frac{1}{p}} \tag{20}$$

for a constant $C_{n,N,p}$ depending only on n, N and p.

Proof. As above $u(x) = R^n \psi(R\cdot) * u = \langle u, R^n \psi(R(x-\cdot)) \rangle$, which can be written as an integral since $(1+|y|)^{-k} u(y)$ is in $L_1(\mathbb{R}^n)$ for a large k; therefore (17) holds. Suppose now that the right-hand side of (20) is finite.

For $1 \le p < \infty$ one can simply use Hölder's inequality for $p + p' = p'p$ in the passage from (17) to (18); then $C_{n,N,p} = (\int (1+|z|)^{Np'} |\psi(z)|^{p'}\,dz)^{1/p'}$ gives (20).

If $0 < p \le 1$ the L_1-norm with respect to z in (17) can be estimated by the L_p-norm, according to (12), for the Fourier transform of $z \mapsto \psi(R(y-z))u(z)$ is supported by $\overline{B}(0,3R)$. Invoking the specific constant in (12) and proceeding as before, this gives

$$\begin{aligned} u^*(x)^p &\le \int \sup_{y \in \mathbb{R}^n} \frac{\|\psi\|_\infty^{(\frac{1}{p}-1)p} (3R)^{(\frac{n}{p}-n)p} C_N^p R^{np} |u(z)|^p}{(1+R|x-y|)^{Np}(1+R|y-z|)^{Np}}\,dz \\ &\le C_{n,N,p}^p \int \frac{R^n |u(x-z)|^p}{(1+R|z|)^{Np}}\,dz, \end{aligned} \tag{21}$$

where C_N is as in (18) and now $C_{n,N,p} = C_N 3^{n/p} \|\psi\|_\infty^{(1-p)/p}$. □

These elementary considerations give a short proof, in the style of (19), of the following important theorem on the L_p-boundedness of the maximal operator $u \mapsto u^*$.

Theorem 2.1. *When $0 < p \le \infty$ and $N > n/p$, then there is a constant $C'_{n,N,p} > 0$ such that the maximal function $u^*(N,R;x)$ in (2) fulfils*

$$\|u^*(N,R;\cdot)\|_p \le C'_{n,N,p}\|u\|_p \tag{22}$$

for every $u \in L_p(\mathbb{R}^n) \cap \mathscr{S}'(\mathbb{R}^n)$ in the closed subspace with $\operatorname{supp}\mathscr{F}u \subset \overline{B}(0,R)$. On this subspace there is Lipschitz continuity

$$\|u^*(N,R;\cdot) - v^*(N,R;\cdot)\|_p \le C'_{n,N,p}\|u-v\|_p. \tag{23}$$

Proof. Lemma 2.1 yields that u^* is finite and consequently continuous as noted after (7), hence measurable. The case $p = \infty$ then follows at once from the lemma. For $0 < p < \infty$ one can integrate both sides of (20), which by Fubini's theorem yields

$$\int |u^*(x)|^p\,dx \le C^p_{n,N,p} \iint \frac{|u(x-z/R)|^p}{(1+|z|)^{Np}}\,dz\,dx = C^p \int |u(x)|^p\,dx \tag{24}$$

for $C^p = C^p_{n,N,p}\int(1+|z|)^{-Np}\,dz$. Since $Np > n$ this gives $u^* \in L_p$ and (22). Now the Lipschitz property follows by integration on both sides of (9). □

Among the further properties there is a Bernstein inequality for u^*, which states that the maximal function of u controls that of the derivatives $\partial^\alpha u$.

Proposition 2.3. *The estimate $(\partial^\alpha u)^*(N,R;x) \le C_N^{(\alpha)} R^{|\alpha|} u^*(N,R;x)$ is valid when $\operatorname{supp}\mathscr{F}u \subset \overline{B}(0,R)$, with a constant $C_N^{(\alpha)}$ independent of u, R.*

While this is known (cf. [27, 1.3.1] for $R = 1$), it is natural to give the short proof here. Writing $u(x-y)(1+R|y|)^{-N}$ in terms of the convolution $R^n\psi(R\cdot) * u$, cf. Example 2.2, it is straightforward to see by applying ∂_x^α that for $N' > 0$,

$$(\partial^\alpha u)^*(N,R;x) \le \sup(1+|\cdot|)^{N'}|\partial^\alpha\psi| \sup_y \int \frac{R^{n+|\alpha|}|u(z)|}{(1+R|y|)^N(1+R|x-y-z|)^{N'}}\,dz. \tag{25}$$

For $N' = N+n+1$ a simple estimate of the denominator, cf. Example 2.2, now shows Proposition 2.3 with the constant $C_N^{(\alpha)} = \sup(1+|\cdot|)^{N+n+1}|\partial^\alpha\psi|\int(1+|z|)^{-n-1}\,dz$.

Remark 2.1. The maximal function u^* was introduced by Peetre [22], inspired by the non-tangential maximal function used by Fefferman and Stein a few years earlier [7]. It has been widely used in the theory of Besov and Lizorkin–Triebel spaces, cf. [27, 28, 23], where the boundedness in Theorem 2.1 has been a main tool since the 1970's; cf. [27, 1.4.1]. Usually its proof has been based on an estimate in terms of the Hardy–Littlewood maximal function, $M_r u(x) = \sup_\rho(\rho^{-n}\int_{|y|<\rho}|u(x+y)|^r\,dy)^{1/r}$, i.e. for $\operatorname{supp}\mathscr{F}u \subset \overline{B}(0,R)$,

$$u^*(N,R;x) \le cM_r u(x), \qquad N \ge n/r. \tag{26}$$

When $N > n/r$ this results from Proposition 2.2 by splitting the integral ($p = r$) in regions with $2^k \le |z| \le 2^{k+1}$. (For $N = n/r$ it was shown by Triebel, cf. [27, 1.3.1 ff], by combining an inequality for u^*, $(\partial_j u)^*$ and $M_r u$, due to Peetre [22], with the Bernstein inequality for u^*; cf. Proposition 2.3.) This gave a proof of (4) by combining (26) with the inequality $\|M_r u\|_p \le c\|u\|_p$ for $p > r$. The present proofs of Proposition 2.2 and Theorem 2.1 are rather simpler.

3. PREPARATIONS

Notation and notions from distribution theory are the same as in Hörmander's book [8], unless otherwise mentioned. E.g. $[t]$ denotes the largest integer $k \le t$ for $t \in \mathbb{R}$. The Fourier transformation is $\mathscr{F}u(\xi) = \int e^{-\mathrm{i}x\cdot\xi} u(x)\,dx$, which will be written as $\mathscr{F}_{x\to\xi} u(x,y)$ when u depends on further variables y. The value of $u \in \mathscr{S}'(\mathbb{R}^n)$ on the Schwartz function $\psi \in \mathscr{S}(\mathbb{R}^n)$ is denoted by $\langle u,\, \psi \rangle$.

