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GENERALISED SHOT NOISE COX PROCESSES

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Abstract

We introduce a new class of Cox cluster processes called generalised shot-noise Cox processes (GSNCPs), which extends the definition of shot noise Cox processes (SNCPs) in two directions: the point process which drives the shot noise is not necessarily Poisson, and the kernel of the shot noise can be random. Thereby a very large class of models for aggregated or clustered point patterns is obtained. Due to the structure of GSNCPs, a number of useful results can be established. We focus first on deriving summary statistics for GSNCPs and next on how to make simulation for GSNCPs. Particularly, results for first and second order moment measures, reduced Palm distributions, the J-function, simulation with or without edge effects, and conditional simulation of the intensity function driving a GSNCP are given. Our results are exemplified for special important cases of GSNCPs, and we discuss the relation to corresponding results for SNCPs.

Keywords: Cluster process; conditional simulation; Cox process; edge effect; generalised shot noise G Cox process; geometric ergodicity; Markov point process; Metropolis-Hastings algorithm; Palm distribution; perfect simulation; shot noise Cox process; spatial point process; summary statistics

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1. Introduction

One of the most important and versatile class of point process models for clustered point patterns is the class of Cox process models, see e.g. [8, 9, 11, 42]. Recently, simulation-based inference for certain families of Cox processes has been studied in great detail [4, 5, 7, 22, 30, 31, 32, 33, 46]. As explained in Section 2 this research has shown the need for extending existing model classes for Cox processes, and the present paper is therefore concerned with a new rich class of Cox process models, which to some extend is tractable for mathematical analysis and particularly for simulation-based inference.

Recall that a point process X on $\mathbb{R}^d$ is a Cox process driven by a random field $Z(\xi) \geq 0$, $\xi \in \mathbb{R}^d$, if $X \mid Z$ is a Poisson process on $\mathbb{R}^d$ with intensity function Z. Throughout this paper X denotes a Cox process driven by a random field of the form

$$Z(\xi) = \sum_j \gamma_j k_{b_j}(c_j, \xi) \tag{1}$$

where

- the (c_j, b_j, γ_j) are the points of a point process Φ on $\mathbb{R}^d \times (0,\infty) \times (0,\infty)$,
- we identify $\sum_j$ and $\sum_{(c_j,bj,\gamma_j)\in\Phi}$,
- and $b_j > 0$ is a bandwidth for the kernel $k_{b_j}(c_j, \cdot)$:

$$k_{b_j}(c_j, \xi) = k_1(c_j/b_j, \xi/b_j)/b_j^d \tag{2}$$

  where $k_1(c_j, \cdot)$ is a density with respect to Lebesgue measure on $\mathbb{R}^d$.

We call X a *generalised shot noise Cox process (GSNCP)*, since a *shot noise Cox process (SNCP)* is the special case with $\{(c_j, \gamma_j)\}$ a Poisson process on $\mathbb{R}^d \times (0,\infty)$ and all b_j equal and fixed, cf. Møller [29] (extending previous work by Brix [3] and Wolpert and Ickstadt [46]). We can also view a GSNCP as a Cox cluster process, since $X \mid \Phi$ is distributed as the superposition $\bigcup_j X_j$ of independent Poisson processes X_j with intensity functions $\gamma_j k_{b_j}(c_j, \cdot)$. In applications of (2) the kernel $k_1(c_j, \cdot)$ is usually concentrated around c_j, so we call $\Phi_{\text{cent}} = \{c_j\}$ the centre (or mother) process and X_j the cluster (or daughter process) with centre (or mother) c_j, intensity γ_j, and dispersion density $k_{b_j}(c_j, \cdot)$. Many of the results in this paper can easily be modified to hold for a general stochastic kernel $k_{b_j}(c_j, \cdot)$ where b_j is a random variable with state space given by an arbitrary metric space E, but for specificity and ease of presentation we let $E = (0, \infty)$ and interpret b_j as a bandwidth.

The focus in this paper is on the probabilistic aspects of GSNCPs, with a view to statistical applications. Although GSNCPs are more complicated than SNCPs, we show that due to the structure of GSNCPs, still a number of useful and general results can be established. Our results are exemplified for special important cases of GSNCPs, and we discuss the relation to corresponding results for SNCPs.

