Second-order analysis of structured inhomogeneous spatio-temporal point processes

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Abstract

Statistical methodology for spatio-temporal point processes is in its infancy. We consider second-order analysis based on pair correlation functions and K-functions for first general inhomogeneous spatio-temporal point processes and second inhomogeneous spatio-temporal Cox processes. Assuming spatio-temporal separability of the intensity function, we clarify different meanings of second-order spatio-temporal separability. One is second-order spatio-temporal independence and relates e.g. to log-Gaussian Cox processes with an additive covariance structure of the underlying spatio-temporal Gaussian process. Another concerns shot-noise Cox processes with a separable spatio-temporal covariance density. We propose diagnostic procedures for checking hypotheses of second-order spatio-temporal separability, which we apply on simulated and real data (the UK 2001 epidemic foot and mouth disease data).

Key words: spatio-temporal functional summary statistics; K-function; pair correlation function; second-order intensity-reweighted stationarity; shot-noise Cox process; spatio-temporal separability.

1 Introduction

While statistical methodology for spatial point processes (Diggle 2003, Møller & Waagepetersen 2004, Møller & Waagepetersen 2007, Illian, Penttinen, Stoyan & Stoyan 2008) and for temporal point processes (Daley & Vere-Jones 2003, Daley & Vere-Jones 2008) is rather well-developed, it is still in its infancy for spatio-temporal point processes (Gabriel & Diggle 2009). We consider a spatio-temporal point process with no multiple points as a random countable subset

X of $\mathbb{R}^2 \times \mathbb{R}$, where a point $(u,t) \in X$ corresponds to an event $u \in \mathbb{R}^2$ occurring at time $t \in \mathbb{R}$. Examples of events include incidence of disease, sightings or births of a species, the occurrences of fires, earthquakes, tsunamis, or volcanic eruptions (Schoenberg, Brillinger & Guttorp 2002). In practice, X is observed within a spatio-temporal window $W \times T$, where $W \subset \mathbb{R}^2$ is a bounded region of area $|W| > 0$, T is a bounded time interval of length $|T| > 0$, and $X \bigcap (W \times T) = \{(u_i, t_i),\ i = 1 \dots n\}$ are the data. Assuming that X has an intensity function and a pair correlation function, the spatial component process X_{space} consisting of those events with times in T and the temporal component process X_{time} consitsting of those times with events in W are then well-defined point processes on $\mathbb{R}^2$ and $\mathbb{R}$, respectively, with well-defined intensity and pair correlation functions (as detailed in Section 2).

The aim of this paper is to study spatio-temporal separability properties and inferential procedures based on first and second-order properties as given by the intensity and pair correlation functions for X, X_{space}, and X_{time} as well as related K-functions and other functional summary statistics. The first part of the paper considers general inhomogeneous spatio-temporal point processes and the second part inhomogeneous spatio-temporal Cox processes (Cox 1955). In the latter case we are given a non-negative stochastic process λ defined on $\mathbb{R}^2 \times \mathbb{R}$ such that X conditional on λ is a Poisson process with intensity function λ. Furthermore, λ is assumed to have the multiplicative structure

$$\lambda(u,t) = \rho(u,t)S(u,t), \quad \mathrm{E}S(u,t) = 1, \quad (u,t) \in \mathbb{R}^2 \times \mathbb{R}, \tag{1}$$

where we refer to S as the residual process. This has unit mean so that ρ becomes the intensity function.

Throughout the paper we assume that X has a spatio-temporal separable intensity function ρ as specified in Section 2. Further, assuming that X is second-order intensity-reweighted stationary (Baddeley, Møller & Waagepetersen 2000, Gabriel & Diggle 2009), Section 2 recalls how second-order properties are specified by pair correlation and K-functions for X, X_{space}, and X_{time}. Section 2 also discusses how such functions are estimated by non-parametric methods.

Section 3 concerns the hypothesis of spatio-temporal separability of the pair correlation function. Diggle, Chetwynd, Häggkvist & Morris (1995) suggested simple diagnostic procedures for this hypothesis in the stationary case of X, i.e., when the distribution of X is invariant under translations in $\mathbb{R}^2 \times \mathbb{R}$. These were also used in the inhomogeneous case (i.e. when X is non-stationary) in connection to Figure 4 in Gabriel & Diggle (2009). Section 3 corrects a mistake in connection to these diagnostic procedures, and discusses the case of a log-Gaussian Cox process, i.e., when the residual process in (1) is a log-Gaussian process.

The pair correlation and K-functions for X, X_{space}, and X_{time} are related to various modifications of intensity-reweighted second-order measures. Section 4 introduces a new kind of modified intensity-reweighted second-order measures and their related densities and K-functions, which can be estimated by non-parametric methods. This becomes e.g. important when the residual process in (1) is a shot-noise process, i.e., when X is a spatio-temporal shot-noise Cox process (Møller 2003, Møller & Torrisi 2005) as studied in Section 5.

So far in the literature on spatio-temporal Cox processes, mainly log-Gaussian Cox processes have been studied (Brix & Møller 2001, Brix & Diggle 2001, Brix & Diggle 2003, Brix & Chadoeuf 2002, Diggle, Rowlingson & Su 2005, Diggle 2007) and to some extent, in a discrete time setting, shot-noise Cox processes (Møller & Diaz-Avalos 2010). In Section 5, which deals with inhomogeneous spatio-temporal shot-noise Cox processes in a continuous-time setting, our definition of spatio-temporal separability implies spatio-temporal separability of the density of the covariance function for the counts $N(A) = \#(X \cap A)$ (where $A \subseteq \mathbb{R}^2 \times \mathbb{R}$ is a Borel set). In this connection a diagnostic procedure is proposed, and a quick parameter estimation procedure based on the second-order properties for a specific type of inhomogeneous spatio-temporal shot-noise Cox process is discussed. Finally, this methodology is investigated for simulated data and for the UK 2001 epidemic foot and mouth disease data previously analyzed in Keeling, Woolhouse, Shaw, Matthews, Chase-Topping, Haydon, Cornell, Kappey, Wilesmith & Grenfell (2001), Diggle (2006), Diggle (2007), and Gabriel, Rowlingson & Diggle (2010).

2 Assumptions and background

This section specifies the setting and recalls the properties and non-parametric estimation procedures of the intensity, pair correlation, and K-functions of the processes X, X_{space}, and X_{time} as needed in the sequel. For statistical background material on spatio-temporal point processes, see Diggle (2007), Diggle & Gabriel (2010), Møller & Diaz-Avalos (2010), and the references therein; for measure theoretical details, see e.g. Daley & Vere-Jones (2003) or Appendix B in Møller & Waagepetersen (2004).

We assume that X has intensity function ρ and pair correlation function g (see e.g. Møller & Waagepetersen (2004)). Then

$$\int\int f((u,s),(v,t))g((u,s),(v,t))\,\mathrm{d}(u,s)\,\mathrm{d}(v,t) = \mathrm{E}\sum_{(u,s),(v,t)\in X}^{\neq} \frac{f((u,s),(v,t))}{\rho(u,s)\rho(v,t)} \tag{2}$$

for any non-negative Borel function f defined on $(\mathbb{R}^2\times\mathbb{R})\times(\mathbb{R}^2\times\mathbb{R})$. Here $\sum^{\neq}$ means that $(u,s)\neq(v,t)$, and we take $a/0=0$ for $a\geq 0$. The pair correlation function is related to the density function c of the covariance function for the counts $N(A)=\#(X\cap A)$ (where $A\subseteq\mathbb{R}^2\times\mathbb{R}$ is a Borel set) by

$$c((u,s),(v,t)) = \rho(u,s)\rho(v,t)(g((u,s),(v,t))-1), \quad (u,s)\neq(v,t), \tag{3}$$

see e.g. Daley & Vere-Jones (2003).