As mentioned in the introduction the paper deals with operators given by

$$a(x,D)u = \mathrm{OP}(a)u(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\eta} a(x,\eta)\mathscr{F}u(\eta)\,d\eta, \qquad u \in \mathscr{S}(\mathbb{R}^n). \tag{27}$$

Hereby the symbol $a(x,\eta)$ is C^∞ on $\mathbb{R}^n \times \mathbb{R}^n$ and is taken to fulfil the Hörmander condition of order $d \in \mathbb{R}$, i.e. for all multiindices α, $\beta \in \mathbb{N}_0^n$ there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|D_\eta^\alpha D_x^\beta a(x,\eta)| \le C_{\alpha,\beta}(1+|\eta|)^{d-\rho|\alpha|+\delta|\beta|}. \tag{28}$$

The space of such symbols is denoted by $S^d_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ or $S^d_{\rho,\delta}$; and $S^{-\infty} := \bigcap S^d_{\rho,\delta}$.

The parameters ρ, $\delta \in [0,1]$ are mainly assumed to fulfil $\delta < \rho$, so that $a(x,D)$ by duality has a continuous extension $a(x,D)\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$. (Type 1,1-operators, i.e. $\delta = 1 = \rho$, are considered briefly in Section 6 below.) If desired the reader may specialise to the classical case $\rho = 1$, $\delta = 0$.

Together with $a(x,D)$ one has the distribution kernel $K(x,y) = \mathscr{F}^{-1}_{\eta\to z} a(x,\eta)\big|_{z=x-y}$ that in the usual way is seen to be C^∞ for $x \ne y$ (also for $a \in S^d_{1,1}$). It fulfils

$$\langle a(x,D)u,\, \varphi \rangle = \langle K,\, \varphi \otimes u \rangle \quad \text{for all} \quad u, \varphi \in \mathscr{S}. \tag{29}$$

As preparations, two special cases are considered: if $u = v + v'$ is any splitting of $u \in \mathscr{S} + \mathscr{F}^{-1}\mathscr{E}'$ with $v \in \mathscr{S}$ and $v' \in \mathscr{F}^{-1}\mathscr{E}'$ then

$$a(x,D)u = a(x,D)v + \mathrm{OP}(a(1\otimes\chi))v', \tag{30}$$

whereby $a(1\otimes\chi)(x,\eta) = a(x,\eta)\chi(\eta)$ and $\chi \in C_0^\infty(\mathbb{R}^n)$ is chosen so that $\chi = 1$ holds in a neighbourhood of $\operatorname{supp}\mathscr{F}v'$, or just on a neighbourhood of the smaller set

$$\bigcup_{x\in\mathbb{R}^n} \operatorname{supp} a(x,\cdot)\mathscr{F}v'(\cdot). \tag{31}$$

Indeed, by linearity on the left-hand side of (30) the identity results, for the term $a(x,D)v'$ equals $\mathrm{OP}(a(1\otimes\chi))v'$ if $v' \in \mathscr{F}^{-1}C_0^\infty(\mathbb{R}^n)$ that extends to $v' \in \mathscr{F}^{-1}\mathscr{E}'$ by mollification of $\mathscr{F}v'$ since $a(1\otimes\chi) \in S^{-\infty}$.

Moreover, for every auxiliary function $\psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighbourhood of the origin, continuity of the adjoint operation $a \mapsto e^{\mathrm{i}D_x\cdot D_\eta}\bar{a}$ yields

$$a(x,D)u = \lim_{m\to\infty} \mathrm{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u. \tag{32}$$

4. POINTWISE ESTIMATES

This section develops a flexible framework for discussion of pseudo-differential operators. These are only for convenience restricted to the classes recalled in Section 3.

4.1. **The factorisation inequality.** The simple result below introduces $u^*(x)$ as a fundamental tool for the proof of (5), hence of (1). It is therefore given as a theorem.

Formally the idea is to proceed as in Example 2.1, cf. (15), now departing from

$$a(x,D)u(x) = \int K(x,x-y)u(x-y)\,dy. \tag{33}$$

This leads to the *factorisation inequality* (34) below, where the dependence on $a(x,\eta)$ is taken out in the symbol factor F_a, also called the "a-factor". This is essentially a weighted L_1-norm of the distribution kernel. (The estimate shows that the case of an operator is not much worse than that of $\varphi * u$ in Example 2.1.)

Theorem 4.1. *Let $a \in S^d_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for $0 \le \delta < \rho \le 1$. When $u \in \mathscr{S}'(\mathbb{R}^n)$ with $\operatorname{supp}\hat{u} \subset \overline{B}(0,R)$, then one has the following pointwise estimate for all $x \in \mathbb{R}^n$:*

$$|a(x,D)u(x)| \le F_a(N,R;x) \cdot u^*(N,R;x). \tag{34}$$

Hereby u^* is as in (2) while F_a is bounded and continuous for $x \in \mathbb{R}^n$ and is given in terms of an auxiliary function $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on a neighbourhood of $\operatorname{supp}\hat{u}$ as

$$F_a(N,R;x) = \int_{\mathbb{R}^n} (1+R|y|)^N |\mathscr{F}^{-1}(a(x,\cdot)\chi(\cdot))|\,dy. \tag{35}$$

The inequality (34) holds for $N > 0$, and remains true if $\chi = 1$ on $\bigcup_{x\in\mathbb{R}^n} \operatorname{supp} a(x,\cdot)\hat{u}(\cdot)$. ∎

Proof. Using formula (30) with $v' = u$, and (31) for the last statement,

$$a(x,D)u(x) = \operatorname{OP}(a(1\otimes\chi))u = \langle u, \mathscr{F}_{\eta\to y}(\tfrac{e^{\mathrm{i}x\cdot\eta}}{(2\pi)^n} a(x,\eta)\chi(\eta))\rangle \tag{36}$$

for the last rewriting is evident from (27) if $u \in \mathscr{F}^{-1}C_0^\infty$ and follows for general $u \in \mathscr{F}^{-1}\mathscr{E}'$ by mollification of $\mathscr{F}u$, since $a(1\otimes\chi)$ is in $S^{-\infty}$.

Now $a(x,\eta)\chi(\eta)$ is in $C_0^\infty(\mathbb{R}^n)$ for fixed x, so $y \mapsto \mathscr{F}^{-1}_{\eta\to y}(a(1\otimes\chi))(x,x-y)$ decays rapidly while $u(y)$ grows polynomially by the Paley–Wiener–Schwartz theorem. Therefore the above scalar product on $\mathscr{S}' \times \mathscr{S}$ is an integral, so by the change of variables $y \mapsto x-y$,

$$\begin{aligned}
|a(x,D)u(x)| &= |\int u(x-y)\mathscr{F}^{-1}_{\eta\to y}(a(1\otimes\chi))(x,y)\,dy| \\
&\le \sup_{z\in\mathbb{R}^n} \frac{|u(x-z)|}{(1+R|z|)^N} \int (1+R|y|)^N |\mathscr{F}^{-1}_{\eta\to y}(a(1\otimes\chi))(x,y)|\,dy \\
&= u^*(x)F_a(x),
\end{aligned} \tag{37}$$

according to the definition of $u^*(x)$ in (2) and that of $F_a(x)$ in (35).