The paper is organised as follows. Section 2 motivates the introduction of GSNCPs and contains some other preliminaries, including examples of GSNCPs. Section 3 concerns results for summary statistics and reduced Palm distributions of GSNCPs. Particularly, results for first and second order properties and the J-function are obtained. Section 4 deals with algorithms for simulation of a GSNCP X within a bounded window W, and for conditional simulation of Φ given the restriction of X to W. We quantify the effect of ignoring edge effects in a straightforward simulation algorithm for X when e.g. Φ_{cent} is restricted to a bounded extended window $W_{\text{ext}} \supset W$. We also describe a perfect simulation algorithm inspired by the work of Brix and Kendall [6]. Further, for a certain Metropolis-Hastings algorithm for conditional simulation of Φ given $X \cap W$, we discuss convergence properties, particularly we establish geometric ergodicity. Finally, Section 5 contains a brief discussion of future research and the importance of our results for statistical inference.

2. Background

2.1. Motivation

Our extension of SNCPs to GSNCPs is motivated by different statistical applications:

(a) For Neyman-Scott processes (see Example 2 in Section 2.3) and many other Cox cluster processes used for statistical analysis (see e.g. [11, 33, 42]), Φ_{cent} is assumed to be a Poisson process and the (b_j, γ_j) are assumed to be equal to an unknown parameter. As illustrated by van Lieshout and Baddeley [22], a repulsive Markov point process model for Φ_{cent} may be more relevant in many situations. Such models may be described by GSNCPs but not by SNCPs.

(b) Wolpert and Ickstadt [46] considered a fully Bayesian analysis for a Poisson-gamma model (a particular case of an SNCP, see Example 3 in Section 2.3): a prior for the parameter in a parametric model for the intensity function of the Poisson process Φ is imposed, whereby X becomes a GSNCP.

(c) Møller and Waagepetersen [33] considered likelihood inference for parametric models of SNCPs, where the likelihood is in general not available in closed form and the underlying random field Z is unobserved. For different fixed values of the bandwidths, they combined Markov chain Monte Carlo (MCMC) methods for calculating maximum partial likelihood estimates and likelihood ratios, whereby an approximate MLE was obtained (see Section 10.3 in [33]). This procedure is rather time consuming, and it would be easier to use a Bayesian MCMC approach, imposing a prior on the bandwidth. Then a GSNCP model (with all b_j equal but random) is obtained for the likelihood term in the posterior density.

(d) SNCPs are often claimed to allow a certain degree of flexibility [28, 31, 38, 46]. In an SNCP, for the random intensity function Z in (1), the random cluster intensity γ_j is scaling the kernel $k_b(c_j, \cdot)$ where $b > 0$ is the fixed bandwidth. However, the same degree of flexibility for modelling Z may be obtained by letting the γ_j be equal but using different random bandwidths b_j. Even more flexibility is of course obtained when both the γ_j and the b_j are random as in a GSNCP.

(e) Kingman [19] (see also Section 5.5 in [11]) considered a model for reproducing individuals, where the $(n+1)$th generation G_{n+1} given the nth generation G_n (and previous generations $G_0, \ldots, G_{n-1}$) is a Poisson process with an intensity measure of the form (1), where the cluster centres are given by G_n, each γ_j is a function of (c_j, G_n), and the bandwidths are equal and fixed. Thus G_{n+1} becomes a GSNCP but not necessarily an SNCP. Furthermore, assuming that G_0 is a Poisson process, the superposition of GSNCPs $G_0, G_1, \ldots$, can be interpreted as a spatial Hawkes process (extending the definition in [15, 16] for $d = 1$ to $d \geq 1$ and allowing a more general structure of the conditional intensity function).

We consider in particular the following models for Φ. Cox process models for Φ appear naturally when we consider a hierarchical model where $\Phi \mid \theta$ is a Poisson process with an intensity measure ζ_θ depending on a random variable θ; see points (b) and (d) above and Example 3 in Section 2.3. The case where Φ_{cent} and $\{(b_j, \gamma_j)\}$ are independent is particularly tractable, and in our opinion the most important models for Φ_{cent} are then Poisson models and Markov point process models; see point (a) above and Example 4 in Section 2.3. Markov point process models provide flexible models for inhibition between the cluster centres, while Poisson models provide a mathematical convenient framework, cf. [29, 33].