It follows from (2) that with probability one, for any pair of distinct points (u,s) and (v,t) from X, we have that $u\neq v$ and $s\neq t$. We can therefore ignore the case where the spatial and temporal component processes X_{space} and X_{time} have multiple points and define them by

$$X_{\mathrm{space}} = \{u : (u,t) \in X,\, t \in T\}, \quad X_{\mathrm{time}} = \{t : (u,t) \in X,\, u \in W\}.$$

We consider X_{space} and X_{time} rather than the marginal processes given by all events respective all times in X, since the later processes may not have well-defined first or second-order properties as studied in this paper. Clearly, though we suppress this in the notation, X_{space} depends on T, and X_{time} depends on W.

2.1 First-order properties

2.1.1 First-order spatio-temporal separability

Throughout this paper we assume first-order spatio-temporal separability, i.e.,

$$\rho(u,t) = \bar{\rho}_1(u)\bar{\rho}_2(t), \quad (u,t) \in \mathbb{R}^2 \times \mathbb{R}, \tag{4}$$

where $\bar{\rho}_1$ and $\bar{\rho}_2$ are non-negative functions. One may be tempted to call this property 'first-order spatio-temporal independence', since intuitively the probability that X has a point in an infinitesimally small region around (u,t) of volume $\mathrm{d}(u,t) = \mathrm{d}u\,\mathrm{d}t$ is

$$\rho(u,t)\,\mathrm{d}(u,t) = [\bar{\rho}_1(u)\,\mathrm{d}u][\bar{\rho}_2(t)\,\mathrm{d}t]$$

which is a product of a function of u and du and a function of t and dt. However, we prefare to avoid this terminology, since (4) does not necessarily mean that for a point (u,t) in X, the event u is independent of its time t. More precisely, (4) means that the intensity measure given by $\mu(A \times B) = \mathrm{E}N(A \times B)$ for Borel sets $A \subseteq \mathbb{R}^2$ and $B \subseteq \mathbb{R}$ is a product measure, since $\mu(A \times B) = \int_A \bar{\rho}_1(u)\,\mathrm{d}u \int_B \bar{\rho}_2(t)\,\mathrm{d}t$.

First-order spatio-temporal separability is a convenient working hypothesis which is hard to check. It implies that

$$\rho_{\mathrm{space}}(u) = \bar{\rho}_1(u) \int_T \bar{\rho}_2(t)\,\mathrm{d}t, \quad \rho_{\mathrm{time}}(t) = \bar{\rho}_2(t) \int_W \bar{\rho}_1(u)\,\mathrm{d}u, \tag{5}$$

are the intensity functions of the spatial and temporal component processes, and

$$\rho(u,t) = \frac{\rho_{\mathrm{space}}(u)\rho_{\mathrm{time}}(t)}{\int_{W\times T} \rho(u,t)\,\mathrm{d}(u,t)}. \tag{6}$$

Note that if X is stationary, ρ, ρ_{space}, and ρ_{time} are all constant. In the sequel, our focus is on the inhomogenuous case.

2.1.2 Non-parametric estimation

In Section 5 we consider semi-parametric models, with a non-parametric model for ρ and a parametric model for g. The present section deals with non-parametric estimation of the spatial and temporal intensity functions ρ_{space} and ρ_{time}.

Suppose we are given estimates $\hat{\rho}_{\text{space}}$ and $\hat{\rho}_{\text{time}}$. If these are unbiased estimates of the expected number of observed points, i.e., $\int_W \hat{\rho}_{\text{space}}(u)\,\mathrm{d}u = \int_T \hat{\rho}_{\text{time}}(s)\,\mathrm{d}s = n$, then the estimate of the spatio-temporal intensity function given by

$$\hat{\rho}(u,t) = \hat{\rho}_{\text{space}}(u)\hat{\rho}_{\text{time}}(t)/n \tag{7}$$

also becomes an unbiased estimate of the expected number of observed points, since by (6), $\int_{W\times T} \hat{\rho}(u,t)\,\mathrm{d}(u,t) = n$.

For non-parametric estimation of ρ_{space}, we may follow Diggle (1985) and Berman & Diggle (1989) in using the kernel estimate

$$\hat{\rho}_{\text{space}}(u) = \sum_{i=1}^{n} \omega_b(u-u_i)/c_{W,b}(u_i), \quad u \in W, \tag{8}$$

where $\omega_b(u) = \omega(u/b)/b^2$ is a kernel with bandwidth $b > 0$, i.e., ω is a given density function. Further,

$$c_{W,b}(u_i) = \int_W \omega_b(u-u_i)\ \mathrm{d}u$$

is an edge correction factor ensuring that $\int_W \hat{\rho}_{\text{space}}(u)\,\mathrm{d}u = n$. However, in practice, for complicated or irregular windows W, this edge correction factor is often ignored.

A similar kernel estimate may be used for non-parametric estimation of ρ_{time}. If the tail of the empirical distribution function of the observed times t_i turns out to be heavy tailed (this is the case for the data in Section 5.3.2), it may be more reasonable to use the log-transform re-transform scheme in Markovich (2007), where first a kernel estimate $\hat{h}$ is obtained for the intensity function of the log-transformed observed times, and next

$$\hat{\rho}_{\text{time}}(t) = \hat{h}(\log(t))/t \tag{9}$$

is used as the non-parametric estimate of ρ_{time}.

Although these non-parametric estimation procedures of the spatial and temporal intensity functions may only lead to approximately unbiased estimates, we will still use (7).

2.2 Second-order properties

2.2.1 The spatio-temporal case

Throughout this paper, following Baddeley et al. (2000) and Gabriel & Diggle (2009), we assume that X is second-order intensity-reweighted stationary, i.e.,

$$g((u,s),(v,t)) = g(u-v, s-t), \quad (u,s),(v,t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{10}$$

Then the spatio-temporal inhomogeneous K-function is defined by

$$K(r,t) = \int 1[\|u\| \le r, |s| \le t] g(u,s)\, \mathrm{d}(u,s), \quad r > 0,\ t > 0, \tag{11}$$

where $\|u\|$ denotes usual distance in $\mathbb{R}^2$ and $|s|$ numerical value (not to be confused with the length $|T|$ or the area $|W|$). In the stationary case of X, $\rho K(r,t)$ is the expected number of further points within distance r and time lag t from the origin given that X has a point at the origin (Ripley 1976, Ripley 1977). In our opinion, (11) is therefore a more natural definition than the one used in Gabriel & Diggle (2009) which differ by a factor $1/2$. Note that if X is a Poisson process, $g = 1$ and $K(r,t) = 2\pi r^2 t$.

Non-parametric estimation of pair correlation functions are usually based on kernel methods (Stoyan & Stoyan 1994, Illian et al. 2008), where the specification of the bandwidth of the kernel is debatable, not at least in the inhomogeneous case (Baddeley et al. 2000). Alternatively, an approximately unbiased non-parametric estimate of the K-function is given by

$$\hat{K}(r,t) = \frac{1}{|W||T|} \sum_{i \neq j} \frac{I\left[\|u_i - u_j\| \le r, |t_i - t_j| \le t\right]}{\hat{\rho}(u_i,t_i)\hat{\rho}(u_j,t_j) w_1(u_i,u_j) w_2(t_i,t_j)} \tag{12}$$

where $\sum_{i \neq j}$ means the sum over all pairs $(u_i,t_i) \neq (u_j,t_j)$ of the data points; $I(\cdot)$ denotes the indicator function; $\hat{\rho}(u,t)$ is as in Section 2.1.2; either $w_1(u_i,u_j)$ is Ripley's isotropic edge correction factor (Ripley 1976, Ripley 1977), that is, the reciprocal of the proportion of the circumference of the circle with center u_i and radius $\|u_i - u_j\|$ that lies within W—or, if W is too complicated, $w_1(u_i,u_j) = 1$; $w_2(t_i,t_j)$ is the temporal edge correction factor which is equal to one if both ends of the interval of length $2|t_i - t_j|$ and center t_i lie within T, and $w_2(t_i,t_j) = 2$ otherwise (Diggle et al. 1995).