That $x \mapsto F_a(x)$ is bounded follows by insertion of $1 = (1+|y|^2)^{N'}(1+|y|^2)^{-N'}$ for $N' > (N+n)/2$ since $\mathscr{F}^{-1}_{\eta\to y}((1-\Delta_\eta)^{N'}[a(x,\eta)\chi(\eta)])$ is bounded with respect to (x,y) because of the compact suppport of χ. These estimates also yield continuity of the symbol factor $F_a(x)$. □

Disregarding the spectral radius R and N, (34) may be written concisely as

$$|a(x,D)u(x)| \le F_a(x) \cdot u^*(x). \tag{38}$$

It is noteworthy that the entire influence of the symbol lies in the a-factor $F_a(x)$, while u itself is mainly felt in $u^*(x)$. It is only in a vague way, i.e. through N and R, that u contributes to $F_a(x)$, so the factorisation inequality is rather convenient.

The theorem is also valid more generally; e.g. Section 6 gives an extension to symbols of type $1,1$ (extensions to other general symbols can undoubtedly be worked out when needed). To give a version for functions without compact spectrum, $\mathscr{O}_M(\mathbb{R}^n)$ will as usual stand for the space of slowly increasing functions, i.e. the $f \in C^\infty(\mathbb{R}^n)$ satisfying the estimates

$$|D^\alpha f(x)| \le C_\alpha (1+|x|)^{N_\alpha}. \tag{39}$$

Analogously to the argument after (8), $f^*(N,R;\cdot)$ is finite for $N \ge N_{(0,\dots,0)}$, any $R > 0$. There is a factorisation inequality for such functions, at the expense of a sum over its derivatives:

Theorem 4.2. *When $a(x,\eta)$ is in $S^d_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $1 \le \delta < \rho \le 1$, and $u \in \mathscr{O}_M(\mathbb{R}^n)$ while $N' > (d+n)/2$ is a non-negative integer, then one has for N, $R > 0$ that*

$$|a(x,D)u(x)| \le cF_a(N,R;x) \sum_{|\alpha| \le 2N'} (D^\alpha u)^*(N,R;x), \tag{40}$$

where F_a is defined by (35) for $\chi(\eta) = (1+|\eta|^2)^{-N'}$ and again is in $C(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$.

Proof. That F_a is in $C(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ can be seen as above, for $a(x,\eta)\chi(\eta) \in S^{d-2N'}_{\rho,\delta}$ is integrable with respect to η. When $a \in S^{-\infty}$ and $u \in \mathscr{S}$,

$$a(x,D)u(x) = \int (1-\Delta)^{N'} u(y) \mathscr{F}_{\eta \to y}\big(\tfrac{e^{\mathrm{i}x\cdot\eta}}{(2\pi)^n} a(x,\eta)\chi(\eta)\big)\, dy. \tag{41}$$

By continuity this extends to all $u \in \mathscr{S}'$, in particular to $u \in \mathscr{O}_M$; and since $S^{-\infty}$ is dense in $S^{d'}_{\rho,\delta}$ for $d' > d$, it extends then to all $a \in S^d_{\rho,\delta}$ since $(1+|y|)^{-N''}(1-\Delta)^{N'}u(y)$ is in L_1 for a large N''. In the same way as in (37) this yields

$$|a(x,D)u(x)| \le F_a(N,R;x)((1-\Delta)^{N'}u)^*(N,R;x). \tag{42}$$

Since $(1-\Delta)^{N'}u = \sum_{|\alpha|\le 2N'} c_{\alpha,N'} D^\alpha u$, subadditivity of the maximal operator gives the rest. □

As a first consequence of the factorisation inequalities, when $u \in \mathscr{O}_M$ then $a(x,D)u$ is of polynomial growth by (40), and continuous by (41); indeed, $F_a(x)$ is bounded and $D^\alpha u \in \mathscr{O}_M$ so $(D^\alpha u)^*(x)$ has such growth for N sufficiently large; cf. (7). Moreover, this applies to the commutator $[D^\beta, a(x,D)]$, say in the class $\mathrm{OP}(S^{d+|\beta|}_{\rho,\delta})$, so also $D^\beta a(x,D)u$ has polynomial growth. Which altogether proves

Corollary 4.1. *$a(x,D)$ is a map $\mathscr{O}_M(\mathbb{R}^n) \to \mathscr{O}_M(\mathbb{R}^n)$ when $a \in S^d_{\rho,\delta}$, $0 \le \delta < \rho \le 1$.*

While this is known for $\rho = 1$, $\delta = 0$ from e.g. [24, Cor. 3.8], the above version for the general case $0 \le \delta < \rho \le 1$ is rather more direct.

Secondly, one may now obtain the L_p-estimate mentioned in the introduction.

Corollary 4.2. *For each $a \in S^d_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \le \delta < \rho \le 1$, and $p \in \,]0,\infty]$ there is to every $R \ge 1$, $N > n/p$ a constant $C(N,R)$ such that*

$$\|a(x,D)u\|_p \le C(N,R)\|u\|_p \tag{43}$$

whenever $u \in L_p(\mathbb{R}^n) \bigcap \mathscr{S}'(\mathbb{R}^n)$, fulfils $\operatorname{supp}\hat{u} \subset \overline{B}(0,R)$.

Proof. By taking L_p-norms on both sides of the factorisation inequality, (43) results with $C(N,R) = C'_{n,N,p} \sup_x |F_a(N;R;x)|$, cf. Theorem 2.1; this is finite according to Theorem 4.1. □

Since the spectral condition on u implies $u \in C^\infty$, it is hardly surprising that the above L_p-result is valid for arbitrary orders $d \in \mathbb{R}$. In fact it may, say for $1 < p < \infty$, $(\rho,\delta) = (1,0)$, be proved simply by observing that $a(x,D)$ has the same action on u as some $b(x,D) \in \mathrm{OP}(S^{-\infty})$ so that boundedness of $b(x,D)$ on L_p gives the rest.

It is noteworthy, however, that the existing proofs of L_p-boundedness use fundamental parts of real analysis, e.g. Marcinkiewicz interpolation and the Calderon–Zygmund lemma. In contrast to this, pointwise estimates lead straightforwardly to Corollary 4.2. This evident efficacy is also clear from the easy extension to the full range $0 < p \le \infty$ and to type 1,1-operators in Section 6.

4.2. **Estimates of the symbol factor.** To utilise Theorem 4.1 it is of course vital to control F_a. This leads directly to integral conditions on a, similarly to the Mihlin–Hörmander theorem.

Theorem 4.3. *Assume $a(x,\eta)$ is in $S^d_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$, $0 \le \delta < \rho \le 1$, and let $F_a(N,R;x)$ be given by* (35) *for parameters $R,N > 0$, whereby the auxiliary function is taken as $\chi = \psi(R^{-1}\cdot)$ for $\psi \in C_0^\infty(\mathbb{R}^n)$ equalling* 1 *in (the closure of) an open set. Then*

$$0 \le F_a(x) \le c_{n,k} \sum_{|\alpha|\le k} \Big(\int_{R\,\mathrm{supp}\,\psi} |R^{|\alpha|} D^\alpha_\eta a(x,\eta)|^2 \frac{d\eta}{R^n}\Big)^{1/2} \tag{44}$$

for all $x \in \mathbb{R}^n$, when k is the least integer satisfying $k > N + n/2$.