2.2. Assumptions and other preliminaries

This section specifies certain conditions which are assumed throughout the text. Also some terminology and notation are introduced.

We assume that Φ is a random locally finite subset of $\Omega = \mathbb{R}^d \times (0,\infty) \times (0,\infty)$, i.e. $\Phi_D \equiv \Phi \cap D$ is finite whenever $D \subset \Omega$ is bounded. Moreover, Z is assumed to be almost surely locally integrable, i.e. with probability 1, $\int_B Z(\xi)\mathrm{d}\xi < \infty$ for bounded Borel sets $B \subset \mathbb{R}^d$. Hence, for any bounded $B \subset \mathbb{R}^d$, $X_B \equiv X \cap B$ is almost surely finite, and therefore only finitely many clusters X_j have points in B. Note that the centre process Φ_{cent} is countable, but not necessarily locally finite (see Example 3 in Section 2.3).

If Φ_{cent} is stationary (i.e. its distribution is invariant under translations in $\mathbb{R}^d$) and $k_1(c,\xi) = k_1(\xi - c)$ is invariant under translations in $\mathbb{R}^d$, then Z and hence X is stationary. If furthermore the distribution of Φ_{cent} is invariant under motions in $\mathbb{R}^d$ and k_1 is isotropic, i.e. $k_1(c,\xi)$ depends only on the distance $\|c-\xi\|$, then the distributions of Z and X are invariant under motions in $\mathbb{R}^d$. Two frequently used choices of isotropic kernels are the Gaussian kernel

$$k_1(\xi) = (2\pi)^{-d/2} \exp\big(-\|\xi\|^2/2\big) \tag{3}$$

and the uniform kernel

$$k_1(\xi) = \mathbf{1}[\|\xi\| \le 1]/\omega_d \tag{4}$$

where $\mathbf{1}[\cdot]$ denotes the indicator function and $\omega_d = \pi^{d/2}/\Gamma(1+d/2)$.

We shall often refer to the intensity measure and the second order reduced moment measure of Φ, which we denote by ζ and $\zeta^{(2)}$, respectively. Recall that for measurable functions $h_1 : \Omega \mapsto [0, \infty)$ and $h_2 : \Omega \times \Omega \mapsto [0, \infty)$,

$$\int h_1(c, b, \gamma)\,\mathrm{d}\zeta(c, b, \gamma) = \mathrm{E}\sum_j h_1(c_j, b_j, \gamma_j) \tag{5}$$

and

$$\int h_2((c, b, \gamma), (c', b', \gamma'))\,\mathrm{d}\zeta^{(2)}((c, b, \gamma), (c', b', \gamma')) = \mathrm{E}\sum_{j \neq j'} h_2((c_j, b_j, \gamma_j), (c_{j'}, b_{j'}, \gamma_{j'})) \tag{6}$$

see e.g. [42]. In applications, ζ is often of the form

$$\zeta(D) = \int \int_D \mathrm{d}c\,\mathrm{d}\chi(b, \gamma) \tag{7}$$

for Borel sets $D \subseteq \Omega$, where χ is a measure on $(0, \infty) \times (0, \infty)$, cf. Examples 2–4 in Section 2.3.

2.3. Examples

The following examples describe important model classes of GSNCPs. As the measures ζ and $\zeta^{(2)}$ introduced above play an important role in Section 3, we specify these measures in the examples.

Example 1. A particular tractable model class is obtained when Φ is a Poisson process with locally finite intensity measure ζ. This class contains the SNCPs (the special case where all bandwidths are equal and fixed). Note that $\zeta^{(2)} = \zeta \times \zeta$ is just a product measure, and (7) is equivalent to stationarity of Φ_{cent}.