2.2.2 Spatial and temporal components

The pair correlation function g_{space} of the spatial component process X_{space} satisfies

$$\int\int f(u,v) g_{\text{space}}(u,v)\, \mathrm{d}u\, \mathrm{d}v = \mathrm{E} \sum_{u,v \in X_{\text{space}}}^{\neq} \frac{f(u,v)}{\rho_{\text{space}}(u)\rho_{\text{space}}(v)} \tag{13}$$

for any non-negative Borel function f defined on $\mathbb{R}^2 \times \mathbb{R}^2$. Defining

$$p_1(u) = \bar{\rho}_1(u) / \int_W \bar{\rho}_1(v)\ \mathrm{d}v, \quad p_2(t) = \bar{\rho}_2(t) / \int_T \bar{\rho}_2(s)\ \mathrm{d}s,$$

and using (2), (4), (5), (10), and (13), we obtain second-order intensity-reweighted stationarity of X_{space}, since we can take

$$g_{\text{space}}(u,v) = g_{\text{space}}(u-v) = \int_T \int_T p_2(s)p_2(t)g(u-v,s-t)\,\mathrm{d}s\,\mathrm{d}t. \tag{14}$$

Similarly, X_{time} is second-order intensity-reweighted stationary with

$$g_{\text{time}}(s,t) = g_{\text{time}}(s-t) = \int_W \int_W p_1(u)p_1(v)g(u-v,s-t)\,\mathrm{d}u\,\mathrm{d}v. \tag{15}$$

The corresponding K-functions are

$$K_{\text{space}}(r) = \int_{\|u\|\le r} g_{\text{space}}(u)\ \mathrm{d}u, \quad r>0, \tag{16}$$

and

$$K_{\text{time}}(t) = \int_{-t}^{t} g_{\text{time}}(s)\ \mathrm{d}s, \quad t>0. \tag{17}$$

Non-parametric estimates of these are given by

$$\hat{K}_{\text{space}}(r) = \frac{1}{|W|}\sum_{i\neq j}\frac{I\left[\|u_i-u_j\|\le r\right]}{w_1(u_i,u_j)\hat{\rho}_{\text{space}}(u_i)\hat{\rho}_{\text{space}}(u_j)}$$

and

$$\hat{K}_{\text{time}}(t) = \frac{1}{|T|}\sum_{i\neq j}\frac{I\left[|t_i-t_j|\le t\right]}{w_2(t_i,t_j)\hat{\rho}_{\text{time}}(t_i)\hat{\rho}_{\text{time}}(t_j)}$$

where we use the same notation as in (12).

3 Spatio-temporal separability of the pair correlation function

The hypothesis of spatio-temporal separability of the pair correlation function states that

$$g(u,t) = \bar{g}_1(u)\bar{g}_2(t), \quad (u,t)\in\mathbb{R}^2\times\mathbb{R}, \tag{18}$$

where $\bar{g}_1$ and $\bar{g}_2$ are non-negative functions. Intuitively, (2), (4), and (18) imply that the probability of observing a pair of points from X occurring jointly in each of two infinitesimally small sets with centers (u,s), (v,t) and volumes $\mathrm{d}u\,\mathrm{d}s$, $\mathrm{d}v\,\mathrm{d}t$ is

$$[\bar{\rho}_1(u)\bar{\rho}_1(v)\bar{g}_1(u-v)\,\mathrm{d}u\,\mathrm{d}v][\bar{\rho}_2(s)\bar{\rho}_2(t)\bar{g}_2(s-t)\,\mathrm{d}s\,\mathrm{d}t]$$

which is a product of a function of the locations (u,v) and the areas $(\mathrm{d}u,\mathrm{d}v)$ and a function depending on the times (s,t) and the lengths $(\mathrm{d}s,\mathrm{d}t)$. Therefore one could be tempted to refer to (18) as the hypothesis of second-order spatio-temporal independence, but we shall avoid this terminology for a similar reason as in Section 2.1.1. However, note that if e.g. the times in X form a second-order intensity-reweighted stationary point process which is independent of the events in X and has intensity function ρ_2 and pair correlation function g_2, and if the events are i.i.d. with density f, then $\rho(u,t) = f(u)\rho_2(t)$ and $g((u,s),(v,t)) = g_2(s-t)$, and so X is second-order intensity-reweighted stationary and satisfies (18).

Observe that (14)-(15) and (18) imply

$$g_{\text{space}}(u) = c_{\text{space}}\,\bar{g}_1(u), \quad \text{with } c_{\text{space}} = \int_T \int_T p_2(s)p_2(t)\bar{g}_2(s-t)\,\mathrm{d}s\,\mathrm{d}t, \tag{19}$$

and

$$g_{\text{time}}(t) = c_{\text{time}}\,\bar{g}_2(t), \quad \text{with } c_{\text{time}} = \int_W \int_W p_1(u)p_1(v)\bar{g}_1(u-v)\,\mathrm{d}u\,\mathrm{d}v. \tag{20}$$

Hence, by (11) and (16)-(20), spatio-temporal separability of g implies that

$$K(r,t) = \frac{K_{\text{space}}(r)}{c_{\text{space}}}\frac{K_{\text{time}}(t)}{c_{\text{time}}}. \tag{21}$$

Diggle et al. (1995) incorrectly expected that the functional summary statistic given by

$$\hat{D}(r,t) = \frac{\hat{K}(r,t)}{\hat{K}_{\text{space}}(r)\hat{K}_{\text{time}}(t)}, \quad r,t > 0, \tag{22}$$

is close to one under the hypothesis of spatio-temporal separability of the pair correlation function. This mistake was repeated in connection to Figure 4 in (Gabriel & Diggle 2009). In fact, under the hypothesis of spatio-temporal separability of g, $\hat{D}$ should be expected to be approximately equal to $c_{\text{space}}c_{\text{time}}$, cf. (21), and this constant is in general different from one (unless $g = 1$). As an illustration, Figure 1 shows a perspective plot of $\hat{D}$ for the UK 2001 epidemic foot and mouth disease dataset which is further analyzed in Section 5.3.2. It seems that $\hat{D}$ is far from being constant, indicating that g is not spatio-temporal separable.