First it is convenient to recall that, for $z \in \mathbb{R}^n$ and $k \in \mathbb{N}$, an expansion yields

$$(1+|z|)^k \le \sum_{j=0}^k \binom{k}{j} (|z_1| + \dots + |z_n|)^j = \sum_{|\alpha|\le k} C_{k,\alpha}|z^\alpha|. \tag{45}$$

Proof. The idea is to pass to the L_2-norm in (35) using Cauchy–Schwarz' inequality and that $(\int R^n(1+|Ry|)^{-n-\varepsilon}\,dy)^{1/2} < \infty$ for $\varepsilon > 0$. Thus, if ε is so small that $k \ge N + (n+\varepsilon)/2$,

$$F_a(N,R;x) \le c_n R^{-n/2}\Big(\int (1+|Ry|)^{2k} |\mathscr{F}^{-1}_{\eta\to y}[a(x,\cdot)\psi(R^{-1}\cdot)]|^2\,dy\Big)^{1/2}. \tag{46}$$

Applying (45) to $z = Ry$ and 'commuting' the resulting polynomials $(Ry)^\alpha$ with the inverse Fourier transformation, it is seen that for fixed $x \in \mathbb{R}^n$,

$$\begin{aligned} F_a(x) &\le c_n R^{-n/2} \sum_{|\alpha|\le k} C_{k,\alpha}\Big(\int |\mathscr{F}^{-1}_{\eta\to y}[(\mathrm{i}R\partial_\eta)^\alpha a(1\otimes\psi(R^{-1}\cdot))](x,y)|^2\,dy\Big)^{1/2} \\ &\le c \sum_{|\alpha+\beta|\le k} \Big(\int_{R\,\mathrm{supp}\,\psi} (R^{|\alpha+\beta|}|D^\alpha_\eta a(x,\eta)||D^\beta(\psi(R^{-1}\eta))|)^2 \frac{d\eta}{R^n}\Big)^{1/2}. \end{aligned} \tag{47}$$

Since $D^\beta(\psi(R^{-1}\cdot)) = R^{-|\beta|}(D^\beta\psi)(R^{-1}\cdot)$ is bounded, the result follows. □

Remark 4.1. As an alternative to the estimate $|a(x,D)u(x)| \le F_a(x)u^*(x)$, it deserves to be mentioned that other useful properties can be obtained in a similar fashion: by defining an a-factor in terms of an L_2-norm, i.e.

$$\tilde F_a(N,R;x)^2 = \int_{\mathbb{R}^n} (1+|Ry|)^{2N} |\mathscr{F}^{-1}_{\eta\to y}(a(x,\cdot)\chi(\cdot))|^2\,dy, \tag{48}$$

the Cauchy–Schwarz inequality gives

$$|a(x,D)u(x)| \le \tilde F_a(N,R;x)\Big(\int_{\mathbb{R}^n} \frac{|u(x-y)|^2}{(1+|Ry|)^{2N}}\,dy\Big)^{1/2} \le c\tilde F_a(N,R;x)u^*(\varepsilon,R;x) \tag{49}$$

where $c = (\int(1+|Ry|)^{-2(N-\varepsilon)}\,dy)^{1/2} < \infty$ whenever $N > n/2+\varepsilon$ for some $\varepsilon > 0$.

For one thing $\tilde{F}_a^2 \in C^\infty(\mathbb{R}^n)$, with bounded derivatives of any order. Secondly, this gives a version of Theorem 4.3 where only estimates with $|\alpha| \le [n/2]+1$ is required, as in the Mihlin–Hörmander theorem. But it would not be feasible in general to replace $u^*(N,R;x)$ by $u^*(\varepsilon,R;x)$ for small ε as above, so $\tilde{F}_a(x)$ is only mentioned in this remark.

Although it is a well-known exercise to control (44) in terms of symbol seminorms, it is important to control the behaviour with respect to R and to verify that it improves when $a(x,\cdot)\hat{u}(\cdot)$ is supported in a corona. Therefore the special case in (51) below is included:

Corollary 4.3. *Assume $a \in S^d_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$, $0 \le \delta < 1$, and let N, R and ψ have the same meaning as in Theorem 4.3. When $R \ge 1$ and $k > N+n/2$, $k \in \mathbb{N}$, then there is a seminorm p on $S^d_{1,\delta}$ and some $c_k > 0$ independent of R such that*

$$0 \le F_a(x) \le c_k p(a) R^{\max(d,k)} \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{50}$$

Moreover, if supp ψ *is contained in a corona*

$$\{\, \eta \mid \theta_0 \le |\eta| \le \Theta_0 \,\}, \tag{51}$$

and $\psi(\eta) = 1$ holds for $\theta_1 \le |\eta| \le \Theta_1$, whereby $0 \ne \theta_0 < \theta_1 < \Theta_1 < \Theta_0$, then

$$0 \le F_a(x) \le c'_k R^d p(a) \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{52}$$

with $c'_k = c_k \max(1, \theta_0^{d-k}, \theta_0^d)$.

Remark 4.2. For general $\rho \in]0,1]$, the asymptotics of $F_a(x)$ for $R \to \infty$ corresponding to (50), (52) will be $\mathscr{O}(R^{\max(d+(1-\rho)k,k)})$ and $\mathscr{O}(R^{d+(1-\rho)k})$, respectively. Details are left out for simplicity's sake.

Proof. Setting $p_{\alpha,\beta}(a) = \sup(1+|\eta|)^{-d+|\alpha|-\delta|\beta|}|D^\beta_x D^\alpha_\eta a(x,\eta)|$ and continuing from the proof of Theorem 4.3, the change of variables $\eta = R\zeta$ gives

$$\begin{aligned} F_a(x) &\le c \sum_{|\alpha|\le k} p_{\alpha,0}(a) \Big(\int_{\operatorname{supp}\psi} |(1+|R\zeta|)^{d-|\alpha|} R^{|\alpha|}|^2 \, d\zeta \Big)^{\frac12} \\ &\le C'_{n,k} R^{\max(d,k)} \sum_{|\alpha|\le k} p_{\alpha,0}(a). \end{aligned} \tag{53}$$

In fact $d \ge k \ge |\alpha|$ gives $R^{|\alpha|}(1+R|\zeta|)^{d-|\alpha|} \le R^d(1+|\zeta|)^{d-|\alpha|}$ since $R \ge 1$; for $d < k$ the crude estimate $R^{|\alpha|}(1+R|\zeta|)^{d-|\alpha|} \le R^k$ applies e.g. for $|\alpha| = k$. This shows (50).