Example 2. A Neyman-Scott process [35] is obtained when Φ_{cent} is a stationary Poisson process on $\mathbb{R}^d$ and the cluster intensities $\gamma_j = \gamma$ and the bandwidths $b_j = b$ are equal and fixed. For the Gaussian kernel (3) we have a (modified) Thomas process [44], and for the uniform kernel (4) we have a Matérn cluster process [24, 25].

Natural extensions of this model include GSNCPs obtained if Φ_{cent} is a stationary point process on $\mathbb{R}^d$ with intensity $\rho_{\mathrm{cent}} < \infty$, Φ_{cent} is independent of the (b_j, γ_j), and either

(i) the $(b_j, \gamma_j) = (b, \gamma)$ are identical with distribution Q, or

(ii) the $b_j = b$ are identical with distribution Q_1, the γ_j are i.i.d. with distribution Q_2, and b and $\{\gamma_j\}$ are independent, or

(iii) the (b_j, γ_j) are i.i.d. with distribution Q.

We call such extensions for *generalised Neyman-Scott processes.* Then (7) holds where

$$\chi = \rho_{\text{cent}} Q \tag{8}$$

with $Q = Q_1 \times Q_2$ in case (ii). This follows by conditioning on Φ_{cent} in the right hand side of (5) and using standard arguments, where we first let $h_1(c, b, \gamma)$ in (5) be an indicator function $\mathbf{1}[c \in A, b \in B, \gamma \in C]$. Furthermore, using similar arguments and (6), we obtain

$$\int h_2((c, b, \gamma), (c', b', \gamma'))\, \mathrm{d}\zeta^{(2)}((c, b, \gamma), (c', b', \gamma')) \tag{9}$$

$$= \begin{cases} \int\int h_2((c, b, \gamma), (c', b, \gamma))\, \mathrm{d}Q(b, \gamma)\, \mathrm{d}\zeta^{(2)}_{\text{cent}}(c, c') & \text{in case (i)} \\ \int\int\int\int h_2((c, b, \gamma), (c', b, \gamma'))\, \mathrm{d}Q_1(b)\, \mathrm{d}Q_2(\gamma)\, \mathrm{d}Q_2(\gamma')\, \mathrm{d}\zeta^{(2)}_{\text{cent}}(c, c') & \text{in case (ii)} \\ \int\int\int h_2((c, b, \gamma), (c', b', \gamma'))\, \mathrm{d}Q(b, \gamma)\, \mathrm{d}Q(b', \gamma')\, \mathrm{d}\zeta^{(2)}_{\text{cent}}(c, c') & \text{in case (iii)} \end{cases}$$

where $\zeta^{(2)}_{\text{cent}}$ denotes the second order reduced moment measure for Φ_{cent}. Especially, if Φ_{cent} is a stationary Poisson process, then $\mathrm{d}\zeta^{(2)}_{\text{cent}}(c, c') = \rho^2_{\text{cent}}\, \mathrm{d}c\, \mathrm{d}c'$ (see e.g. [42]).

Example 3. Suppose that $\{(c_j, \gamma_j)\}$ is a Poisson process on $\mathbb{R}^d \times (0, \infty)$ with intensity function

$$\lambda_\theta(c, \gamma) = \lambda_\theta(\gamma) = \kappa\gamma^{-\alpha-1} \exp(-\tau\gamma)/\Gamma(1 - \alpha) \tag{10}$$

where $\theta = (\alpha, \kappa, \tau)$ is a parameter with $\alpha < 1$, $\kappa > 0$, and $\tau > 0$ (these restrictions are equivalent to local integrability of λ_θ). When all the b_j in (1) are equal and fixed, X is called a shot noise G Cox process (SNGCP) [3, 29]; a Poisson-gamma process [46] is the special case $\alpha = 0$. In the stationary case, a SNGCP has intensity equal to $\kappa\tau^{\alpha-1}$.

We shall later refer to the following properties of the Poisson process $\{(c_j, \gamma_j)\}$. The two point processes $\{c_j\}$ and $\{\gamma_j\}$ are independent, and their distributions depend on the value of α as follows.

(i) $\alpha < 0$: Then Φ_{cent} is a stationary Poisson process with intensity $-\kappa\tau^\alpha/\alpha$, and the γ_j are independent and gamma distributed with shape parameter $-\alpha$ and inverse scale parameter τ.