Cox point process models with a spatio-temporal separable pair correlation function are rather uncommon in the literature. For example, consider a spatio-temporal log-Gaussian Cox process, i.e., when $\log S$ in (1) is a Gaussian process on $\mathbb{R}^2 \times \mathbb{R}$. Then

$$g((u,s),(v,t)) = \exp(C((u,s),(v,t))) \tag{23}$$

where C is the covariance function of the Gaussian process (Møller, Syversveen & Waagepetersen 1998), and so second-order intensity-reweighted stationarity means that

$$C((u,s),(v,t)) = C(u-v, s-t). \tag{24}$$

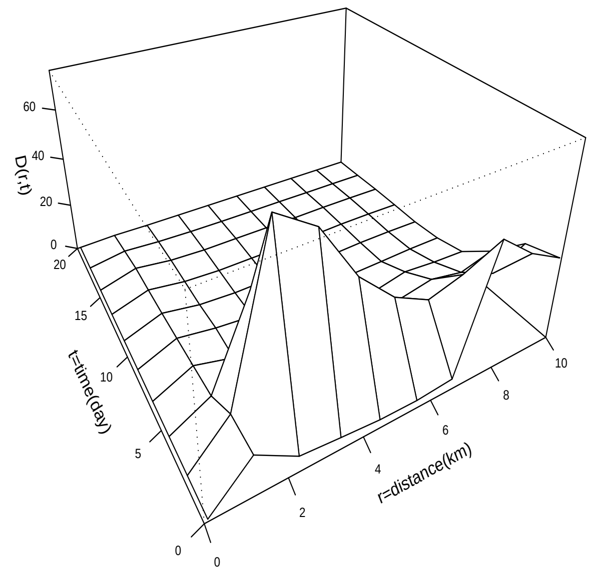

Figure 1: $\hat{D}$ for the UK 2001 epidemic foot and mouth disease dataset.

Hence spatio-temporal separability of g means an additive spatio-temporal covariance structure $C(u,t) = \log \bar{g}_1(u) + \log \bar{g}_2(t)$, cf. (18) and (23)-(24). However, often in the literature the covariance function is instead assumed to be spatio-temporal separable meaning that $C(u,t) = C_1(u)C_2(t)$ is multiplicative, see e.g. Diggle (2007). Another example with spatio-temporal dependence is a shot-noise Cox process as shown in Section 5.

4 Intensity-reweighted second-order measures

The pair correlation functions g, g_{space}, and g_{time} are all densities of modified second-order factorial moment measures obtained by reweighting the points of X, X_{space}, and X_{time} with their respective intensities ρ, ρ_{space}, and ρ_{time}, cf. (2) and (13). In the sequel, we need the functions

$$g_1(u) = \frac{1}{|T|^2}\int_T\int_T g(u, s-t)\,\mathrm{d}s\,\mathrm{d}t, \quad g_2(t) = \frac{1}{|W|^2}\int_W\int_W g(u-v, t)\,\mathrm{d}u\,\mathrm{d}v, \tag{25}$$

where $u \in \mathbb{R}^2$ and $t \in \mathbb{R}$. These are related to more complicated modifications of intensity-reweighted measures, since by (2) and (25),

$$\int\int f_1(u,v) g_1(u-v)\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{|T|^2}\mathrm{E} \sum_{(u,s),(v,t)\in X:\, s,t\in T}^{\neq} \frac{f_1(u,v)}{\rho(u,s)\rho(v,t)}$$

for any non-negative Borel function f_1 defined on $\mathbb{R}^2 \times \mathbb{R}^2$, and

$$\int\int f_2(s,t) g_2(s-t)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{|W|^2}\mathrm{E} \sum_{(u,s),(v,t)\in X:\, u,v\in W}^{\neq} \frac{f_2(s,t)}{\rho(u,s)\rho(v,t)}$$

for any non-negative Borel function f_2 defined on $\mathbb{R}\times\mathbb{R}$. Furthermore, we define corresponding K-functions

$$K_1(r) = \int_{\|u\|\le r} g_1(u)\,\mathrm{d}u, \quad K_2(t) = \int_{-t}^{t} g_2(s)\,\mathrm{d}s, \tag{26}$$

where $r, t > 0$.

In the following special cases we have simple relationships between g_1 and g_{space}, and between g_2 and g_{time}. If X is a Poisson process, these functions are all equal to one. If $\rho(u,t) = \bar{\rho}_1(u)$ does not depend on $t \in T$, then by (14) and (25), $g_1 = g_{\mathrm{space}}$. Similarly, if $\rho(u,t) = \bar{\rho}_2(t)$ does not depend on $u \in W$, then by (15) and (25), $g_2 = g_{\mathrm{time}}$. Moreover, if we have spatio-temporal separability of g, then g_1 and g_{space} are proportional, and g_2 and g_{time} are proportional, since by (18)-(20) and (25),

$$g_1(u) = \frac{\bar{c}_1}{c_{\mathrm{space}}}\bar{g}_{\mathrm{space}}(u), \quad \text{with } \bar{c}_1 = \frac{1}{|T|^2}\int_T\int_T \bar{g}_2(s-t)\,\mathrm{d}s\,\mathrm{d}t,$$

and

$$g_2(t) = \frac{\bar{c}_2}{c_{\mathrm{time}}}\bar{g}_{\mathrm{time}}(t), \quad \text{with } \bar{c}_2 = \frac{1}{|W|^2}\int_W\int_W \bar{g}_1(u-v)\,\mathrm{d}u\,\mathrm{d}v.$$

We also need the following approximately unbiased non-parametric estimates of K_1 and K_2,

$$\hat{K}_1(r) = \frac{1}{|W||T|^2}\sum_{i\neq j}\frac{I\left[\|u_i - u_j\| \le r\right]}{w_1(u_i,u_j)\hat{\rho}(u_i,t_i)\hat{\rho}(u_j,t_j)}$$

and

$$\hat{K}_2(t) = \frac{1}{|W|^2|T|}\sum_{i\neq j}\frac{I\left[|t_i - t_j| \le t\right]}{w_2(t_i,t_j)\hat{\rho}(u_i,t_i)\hat{\rho}(u_j,t_j)},$$

where we use the same notation as in (12).

5 Spatio-temporal shot-noise Cox processes

As noticed in Section 1, so far in the literature on spatio-temporal Cox processes, mainly log-Gaussian Cox processes have been studied. Spatio-temporal shot-noise Cox processes (SNCP) provide an alternative, tractable, and flexible model class, as discussed in Møller & Diaz-Avalos (2010) but in a discrete time setting.

In this section, we consider a continuous time setting and let X be a spatio-temporal SNCP with the multiplicative structure (1) and assume second-order intensity-reweighted stationarity. The residual is then given by

$$S(u,t) = \frac{1}{\nu} \sum_{(v,s)\in\Phi} \kappa(u-v,t-s)$$

where Φ is a stationary Poisson process on $\mathbb{R}^2 \times \mathbb{R}$ with intensity $\nu > 0$, and κ is a density function on $\mathbb{R}^2 \times \mathbb{R}$.

Observe that X has an interpretation as a Poisson cluster process, i.e., as a superposition of independent clusters with centres given by Φ and so that conditional on Φ, the clusters are independent Poisson processes with intensity functions $\lambda_{(v,s)}(u,t) = \rho(u,t)\kappa(u-v,t-s)/\nu$ for $(v,s) \in \Phi$. We therefore refer to $\lambda_{(v,s)}$ as the 'offspring intensity' (associated to the cluster centre (v,s)). Assuming ρ is bounded on $W \times T$ by a positive constant $\rho_{\max}$, we can obtain a simulation of $X \cap (W \times T)$ by first simulating a stationary SNCP $X_{\max}$ within $W \times T$ and where ρ is replaced by $\rho_{\max}$, and second make an independent thinning of $X_{\max} \cap (W \times T)$ where the retention probability of a point $(u,t) \in X_{\max} \cap (W \times T)$ is given by $p(u,t) = \rho(u,t)/\rho_{\max}$. Simulation of a homogeneous SNCP is discussed in Brix & Kendall (2002), Møller (2003), and Møller & Waagepetersen (2004).