In case ψ is supported in a corona as described, $d-|\alpha| < 0$ and $\zeta \in \operatorname{supp}\psi$ entail

$$(1+|R\zeta|)^{d-|\alpha|}R^{|\alpha|} \le (R\theta_0)^{d-|\alpha|}R^{|\alpha|} \le \max(\theta_0^{d-k}, \theta_0^d) R^d. \tag{54}$$

This yields an improvement of (53) for terms with $|\alpha| > d$; thence (52). ☐

As desired Corollary 4.3 shows that the a-factor $F_a(x)$ has its sup-norm bounded by a symbol seminorm. This applies of course in $|a(x,D)u(x)| \le F_a(x)u^*(x)$.

In this connection, one could simply take R equal to the spectral radius of u, or if possible R so large that the corona $\{\, \eta \mid \theta_1 R \le |\eta| \le \Theta_1 R \,\}$ is a neighbourhood of $\operatorname{supp} a(x,\cdot)\hat{u}(\cdot)$ for all x; cf (50) and (52).

However, a good choice of N is a more delicate question, which in general involves the order of $\mathscr{F}u$ as a distribution. E.g. $N \ge \operatorname{order}(\mathscr{F}u)$ was seen in Section 2 to imply that $u^*(N,R;x)$ is finite everywhere. This was relaxed completely to $N > 0$ for

$u \in L_p \cap \mathscr{F}^{-1}\mathscr{E}'$ in Lemma 2.1; moreover, for arbitrary $u \in L_p$ with $1 \le p \le 2$, the order of $\mathscr{F}u$ is 0, so u^* is finite regardless of $N > 0$.

Especially for functions u in Sobolev spaces H^s the function u^* is always finite for $N > 0$. Therefore it is harmless that the estimates in Corollary 4.3 depend on N, for only seminorms $p_{\alpha,0}(a)$ with $|\alpha| \le 1 + [n/2 + N]$ enters there, and by taking $0 < N < 1/2$ in both odd and even dimensions estimates of $a(x,D)u(x)$ with $u \in \bigcup H^s$ only requires the well-known estimates of $D^\alpha_\eta a(x,\eta)$ for $|\alpha| \le [n/2]+1$.

However, in connection with L_p-bounds of $u^*(x)$, one is often forced to take $N > n/p$ in the L_p-estimates of $a(x,D)u$; cf. Theorem 2.1.

In addition to high frequencies removed by the spectral cut-off function χ in Theorem 4.1, the symbols dependence on x may be frequency modulated by means of a Fourier multiplier $\varphi(Q^{-1}D_x)$, which depends on a second spectral quantity $Q > 0$. For the modified symbol

$$a_Q(x,\eta) = \varphi(Q^{-1}D_x)a(x,\eta) \tag{55}$$

and the corresponding symbol factor one can as shown below find its asymptotics for $Q \to \infty$. In Littlewood–Paley theory, this is a frequently asked question for F_{a_Q}:

Corollary 4.4. *When $a \in S^d_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \le \delta < 1$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi = 0$ in a neighbourhood of $\xi = 0$, then there is a seminorm p on $S^d_{1,\delta}$ and constants c_M, depending only on M, n, N, ψ and φ, such that for $R \ge 1$, $M > 0$, $Q > 0$,*

$$0 \le F_{a_Q}(N,R;x) \le c_M p(a) Q^{-M} R^{\max(d+\delta M,[N+n/2]+1)}. \tag{56}$$

Here $d + \delta M$ can replace the maximum when the auxiliary function ψ in F_{a_Q} fulfils the corona condition in Corollary 4.3.

Proof. Because $a_Q(x,\eta) = \int Q^n \check{\varphi}(Qz)a(x-z,\eta)\,dz$, where $\check{\varphi}$ has vanishing moments of every order, Taylor's formula with remainder gives for any $M \in \mathbb{N}$

$$a_Q(x,\eta) = \sum_{|\beta|=M} \tfrac{M}{\beta!} \int (-z)^\beta Q^n \check{\varphi}(Qz) \int_0^1 (1-\tau)^{M-1} \partial_x^\beta a(x-\tau z,\eta)\,d\tau\,dz. \tag{57}$$

Letting z^β absorb Q^M before substitution of z by z/Q, one finds

$$\begin{aligned} Q^M F_{a_Q}(N,R;x) \le \sum_{|\beta|=M} \tfrac{M}{\beta!} \iiint &(1-\tau)^{M-1}(1+|z|)^M |\check{\varphi}(z)|(1+|Ry|)^N \\ &\times |\mathscr{F}^{-1}_{\eta\to y}(\partial_x^\beta a(x-\tau z/Q,\eta)\psi(\eta/R))|\,d\tau\,dz\,dy. \end{aligned} \tag{58}$$

Integrating first with respect to y it follows by applying Corollary 4.3 to $\partial_x^\beta a \in S^{d+\delta M}_{1,\delta}$ that, by setting $p(a) = \sum p_{\alpha,\beta}(a)$ where $|\alpha| \le [N+n/2]+1$ and $|\beta| = M$,

$$F_{a_Q}(N,R;x) \le c_M p(a) R^{d+\delta M} Q^{-M}. \tag{59}$$

This is in case ψ satisfies the corona condition. Otherwise the stated inequality (56) results. □

Remark 4.3. In comparison with Remark 4.2, the asymptotics for $R \to \infty$ are here $\mathscr{O}(R^{\max(d+\delta M+(1-\rho)k,k)})$ and $\mathscr{O}(R^{d+\delta M+(1-\rho)k})$, respectively, for $k = [N+n/2]+1$.

Remark 4.4. As an alternative to the techniques in this section, Marschall's inequality gives a pointwise estimate for symbols $b(x,\eta)$ in $L_{1,\mathrm{loc}}(\mathbb{R}^{2n})\cap\mathscr{S}'(\mathbb{R}^{2n})$ with support in $\mathbb{R}^n\times\overline{B}(0,2^k)$ and $\operatorname{supp}\mathscr{F}u\subset\overline{B}(0,2^k)$, $k\in\mathbb{N}$:

$$|b(x,D)v(x)|\le c\big\|b(x,2^k\cdot)\big\|_{\dot{B}^{n/t}_{1,t}}M_tu(x),\qquad 0<t\le 1. \tag{60}$$

This goes back to [16, p.37] and was exploited in e.g. [17, 18, 19]. In the above form it was proved in [12] under the natural condition that the right-hand side is in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$. While M_tu is as in Remark 2.1, the norm of the homogeneous Besov space $\dot{B}^{n/t}_{1,t}$ on the symbol is defined analogously to that of $B^s_{p,q}$ in (75) below in terms of a partition of unity, though here with $1=\sum_{j=-\infty}^{\infty}(\varphi(2^{-j}\eta)-\varphi(2^{1-j}\eta))$, $\eta\ne 0$ so that (75) should be read with ℓ_q over $\mathbb{Z}$. This yields the well-known dyadic scaling property that

$$\big\|b(x,2^k\cdot)\big\|_{\dot{B}^{n/t}_{1,t}}=2^{k(\frac{n}{t}-n)}\big\|b(x,\cdot)\big\|_{\dot{B}^{n/t}_{1,t}}. \tag{61}$$

While this can be useful, and indeed fits well into the framework of the next section, cf. [17, 18, 19], it is often simpler to use the factorisation inequality with F_b and u^* etc.