(ii) $0 \le \alpha < 1$: Then Φ_{cent} is not locally finite. However, $\{(c_j, \gamma_j) : c_j \in A\}$ and $\{(c_j, \gamma_j) : c_j \in B\}$ are independent for disjoint Borel sets $A, B \subset \mathbb{R}^d$. If A has finite Lebesgue measure $|A|$, then the points in $\Phi_{\mathrm{cent}} \cap A$ are independent and uniformly distributed on A, and the corresponding γ_j form an inhomogeneous Poisson process on $(0,\infty)$ with intensity function $|A|\lambda_\theta(\gamma)$.

Generalised shot noise G Cox processes can naturally be obtained in different ways. For specificity, let π be a distribution imposed on the parameter $\theta = (\alpha, \kappa, \tau)$, and let $\{(c_j, \gamma_j)\} \mid \theta$ be a Poisson process with intensity function λ_θ, cf. [46]. Further, consider (1) with $\{b_j\}$ and $(\theta, \{(c_j, \gamma_j)\})$ independent, and the b_j either identical or i.i.d. with distribution ν. Then we obtain a GSNCP where (7) holds with

$$\chi(A) = \int \int_A \mathrm{E}\lambda_\theta(\gamma)\, \mathrm{d}\nu(b)\, \mathrm{d}\gamma \tag{11}$$

for Borel sets $A \subseteq (0,\infty) \times (0,\infty)$, where the expectation is with respect to π (this follows by similar arguments as in Example 2). Furthermore,

$$\begin{aligned}&\int h_2((c,b,\gamma),(c',b',\gamma'))\, \mathrm{d}\zeta^{(2)}((c,b,\gamma),(c',b',\gamma'))\\ &= \int\int\int\int\int h_2((c,b,\gamma),(c',b,\gamma'))\mathrm{E}\big(\lambda_\theta(\gamma)\lambda_\theta(\gamma')\big)\, \mathrm{d}c\, \mathrm{d}c'\, \mathrm{d}\nu(b)\, \mathrm{d}\gamma\, \mathrm{d}\gamma'\end{aligned} \tag{12}$$

when the b_j are identical, and

$$\begin{aligned}&\int h_2((c,b,\gamma),(c',b',\gamma'))\, \mathrm{d}\zeta^{(2)}((c,b,\gamma),(c',b',\gamma'))\\ &= \int\int\int\int\int\int h_2((c,b,\gamma),(c',b',\gamma'))\mathrm{E}\big(\lambda_\theta(\gamma)\lambda_\theta(\gamma')\big)\, \mathrm{d}c\, \mathrm{d}c'\, \mathrm{d}\nu(b)\, \mathrm{d}\nu(b')\, \mathrm{d}\gamma\, \mathrm{d}\gamma'\end{aligned} \tag{13}$$

when the b_j are i.i.d.

Example 4. As in van Lieshout and Baddeley [22] suppose that Φ_{cent} is a finite Markov (or Gibbs) point process [1, 20, 33, 39, 40] defined on a bounded Borel set $B \subset \mathbb{R}^d$. This means that Φ_{cent} has a density p with respect to the unit rate Poisson process on B so that for finite subsets $\mathbf{c} \subset B$,

$$p(\mathbf{c}) = \prod_{y \subseteq \mathbf{c}} \varphi(y). \tag{14}$$

Here $\varphi(y) \ge 0$ is a so-called interaction function such that $\varphi(y) = 1$ whenever y contains two points larger than R units apart, where $R < \infty$ is a parameter specifying the range of interaction. Note that if $p(\mathbf{c}) > 0$ and we define the so-called Papangelou conditional intensity by $\lambda^*(\mathbf{c}, \xi) \equiv p(\mathbf{c} \cup \{\xi\})/p(\mathbf{c})$, then

$$\lambda^*(\mathbf{c}, \xi) = \prod_{y \subseteq \mathbf{c} \cap b(\xi, R)} \varphi(y \cup \{\xi\}) \tag{15}$$

where $b(\xi, R)$ denotes the ball in $\mathbb{R}^d$ with centre ξ and radius R.