We have

$$g(u,t) = 1 + \kappa * \tilde{\kappa}(u,t)/\nu \tag{27}$$

where $*$ denotes convolution and $\kappa(u,t) = \kappa(-u,-t)$. This follows using the Slivnyak-Mecke's formula, see Møller (2003) and Møller & Waagepetersen (2004). Note that $\kappa * \tilde{\kappa}$ is the density of the vector given by the difference between two offspring within a cluster of $X_{\max}$ (but not of X unless ρ is constant).

5.1 Spatio-temporal separability

Assume that the kernel κ is spatio-temporal separable,

$$\kappa(u,t) = \kappa_1(u)\kappa_2(t) \tag{28}$$

where κ_1 is a density on $\mathbb{R}^2$ and κ_2 is a density on $\mathbb{R}$. In general, (27)-(28) imply spatio-temporal dependence. However,

$$\lambda_{(v,s)}(u,t) = \left[\bar{\rho}_1(u)\kappa_1(u-v)\right]\left[\bar{\rho}_2(t)\kappa_2(t-s)\right]/\nu$$

is a product of a function depending on u and a function depending on t. Thus the offspring intensities are spatio-temporal separable, i.e., (28) is equivalent to spatio-temporal independence within the clusters. Furthermore, the covariance density (3) is spatio-temporal separable, with

$$c((u,s),(v,t)) = \left[\bar{\rho}_1(u)\bar{\rho}_1(v)\kappa_1 * \tilde{\kappa}_1(u-v)\right]\left[\bar{\rho}_2(s)\bar{\rho}_2(t)\kappa_2 * \tilde{\kappa}_2(s-t)\right]/\nu.$$

Defining

$$c_1(W) = 1 \Big/ \int_W \int_W \kappa_1 * \tilde{\kappa}_1(u-v)\,\mathrm{d}u\,\mathrm{d}v, \quad c_2(T) = 1 \Big/ \int_T \int_T \kappa_2 * \tilde{\kappa}_2(s-t)\,\mathrm{d}s\,\mathrm{d}t,$$

and

$$\nu_1 = \nu_1(T) = \nu c_2(T)|T|^2, \quad \nu_2 = \nu_2(W) = \nu c_1(W)|W|^2,$$

we obtain from (25) and (27)-(28),

$$g_1(u) = 1 + \kappa_1 * \tilde{\kappa}_1(u)/\nu_1, \quad g_2(t) = 1 + \kappa_2 * \tilde{\kappa}_2(t)/\nu_2, \tag{29}$$

and so

$$\nu[g(u,t) - 1] = \nu_1\nu_2\,[g_1(u) - 1]\,[g_2(t) - 1]\,.$$

Consequently,

$$\nu\left[K(r,t) - 2\pi r^2 t\right] = \nu_1\nu_2\left[K_1(r) - \pi r^2\right][K_2(t) - 2t]\,.$$

Under the separability hypothesis (28) we therefore expect the functional summary statistic

$$\hat{F}(r,t) = \frac{\left[\hat{K}(r,t) - 2\pi r^2 t\right]}{\left[\hat{K}_1(r) - \pi r^2\right]\left[\hat{K}_2(t) - 2t\right]}, \quad r,t > 0,$$

to be approximately constant.

5.2 Further model assumptions and parameter estimation

In the reminder of this paper, we assume that κ_1 is the density of a zero-mean bivariate radially symmetric normal distribution $N_2(0, \sigma^2 I)$ with variance σ^2, and κ_2 is the density of an exponential distribution with rate α and restricted to a bounded interval $[0, t^*]$, i.e.,

$$\kappa_2(t) = \frac{\alpha}{1 - \exp(-\alpha t^*)}\exp(-\alpha t), \quad 0 \le t \le t^*. \tag{30}$$

This section briefly discusses a simple procedure for estimating the positive parameters ν, σ^2, α, and t^* based on the second-order properties. Further methods are discussed in Møller & Waagepetersen (2007) and the references therein.

By (26) and (29), since $\kappa_1 * \tilde{\kappa}_1$ is the density of $N_2(0, 2\sigma^2 I)$, K_1 agrees with the K-function for a planar Thomas process, see e.g. Møller & Waagepetersen (2004). Thus the Spatstat software package (Baddeley & Turner 2005, Baddeley & Turner 2006) provides an estimate $(\hat{\nu}_1, \hat{\sigma}^2)$ based on a minimum contrast estimation procedure such that the theoretical K_1-function becomes close to its non-parametric estimate $\hat{K}_1$. We use another minimum contrast procedure for estimating α. First, we obtain an estimate $\widehat{t^*}$ by considering a plot of $\hat{K}_2(t)-2t$ and using the fact that $K_2(t)-2t$ is constant for $t \geq t^*$. Next, define

$$R(t;\alpha,t^*) = \frac{K_2(t)-2t}{K_2(t^*)-2t^*}, \quad \hat{R}(t) = \frac{\hat{K}_2(t)-2t}{\hat{K}_2(\widehat{t^*})-2\widehat{t^*}}, \quad 0 < t \leq t^*.$$

By (26) and (29)-(30),

$$\begin{aligned} R(t;\alpha,t^*) &= \frac{\int_{-t}^{t} \kappa_2 * \tilde{\kappa}_2(s)\,\mathrm{d}s}{\int_{-t^*}^{t^*} \kappa_2 * \tilde{\kappa}_2(s)\,\mathrm{d}s} = 2\int_0^t \kappa_2 * \tilde{\kappa}_2(s)\,\mathrm{d}s \\ &= \frac{1+\exp(-2\alpha t^*) - \exp(-\alpha t) - \exp(\alpha t - 2\alpha t^*)}{[1-\exp(-\alpha t^*)]^2} \end{aligned}$$

since $\kappa_2 * \tilde{\kappa}_2$ is a symmetric density concentrated on $[-t^*, t^*]$, with

$$\kappa_2 * \tilde{\kappa}_2(t) = \frac{\alpha \exp(\alpha t)}{2\,[1-\exp(-\alpha t^*)]^2}\left[\exp(-2\alpha t) - \exp(-2\alpha t^*)\right], \quad 0 \leq t \leq t^*. \quad (31)$$

Then the minimum contrast procedure is such that $R(t;\alpha,\widehat{t^*})$ becomes close to its non-parametric estimate $\hat{R}$. Specifically,

$$\hat{\alpha} = \arg\min \int_0^{\widehat{t^*}} \left(R(t;\alpha,\widehat{t^*}) - \hat{R}(t)\right)^2 \,\mathrm{d}t. \quad (32)$$

Finally, since $\nu_1 = \nu c_2(T)|T|^2$, we estimate ν by

$$\hat{\nu} = \hat{\nu}_1 \int_T \int_T \widehat{\kappa_2 * \tilde{\kappa}_2}(s-t)\,\mathrm{d}s\,\mathrm{d}t/|T|^2 \quad (33)$$

where $\widehat{\kappa_2 * \tilde{\kappa}_2}$ is obtained by replacing α and t^* by $\hat{\alpha}$ and $\widehat{t^*}$ in (31).

5.3 Applications

5.3.1 Simulation study

The aim of this section is first to compare results obtained using either the true intensity function ρ or its non-parametric estimate $\hat{\rho}$ from Section 2.1.2, and second to discuss the sensitivity of such results using different bandwidths of the kernel estimate. We let $W \times T = [0,1]^2 \times [0,1]$ be the unit cube, and consider 100 simulated point patterns from a SNCP within $W \times T$, where

$$\rho(u,t) = \frac{200}{(1-\mathrm{e}^{-1})(\mathrm{e}-1)(\mathrm{e}^2-1)}\,\mathrm{e}^{-x+y+2t} \quad \text{with } u=(x,y),$$

and where $\sigma = 0.025$, $\alpha = 20$, $t^* = 0.1$, and $\nu = 10$. Then the expected number of points per simulation is 100.