5. Littlewood–Paley analysis

In order to obtain L_p-estimates, it is convenient to depart from the limit in (32). As usual the test function ψ there gives rise to a Littlewood–Paley decomposition $1=\psi(\eta)+\sum_{j=1}^{\infty}\varphi(2^{-j}\eta)$ by setting $\varphi=\psi-\psi(2\cdot)$. Note here that if $\psi\equiv 1$ for $|\eta|\le r$ while $\psi\equiv 0$ for $|\eta|\ge R$, one can fix an integer $h\ge 2$ so that $2R<r2^h$.

Inserting twice into (32) that $\psi(2^{-m}\eta)=\psi(\eta)+\varphi(2^{-1}\eta)+\dots+\varphi(2^{-m}\eta)$, the so-called paradifferential splitting from the 1980's is recovered: whenever $a(x,\eta)$ is in $S^d_{\rho,\delta}$, $0\le\delta<\rho\le 1$, and $u\in\mathscr{S}'(\mathbb{R}^n)$,

$$a_\psi(x,D)u=a^{(1)}_\psi(x,D)u+a^{(2)}_\psi(x,D)u+a^{(3)}_\psi(x,D)u, \tag{62}$$

whereby the expressions are given by the three series below (they converge in $\mathscr{S}'$),

$$a^{(1)}_\psi(x,D)u=\sum_{k=h}^{\infty}\sum_{j\le k-h}a_j(x,D)u_k=\sum_{k=h}^{\infty}a^{k-h}(x,D)u_k \tag{63}$$

$$\begin{aligned}a^{(2)}_\psi(x,D)u=\sum_{k=0}^{\infty}\big(&a_{k-h+1}(x,D)u_k+\dots+a_{k-1}(x,D)u_k+a_k(x,D)u_k\\&+a_k(x,D)u_{k-1}+\dots+a_k(x,D)u_{k-h+1}\big)\end{aligned} \tag{64}$$

$$a^{(3)}_\psi(x,D)u=\sum_{j=h}^{\infty}\sum_{k\le j-h}a_j(x,D)u_k=\sum_{j=h}^{\infty}a_j(x,D)u^{j-h}. \tag{65}$$

Here $u_k=\varphi(2^{-k}D)u$ while $a_k(x,\eta)=\varphi(2^{-k}D_x)a(x,\eta)$; by convention φ is replaced by ψ for $k=0$ and $u_k\equiv 0\equiv a_k$ for $k<0$. In addition superscripts are used for the convenient shorthands u^{k-h} and $a^{k-h}(x,D)$; e.g. the latter is given by $a^{k-h}(x,D)=\sum_{j\le k-h}a_j(x,D)=\mathrm{OP}(\psi(2^{h-k}D_x)a(x,\eta))$. Using this, there is a brief version of (64),

$$a^{(2)}_\psi(x,D)u=\sum_{k=0}^{\infty}((a^k-a^{k-h})(x,D)u_k+a_k(x,D)(u^{k-1}-u^{k-h})). \tag{66}$$

The main point here is that the series in (63)–(66) are easily treated with the tools of the present paper. First of all, one has the following inclusions for the spectra of the summands in (63), (65) and (66), with $R_h = \frac{r}{2} - R2^{-h}$:

$$\operatorname{supp}\mathscr{F}(a^{k-h}(x,D)u_k) \subset \{\, \xi \mid R_h 2^k \le |\xi| \le \tfrac{5R}{4}2^k \,\}, \tag{67}$$

$$\operatorname{supp}\mathscr{F}(a_k(x,D)u^{k-h}) \subset \{\, \xi \mid R_h 2^k \le |\xi| \le \tfrac{5R}{4}2^k \,\}, \tag{68}$$

$$\operatorname{supp}\mathscr{F}\big(a_k(x,D)(u^{k-1}-u^{k-h}) + (a^k - a^{k-h})(x,D)u_k\big) \subset \{\, \xi \mid |\xi| \le 2R2^k \,\}. \tag{69}$$

Such spectral corona and ball properties have been known since the 1980's (e.g. [29, (5.3)]) although they were verified then only for elementary symbols $a(x,\eta)$, in the sense of Coifman and Meyer [5]. However, this is now a redundant restriction because of the *spectral support rule*, which for $u \in \mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^n)$ states that

$$\operatorname{supp}\mathscr{F}(a(x,D)u) \subset \{\, \xi + \eta \mid (\xi,\eta) \in \operatorname{supp}\mathscr{F}_{x\to\xi}\, a,\ \eta \in \operatorname{supp}\mathscr{F}u \,\}, \tag{70}$$

A short proof of this can be found in [14, App. B] (cf. [12, 13, 14] for the full version). Since (67)–(69) follow easily from (70), as shown in [12, 14], details are omitted.

The novelty in relation to pointwise estimates is that the summands in the decomposition (63)–(65) can be controlled thus: for $a^{(1)}_\psi(x,D)u$ the fact that $k \ge h \ge 2$ allows the corona condition of Corollary 4.3 to be fulfilled for $\Theta_0 = r/2$ and $\Theta_1 = R$ (i.e. the auxiliary function there is 1 on $\operatorname{supp}\hat{u}$), so (63) and the factorisation inequality simply give the first estimate:

$$|a^{k-h}(x,D)u_k(x)| \le F_{a^{k-h}}(N,R2^k;x)u_k^*(N,R2^k;x) \le cp(a)(R2^k)^d u_k^*(x). \tag{71}$$

Hereby the convolution estimate $p(a^{k-h}) \le \|\mathscr{F}^{-1}\psi\|_1 p(a)$ is utilised to get a constant independent of k.

In $a^{(2)}_\psi(x,D)u$ the terms may be treated similarly: in (66) it is for $k \ge 1$ clear that $(a^k - a^{k-h})(x,D)u_k$ only requires the constant to have $\|\mathscr{F}^{-1}(\psi - \psi(2^h\cdot))\|_1$ as a factor instead of $\|\mathscr{F}^{-1}\psi\|_1$, cf. the above; while for $k=0$ it may just be increased by a fixed power of R using the full generality of Corollary 4.3. The remainders in (66) have $k>0$ and can be written as in (64). Hence one obtains the second estimate:

$$\begin{aligned}
&|(a^k - a^{k-h})(x,D)u_k(x) + a_k(x,D)[u^{k-1} - u^{k-h}](x)| \\
&\quad \le F_{a^k - a^{k-h}}(N,R2^k;x)u_k^*(N,R2^k;x) + \sum_{l=1}^{h-1} F_{a_k}(N,R2^{k-l},x)u_{k-l}^*(N,R2^{k-l};x) \\
&\quad \le cp(a)(R2^k)^d \sum_{l=0}^{h-1} 2^{-ld} u_{k-l}^*(N,R2^{k-l};x).
\end{aligned} \tag{72}$$

Here the sum over l is harmless, because the number of terms is independent of k.