For $\hat{\rho}_{\text{space}}$, we use (8) with a bivariate Gaussian kernel where a bandwidth of 0.067 is obtained by the command msd2d of the splancs package (Rowlingson

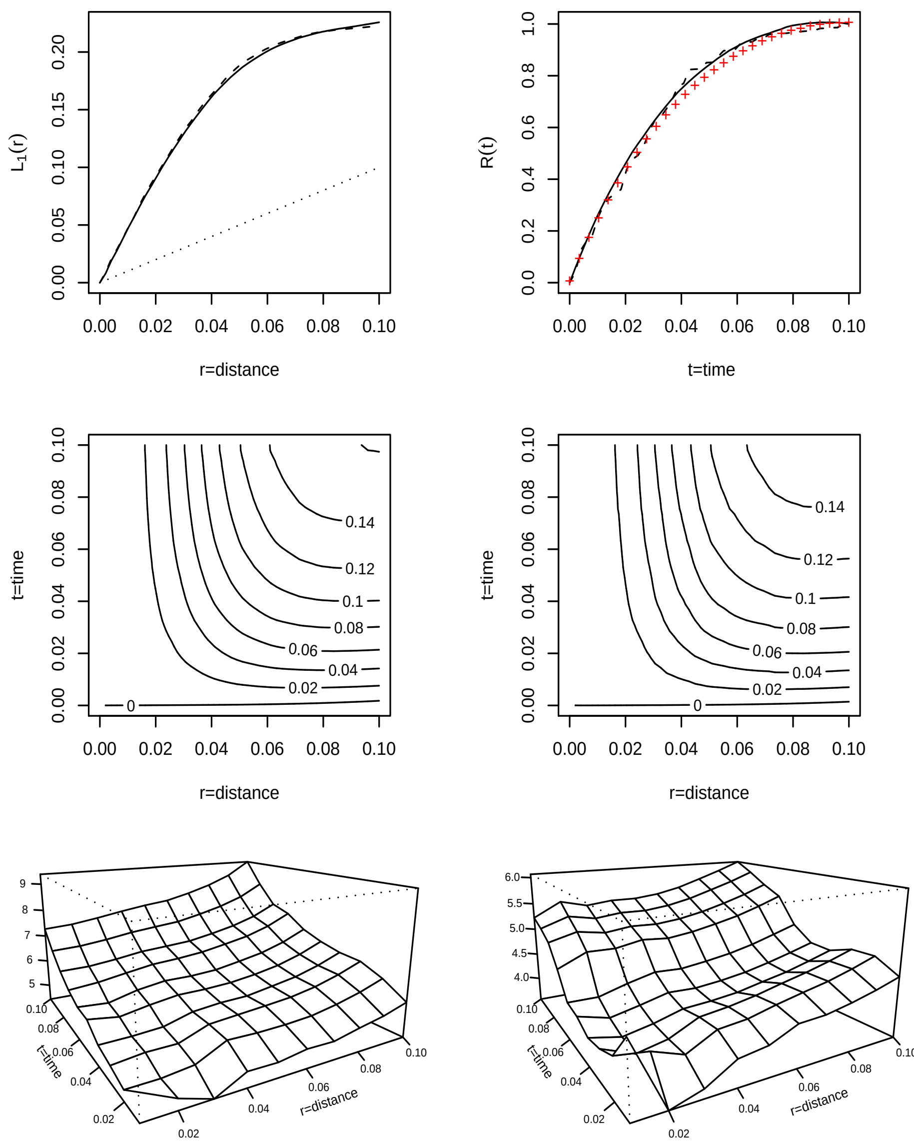

Figure 2: Means of various functional summary statistics based on 100 simulations of the SNCP specified in Section 5.3.1. Top left panel: $\hat{L}_1$ using ρ (solid line) or $\hat{\rho}$ (dashed line), and theoretical L_1 for a Poisson process (dotted line). Top right panel: $\hat{R}$ using ρ (solid line) or $\hat{\rho}$ (dashed line), and theoretical R for the SNCP (pluses). Middle left panel: $\hat{K}(r,t) - 2\pi r^2 t$ using ρ. Middle right panel: $\hat{K}(r,t) - 2\pi r^2 t$ using $\hat{\rho}$. Bottom left panel: $\hat{F}$ using ρ. Bottom right panel: $\hat{F}$ using $\hat{\rho}$.

& Diggle 2010) (briefly, this is based on minimizing a mean square error given in Diggle (1985)). For $\hat{\rho}_{\text{time}}$, we use instead a univariate Gaussian kernel where a bandwidth of 0.6 is first used. This bandwidth may appear to be large as compared to the unit square W, but it was chosen after some experimentation to obtain similar results when ρ or $\hat{\rho}$ is used. This is illustrated in Figure 2. The top left panel shows the mean of the 100 simulated $\hat{L}_1$-functions when using ρ (solid line) or $\hat{\rho}$ (dashed line) and where $\hat{L}_1(r) = \sqrt{\hat{K}_1(r)/\pi}$ (Besag 1977); the two curves are very close. For comparison, the theoretical value $L_1(r) = r$ for a Poisson process is also shown (dotted line); as expected this curve is much below the two other curves. The top right panel shows the mean of the 100 simulated $\hat{R}$-functions when using ρ (solid line) or $\hat{\rho}$ (dashed line), and the theoretical R-function with the true parameters $\alpha = 20$ and $t^* = 0.1$ (pluses). Notice that these are the functions appearing in the minimum contrast estimate $\hat{\alpha}$ given by (32) (when t^* agrees with its estimate $\widehat{t^*}$), and such estimates $\hat{\alpha}$ turn out to be rather similar to the true α no matter if ρ or $\hat{\rho}$ is used. The middle panels show the mean of the 100 simulated $\hat{K}(r,t) - 2\pi r^2 t$ functions. They do not depend much on whether ρ (middle left panel) or $\hat{\rho}$ (middle right panel) is used, and they clearly show the spatio-temporal clustering of the SNCP. The lower panels show the mean of the 100 simulated $\hat{F}$-functions. Now the values are a bit higher when ρ is used (lower left panel) than if $\hat{\rho}$ (lower right panel), but both surfaces are rather flat, indiacting the spatio-temporal separability of the kernel κ.

The discrepancy between results obtained using ρ or $\hat{\rho}$ is more pronounced if a much smaller (or larger) bandwidth than 0.6 is used for $\hat{\rho}_{\text{time}}$. Figure 3 illustrates this when the bandwidth is 0.2 and we consider the $\hat{K}(r,t) - 2\pi r^2 t$ function. The values in the right panel are now about twice as large as in the left panel. This is not surprising since, as the bandwidth decreases, $\hat{\rho}_{\text{time}}$ gets more concentrated around the observed times and hence $\hat{K}$ decreases.

5.3.2 The UK 2001 epidemic foot and mouth disease

This section applies our SNCP model to the data of the UK 2001 epidemic foot and mouth disease in Cumbria, which were previously analyzed in Keeling et al. (2001) and in Diggle (2006), Diggle (2007), and Gabriel et al. (2010). Cumbria is the county in the North-West of England which was most severely affected by the epidemic in 2001. The data analyzed in this section is taken from the R package STPP (Gabriel et al. 2010) by converting the measurement scale of spatial coordinates from meter to kilometer. We refer to these data as the FMD data.