The improved asymptotics of Corollary 4.4 come into play as reinforcements for the series for $a^{(3)}_\psi(x,D)u$. Indeed, for $Q = 2^j$ the first part of (65) gives, for $M>0$, the third estimate

$$\begin{aligned}
|a_j(x,D)u^{j-h}(x)| &\le \sum_{k=0}^{j-h} |a_j(x,D)u_k(x)| \le \sum_{k=0}^{j} F_{a_j}(N,R2^k;x)u_k^*(N,R2^k;x) \\
&\le c'_M p(a) 2^{-jM} \sum_{k=0}^{j} (R2^k)^{d+\delta M} u_k^*(N,R2^k;x).
\end{aligned} \tag{73}$$

Here the number of terms on the right-hand side depends on j, but this is manageable due to 2^{-jM}, which serves as a summation factor. Altogether this proves

Theorem 5.1. *For each symbol a in* $S^d_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, *the paradifferential decomposition* (62) *is valid with the terms in* (63)–(65) *having the spectral relations* (67),(68), (69) *and the pointwise estimates* (71), (72), (73).

Not surprisingly, Theorem 5.1 yields boundedness in several scales. Perhaps this is most transparent for the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$. These generalise both the Sobolev spaces $H^s(\mathbb{R}^n)$ and the Hölder spaces $C^s(\mathbb{R}^n)$ (with $0 < s < 1$, cf Proposition 2.1) as

$$H^s = B^s_{2,2}, \qquad C^s = B^s_{\infty,\infty}. \tag{74}$$

The spaces $B^s_{p,q}$ are for $s \in \mathbb{R}$, $p,q \in]0,\infty]$ defined by means of the Littlewood–Paley decomposition as the $u \in \mathscr{S}'$ for which the following (quasi-)norm is finite,

$$\|u\|_{B^s_{p,q}} = (\sum_{j=0}^{\infty} 2^{sjq} \|\varphi(2^{-j}D)u\|_p^q)^{1/q}; \tag{75}$$

hereby the norm in ℓ_q should be read as the supremum over j for $q = \infty$. (Often a specific choice of the function ψ is stipulated, but this is immaterial as they all lead to equivalent norms on the spaces). For $p,q \in [1,\infty]$ the space $B^s_{p,q}$ is a Banach space. Note that the first part of (74) follows easily from (75); cf. [11, 27] for the second.

Now Theorem 5.1 gives the following continuity result:

Theorem 5.2. *When* $a(x,\eta)$ *belongs to* $S^d_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $0 \le \delta < 1$, *then*

$$a(x,D)\colon H^{s+d}(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \tag{76}$$

$$a(x,D)\colon B^{s+d}_{p,q}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^n) \tag{77}$$

is continuous for every $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$.

Proof. Taking L_p- and ℓ_q-norms on both sides of (71), Theorem 2.1 gives for $N > n/p$,

$$(\sum_{k=0}^{\infty} 2^{skq} \|a^{k-h}(x,D)u_k\|_p^q)^{1/q} \le cp(a)(\sum_{k=0}^{\infty} 2^{(s+d)kq} \|u_k\|_p^q)^{1/q} = cp(a)\|u\|_{B^{s+d}_{p,q}}. \tag{78}$$

Because of the dyadic corona property (67), the above estimate implies convergence of $a^{(1)}_\psi(x,D)u = \sum a^{k-h}(x,D)u_k$ to an element in $B^s_{p,q}$, the norm of which is estimated by the right-hand side (this is well known, cf. [29],[23, 2.3.2] or [14]). So for $m = 1$,

$$\|a^{(m)}_\psi(x,D)u\|_{B^s_{p,q}} \le c'' p(a)\|u\|_{B^{s+d}_{p,q}}. \tag{79}$$

The contribution $a^{(3)}(x,D)$ in (62) is treated similarly, except for the sum over k. This is handled with a small lemma, namely $\sum_{j=0}^{\infty} 2^{sjq}(\sum_{k=0}^{j} |b_k|)^q \le c\sum_{j=0}^{\infty} 2^{sjq}|b_j|^q$, valid for all $b_j \in \mathbb{C}$ and $0 < q \le \infty$ provided $s < 0$; cf. [29]. Thus (73) implies

$$\begin{aligned}\sum_{j=0}^{\infty} 2^{sjq} \|a_j(x,D)u^{j-h}\|_p^q &\le \sum_{j=0}^{\infty} 2^{(s-M)jq} (\sum_{k=0}^{j} cp(a) 2^{k(d+\delta M)} \|u_k^*(N,R2^k;\cdot)\|_p)^q \\ &\le cp(a)^q \sum_{j=0}^{\infty} 2^{(s+d-(1-\delta)M)jq} \|u_j\|_p^q \\ &= cp(a)^q \|u\|_{B^{s+d-(1-\delta)M}_{p,q}}\end{aligned} \tag{80}$$

provided $M > 0$ and $M > s$. This implies (79) for $m = 3$.

For $a^{(2)}(x,D)u$ the estimate is a little simpler, for in (72) one only needs to apply norms of L_p and ℓ_q with respect to x and k, respectively, and use the (quasi-)triangle inequality. Because (69) is a dyadic ball property, the resulting estimate gives (79) with $m = 2$ only in case $s > \max(0, \frac{n}{p} - n)$. But then, via (62), this shows (77).

However, one can reduce to such s by writing $a(x,D) = \Lambda^t(\Lambda^{-t}a(x,D))$ with $t = 2|s|+1$ (or $t = 2|s|+1+\frac{n}{p}-n$ if $0 < p < 1$), for $\Lambda^t = \mathrm{OP}((1+|\eta|^2)^{t/2})$ is of order t in the $B^s_{p,q}$-scale. This shows (77), hence (76) as a special case. □

Theorem 5.2 is well established, of course. For example [8, Thm. 18.1.13] or [24, Thm. 3.6] gives the H^s-part with a classical reduction to Schur's lemma. The present proof should be interesting because it combines Littlewood–Paley theory with the factorisation inequality etc.

The flexibility of this method is apparent from the fact that it extends *at once* to the $B^s_{p,q}$ with arbitrary $p,q \in]0,\infty]$. The previous proofs for $B^s_{p,q}$ in e.g. [1, 29] are cumbersome due to the use of elementary symbols and multiplier results.

6. The case of type 1,1-operators

The above results carry over to type $1,1$-operators, ie to $\delta = 1$ with almost no changes. Previously type $1,1$-operators have been treated in fundamental contributions of Ching [4], Stein 1972 (cf. [25]), Parenti and Rodino [21], Meyer [20], Bourdaud [2, 3], Hörmander [9, 10, 11] and Torres [26].

The present methods were in fact developed for such operators, which emphasizes the efficacy of pointwise estimates. But since the topic is specialised, only brief remarks on the outcome will be given here.

The reader may consult [13, 14] for a review of *operators* of type $1,1$ and a systematic treatment. Here it suffices to recall from [13] that for $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ the identity (32) is used as the *definition:* when the limit there exists in $\mathscr{D}'(\mathbb{R}^n)$ and is independent of ψ, then u belongs to the domain $\in D(a(x,D))$ and the action of $a(x,D)$ on u is set equal to the limit in (30); cf. [13].