The area of Cumbria is 5556.298 km^2 and data have been collected for 200 days starting at February 1, 2001, so we let $T = [0, 200]$. Figure 4 shows the irregular region W defined by Cumbria (the smallest box surrounding W is of size about 100×110 km^2) and the spatial point pattern of 648 infected animals (upper left panel), and the daily number of infected animals (lower panel). The upper right panel shows $\hat{\rho}_{\text{space}}$ given by (8) with bandwidth b=3.83 km (obtained

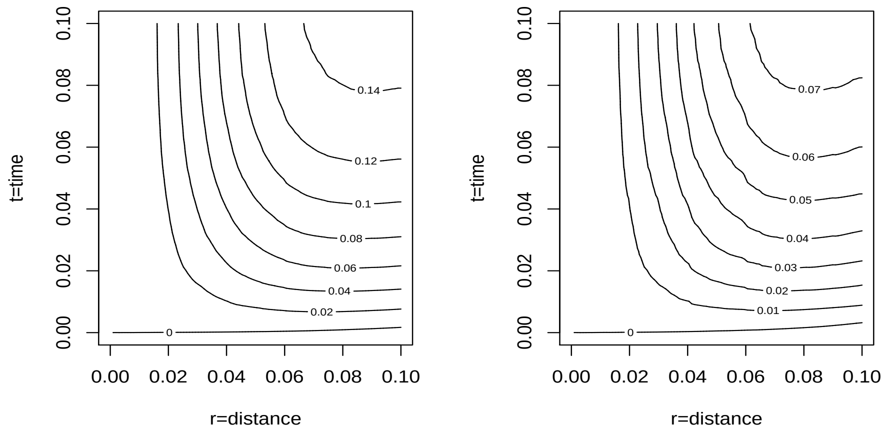

Figure 3: As the middle panels in Figure 2 but when the bandwidth for $\hat{\rho}_{\text{time}}$ is 0.2.

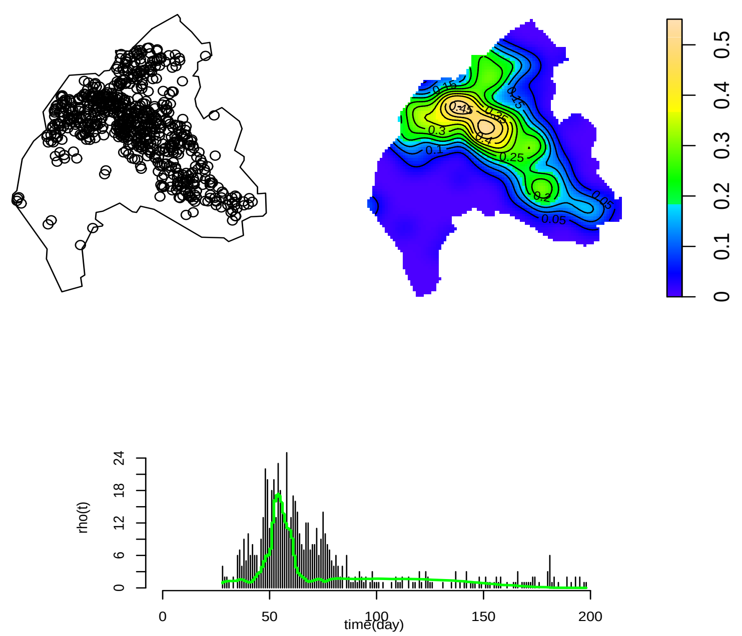

Figure 4: Spatial point pattern of infected animals (top left panel), $\hat{\rho}_{\text{space}}(r)$ (top right panel), and $\hat{\rho}_{\text{time}}(t)$ together with the daily number of infected animals (bottom panel) for the FDM data.

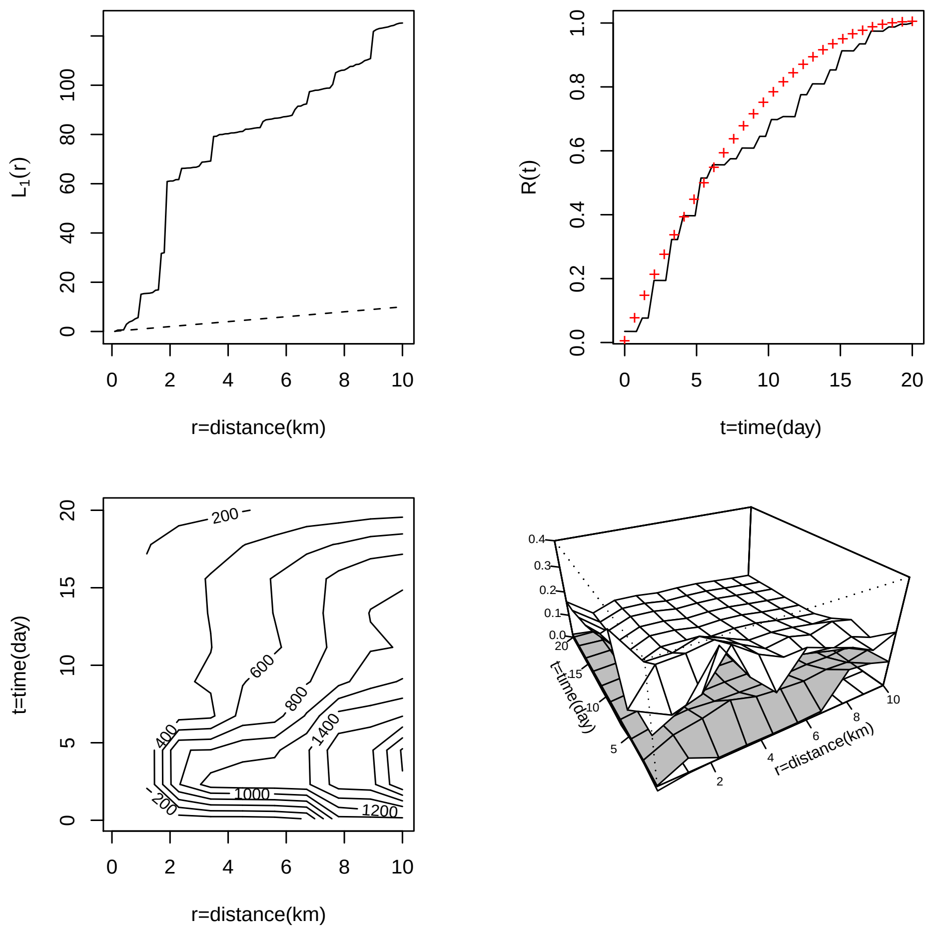

Figure 5: Estimated functional summary statistics for the FDM data. Upper left panel: $\hat{L}_1$ (solid line) and theoretical L_1 for a Poisson process (dashed line). Upper right panel: $\hat{R}$ (solid line) and the parametric estimate of R given by the fitted SNCP (pluses). Lower left panel: $\sqrt{\hat{K}(r,t) - 2\pi r^2 t}$. Lower right panel: $\hat{F}$ for the data and simulated 95%-envelopes under the fitted SNCP model.

by the command msd2d of the splancs package (Rowlingson & Diggle 2010)), and the lower panel shows $\hat{\rho}_{\text{time}}$ given by (9) with bandwidth 0.05 (chosen after some experimentation so that ρ_{time} appears to be in good agreement with the temporal data). Clearly, data presence is more intense in the North-Western to South-Eastern belt of Cumbria and within the first 100 days.