For example, if u is in $\mathscr{S} + \mathscr{F}^{-1}\mathscr{E}'$ the limit in (32) exists and equals the right-hand side of (30). Since the latter does not depend on ψ, nor on v, v', one has by definition that $\mathscr{S} + \mathscr{F}^{-1}\mathscr{E}' \subset D(a(x,D))$, and (30) holds. Cf. [13, Cor. 4.7].

Therefore the proof of Theorem 4.1, which departs from (30), can be repeated for type $1,1$-operators:

Theorem 6.1. *The factorisation inequality $a(x,D)u(x) \le F_a(x)u^*(x)$ is valid* verbatim *for symbols $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$; cf. Theorem 4.1.*

That $a(x,D)\colon \mathscr{O}_M \to \mathscr{O}_M$ is also true for type $1,1$-operators, but the proof of Corollary 4.1 needs to be changed to obtain the decisive inclusion $\mathscr{O}_M \subset D(a(x,D))$ (cf. [14] for more details on this).

However, the proof of Corollary 4.2 gives without changes

Theorem 6.2. *For $a \in S^d_{1,1}$, $d \in \mathbb{R}$, $p \in]0,\infty]$, $R \ge 1$ and $N > n/p$ one has*

$$\|a(x,D)u\|_p \le C(N,R)\|u\|_p \tag{81}$$

whenever $u \in L_p(\mathbb{R}^n) \bigcap \mathscr{S}'(\mathbb{R}^n)$ fulfils $\operatorname{supp}\hat{u} \subset \overline{B}(0,R)$.

This result is a novelty in the type $1,1$-context. It is noteworthy because some operators in $\mathrm{OP}(S^0_{1,1})$ are unbounded on L_p, even for $p = 2$, by a construction of Ching [4] — and therefore pointwise estimates seem indispensable for Theorem 6.2.

It is also straightforward to see that one has

Theorem 6.3. *For the symbol factor $F_a(x)$, the estimates by integrals in Theorem 4.3 are valid* verbatim *for $\delta = 1 = \rho$. Similarly the estimates by symbol seminorms in Corollaries 4.3 and 4.4 carry over. In particular the corona condition yields*

$$F_a(N,R;x) = \mathscr{O}(R^d), \qquad F_{a_Q}(N,R;x) = \mathscr{O}(Q^{-M}R^{d+M}) \tag{82}$$

for $R \to \infty$ and $Q \to \infty$ (fixed R), respectively.

Moreover, the paradifferential decomposition in Section 5 is unchanged, although for type $1,1$-operators it has to be made for arbitrary ψ because of their definition.

As a difference it holds for type $1,1$-operators that the series for $a^{(2)}(x,D)u$ in (64) converges if and only if $u \in D(a(x,D))$. This results at once from the fact that the series for $a^{(1)}(x,D)u$ and $a^{(3)}(x,D)u$ in (63), (65) converge for every $u \in \mathscr{S}'$, which was proved in [14] by combining a lemma of Coifman and Meyer [6, Ch. 15] with the pointwise estimates summed up in Theorem 6.3.

Theorem 5.1 also carries over to arbitrary type $1,1$-operators, whence the boundedness in Theorem 5.2 does so for large s:

Theorem 6.4. *For each a in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ the operator is continuous ($p,q \in]0,\infty]$)*

$$a(x,D)\colon H^{s+d}(\mathbb{R}^n) \to H^s(\mathbb{R}^n), \quad \text{for } s > 0, \tag{83}$$

$$a(x,D)\colon B^{s+d}_{p,q}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^n), \quad \text{for } s > \max(0, \tfrac{n}{p} - n). \tag{84}$$

The proof is the same, except that the lift operator Λ^t is redundant. The boundedness was essentially shown in [12], though the formal definition of type $1,1$-operators first appeared in [13].

However, Hörmander's condition in [9, 10, 11], that $\mathscr{F}_{x\to\xi}a(x,\eta)$ be small along the twisted diagonal $\xi = -\eta$, allows the conditions on s in (83) and (84) to be removed.

Indeed, in terms of a specific localisation to the twisted diagonal, namely the symbol $a_{\chi,\varepsilon}(x,\eta) = \mathscr{F}^{-1}_{x\to\xi}(\chi(\xi+\eta,\varepsilon\eta)\mathscr{F}_{x\to\xi}a(x,\eta))$ defined in [9, 10, 11] for a suitable $\chi \in C^\infty$ supported where $|\eta| > |\xi|$, $|\eta| > 1$, Hörmander introduced the fundamental condition that for every $\sigma > 0$ there is an estimate for $\varepsilon > 0$:

$$\sup_{R>0,\, x\in\mathbb{R}^n} R^{-d}\Big(\int_{R\le|\eta|\le 2R} |R^{|\alpha|}D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|^2 \frac{d\eta}{R^n}\Big)^{1/2} \le c_{\alpha,\sigma}\varepsilon^{\sigma+n/2-|\alpha|}. \tag{85}$$

This is first of all interesting because of the obvious similarity with the Mihlin–Hörmander type estimates of the symbol factor in Theorems 4.3 and 6.3.

Secondly, for $a(x,\eta)$ fulfilling (85) the conditions on s in Theorem 6.4 were removed in [14] (with an arbitrarily small loss of smoothness for $p < 1$). The proof consisted in a refinement of that of Theorem 6.4, in which the necessary improvements for $a^{(2)}(x,D)u$ were obtained by skipping (82) and controlling the F_a-estimates of Mihlin-Hörmander type directly in terms of Hörmander's condition (85).

Furthermore, Theorem 6.3 was used in [14] to bridge the gap between the general Littlewood–paley theory, i.e. the paradifferential splitting (62), and symbols fulfilling (85), again by controlling the Mihlin–Hörmander type estimates in terms of (85). Thus it was proved explicitly in [14] that the series for $a^{(2)}(x,D)u$, for $u \in \mathscr{S}'$, converges in the topology of $\mathscr{S}'(\mathbb{R}^n)$ whenever (85) holds.

In this connection, it deserves to be mentioned that the approach of Marschall, recalled in Remark 4.4, would be insufficient, since application of (60) to the terms in

$a^{(2)}(x,D)u$ would result in estimates involving Mu_k: for general $u \in \mathscr{S}'$ the Hardy–Littlewood maximal function Mu_k would not be finite, unlike $u_k^*(N,R;x)$ that is so for all sufficiently large N.

The boundedness results extend to the Sobolev spaces $H_p^s = \Lambda^{-s}L_p$, with $s \in \mathbb{R}$ and $1 < p < \infty$, cf. [13], or more generally to the Lizorkin–Triebel scale $F_{p,q}^s$ with $0 < p < \infty$, $0 < q \leq \infty$; cf. [14]. The proofs follow the lines indicated above.

It would be outside of the topic here to give the full statements, so the reader is referred to [13, 14] for more details on the results for operators of type $1,1$.

7. Final remarks

Pointwise estimates in terms of the maximal function u^* were crucial for the author's work on type $1,1$-operators [14] and developed for that purpose, but they were used there with only brief explanations. A detailed presentation has been postponed to the present paper, because the techniques should be of interest in their own right. This is illustrated by the proofs of Corollary 4.1, Theorem 5.2 and Theorem 6.2 e.g.

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