Spatial, temporal, and spatio-temporal clustering is also indicated by the three first panels in Figure 5 showing respectively $\hat{L}_1$ (defined as in Section 5.3.1), $\hat{R}$ with $\widehat{t^*} = 20$, and $\hat{K}(r,t) - 2\pi r^2 t$. The upper right panel also shows the parametric estimate $R(t; \hat{\alpha}, \widehat{t^*})$ with $\hat{\alpha} = 0.0478$ obtained by the minimum contrast method. The two curves in the upper right panel are in close agreement. Furthermore, using the minimum contrast estimation procedure based on K_1, we obtain $(\widehat{\nu_1}, \hat{\sigma}) = (0.0000207, 3.23)$, and hence using (33), $\hat{\nu} = 0.000163$ is obtained. This correcponds to about 182 clusters in $W \times T$. The final panel in Figure 5 shows $\hat{F}$ for the data together with simulated pointwise 95%-envelopes obtained from 39 simulations of the fitted SNCP (such envelopes are obtained for each value of (r,t) by calculating the smallest and largest simulated values of $\hat{F}(r,t)$; see Section 4.3.4 in Møller & Waagepetersen (2004)). For all (r,t), $\hat{F}(r,t)$ for the data is between the envelopes, so the plot is in favour of the hypothesis of spatio-temporal independence in the clusters.

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References

Baddeley, A., Møller, J. & Waagepetersen, R. (2000). Non- and semi-parametric estimation of interaction in inhomogeneous point patterns, *Statistica Neerlandica* **54**: 329–350.

Baddeley, A. & Turner, R. (2005). Spatstat: an R package for analyzing spatial point patterns, *Journal of Statistical Software* **12**: 1–42.

Baddeley, A. & Turner, R. (2006). Modelling spatial point patterns in R, *in* A. Baddeley, P. Gregori, J. Mateu, R. Stoica & D. Stoyan (eds), *Case Studies in Spatial Point Process Modeling*, Springer Lecture Notes in Statistics 185, Springer-Verlag, New York, pp. 23–74.

Berman, M. & Diggle, P. (1989). Testing for spatial association between a point pattern and another stochastic process, *Journal of Royal Statistical Society Series B* **51**: 81–92.

Besag, J. E. (1977). Discussion of the paper by Ripley (1977), *Journal of the Royal Statistical Society Series B* **39**: 193–195.

Brix, A. & Chadoeuf, J. (2002). Spatio-temporal modeling of weeds and shot-noise G Cox processes, *Biometrical Journal* **44**: 83–99.

Brix, A. & Diggle, P. J. (2001). Spatio-temporal prediction for log-Gaussian Cox processes, *Journal of the Royal Statistical Society Series B* **63**: 823–841.

Brix, A. & Diggle, P. J. (2003). Corrigendum: Spatio-temporal prediction for log-Gaussian Cox processes, *Journal of the Royal Statistical Society Series B* **65**: 946.

Brix, A. & Kendall, W. S. (2002). Simulation of cluster point processes without edge effects, *Advances in Applied Probability* **34**: 267–280.

Brix, A. & Møller, J. (2001). Space-time multitype log Gaussian Cox processes with a view to modelling weed data, *Scandinavian Journal of Statistics* **28**: 471–488.

Cox, D. R. (1955). Some statistical models related with series of events, *Journal of the Royal Statistical Society Series B* **17**: 129–164.

Daley, D. J. & Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes. Volume I: Elementary Theory and Methods*, second edn, Springer-Verlag, New York.

Daley, D. J. & Vere-Jones, D. (2008). *An Introduction to the Theory of Point Processes. Volume II: General Theory and Structure*, second edn, Springer-Verlag, New York.

Diggle, P. (1985). A kernel method for smoothing point process data, *Applied Statistics* **34**: 138–147.

Diggle, P. J. (2003). *Statistical Analysis of Spatial Point Patterns*, second edn, Arnold, London.

Diggle, P. J. (2006). Spatio-temporal point processes, partial likelihood, foot and mouth disease, *Statistical Methods in Medical Research* **15**: 325–336.

Diggle, P. J. (2007). Spatio-temporal point processes: methods and applications, *in* B. Finkenstädt, L. Held & V. Isham (eds), *Statistical Methods for Spatio-Temporal Systems*, Chapman & Hall/CRC, Boca Raton, pp. 1–45.

Diggle, P. J., Chetwynd, A. G., Häggkvist, R. & Morris, S. (1995). Second-order analysis of space-time clustering, *Statistical Methods in Medical Research* **4**: 124–136.

Diggle, P. J. & Gabriel, E. (2010). Spatio-temporal point processes, *in* A. E. Gelfand, P. J. Diggle, P. Guttorp & M. Fuentes (eds), *Handbook of Spatial Statistics*, CRC Press, Boca Raton, pp. 451–464.

Diggle, P. J., Rowlingson, B. & Su, T. (2005). Point process methodoly for on-line spatio-temporal disease surveillance, *Environmetrics* **16**: 423–434.

Gabriel, E. & Diggle, P. J. (2009). Second-order analysis of inhomogeneous spatio-temporal point process data, *Statistica Neerlandica* **63**: 43–51.

Gabriel, E., Rowlingson, B. & Diggle, P. J. (2010). Plotting and simulating spatio-temporal point patterns. Manuscript.

Illian, J., Penttinen, A., Stoyan, H. & Stoyan, D. (2008). *Statistical Analysis and Modelling of Spatial Point Patterns*, John Wiley and Sons, Chichester.

Keeling, M., Woolhouse, M. E. J., Shaw, D. J., Matthews, L., Chase-Topping, M., Haydon, D. T., Cornell, S. J., Kappey, J., Wilesmith, J. & Grenfell, B. T. (2001). Dynamics of the 2001 UK foot-and-mouth epidemic dispersal in a heterogeneous landscape, *Science* **294**: 813–817.

Markovich, N. (2007). *Nonparametric Analysis of Univariate Heavy-Tailed Data: Research and Practice*, John-Wiley and Sons, Chichester.

Møller, J. (2003). Shot noise Cox processes, *Advances in Applied Probability* **35**: 4–26.

Møller, J. & Diaz-Avalos, C. (2010). Structured spatio-temporal shot-noise Cox point process models, with a view to modelling forest fires, *Scandinavian Journal of Statistics* **37**: 2–25.

Møller, J., Syversveen, A. R. & Waagepetersen, R. P. (1998). Log Gaussian Cox processes, *Scandinavian Journal of Statistics* **25**: 451–482.

Møller, J. & Torrisi, G. L. (2005). Generalised shot noise Cox processes, *Advances in Applied Probability* **37**: 48–74.

Møller, J. & Waagepetersen, R. P. (2004). *Statistical Inference and Simulation for Spatial Point Processes*, Chapman and Hall/CRC, Boca Raton.

Møller, J. & Waagepetersen, R. P. (2007). Modern spatial point process modelling and inference (with discussion), *Scandinavian Journal of Statistics* **34**: 643–711.

Ripley, B. D. (1976). The second-order analysis of stationary point processes, *Journal of Applied Probability* **13**: 255–266.

Ripley, B. D. (1977). Modelling spatial patterns (with discussion), *Journal of the Royal Statistical Society Series B* **39**: 172–212.

Rowlingson, B. & Diggle, P. J. (2010). Splancs: spatial and space-time point pattern analysis. austria: R Development Core Team (R package version 2.01-27).

Schoenberg, F. P., Brillinger, D. R. & Guttorp, P. M. (2002). Point processes, spatial-temporal, *in* A. El-Shaarawi & W. Piegorsch (eds), *Encyclopedia of Environmetrics*, Vol. 3, Wiley, New York, pp. 1573–1577.

Stoyan, D. & Stoyan, H. (1994). *Fractals, Random Shapes and Point Fields*, Wiley, Chichester.