In [22] it is also assumed that $Z(\xi) = \epsilon + \sum_j h(\xi|c_j)$ where $h(\xi|c_j)$ is a nonnegative integrable function and $\epsilon > 0$ is a parameter. If instead $\epsilon = 0$, we have a GSNCP with non-random (γ_j, b_j), but clearly (7) does not hold, since Φ_{cent} is contained in B. If moreover Φ_{cent} is independent of the (b_j, γ_j), which are either i.i.d. or identical with distribution Q, then

$$\int h_1(c, b, \gamma)\, \mathrm{d}\zeta(c, b, \gamma) = \int \int h_1(\xi, b, \gamma) \mathrm{E}\lambda^*(\Phi_{\text{cent}}, \xi)\, \mathrm{d}\xi\, \mathrm{d}Q(b, \gamma). \tag{16}$$

This follows from the fact that Φ_{cent} has intensity function $\rho_{\text{cent}}(\xi) = \mathrm{E}\lambda^*(\Phi_{\text{cent}}, \xi)$ (see Proposition 6.2 in [33]), but a closed form expression for ρ_{cent} is in general unknown (except in the Poisson case where $\lambda^*(\Phi_{\text{cent}}, \xi)$ does not depend on Φ_{cent}). Furthermore,

$$\int h_2((c, b, \gamma), (c', b', \gamma'))\, \mathrm{d}\zeta^{(2)}((c, b, \gamma), (c', b', \gamma')) = \int \int \int \int h_2((c, b, \gamma), (c', b, \gamma'))$$
$$\mathrm{E}\Bigg(\prod_{y \subseteq \Phi_{\text{cent}} \cap (b(c,R) \cup b(c',R))} \varphi(y \cup \{c\}) \varphi(y \cup \{c'\}) \varphi(y \cup \{c, c'\}) \Bigg) \mathrm{d}c\, \mathrm{d}c'\, \mathrm{d}Q(b, \gamma)\, \mathrm{d}Q(b', \gamma') \tag{17}$$

see e.g. Proposition 6.2 in [33]. Also $\zeta^{(2)}$ is not known on closed form.

Suppose we instead consider a stationary Markov (or Gibbs) point process Φ_{cent} defined on $\mathbb{R}^d$, with an interaction function φ of finite range of interaction R, where φ is invariant under translations in $\mathbb{R}^d$ [12, 33, 36, 37, 41]. Briefly, such a point process can be specified by a Papangelou conditional intensity $\lambda^*(\mathbf{c}, \xi)$ of the form (15) but now defined for locally finite subsets $\mathbf{c} \subset \mathbb{R}^d$ and points $\xi \in \mathbb{R}^d \setminus \mathbf{c}$ (for details, see the abovementioned references). If Φ_{cent} is independent of the (b_j, γ_j), which are either i.i.d. or identical with distribution Q, then we have a GSNCP. Here (7) holds with χ of the form (8) provided

$$\rho_{\text{cent}} = \mathrm{E}\lambda^*(\Phi_{\text{cent}}, 0) = \mathrm{E} \prod_{y \subseteq \Phi_{\text{cent}} \cap b(0,R)} \varphi(y \cup \{0\}) \tag{18}$$

is finite. However, a closed form expression for ρ_{cent} is in general not known. Similarly for $\zeta^{(2)}$, which is still given by (17).

3. Summary statistics

This section deals with summary statistics such as the intensity function, the pair correlation function, and the J-function for GSNCPs.

3.1. First- and second-order characteristics

Expressions for the product moments $\mathrm{E}[Z(\xi_1)\cdots Z(\xi_n)]$ in terms of the moment measures for Φ can be obtained in a similar way as in [29] by using the Slivnyak-Mecke theorem for the Poisson process $X \mid \Phi$. In this paper we concentrate on the two most fundamental summary statistics, namely the *intensity function* $\rho(\xi) = \mathrm{E}Z(\xi)$ and the *pair correlation function* $g(\xi, \eta) = \mathrm{E}[Z(\xi)Z(\eta)]/[\rho(\xi)\rho(\eta)]$ (provided that the means exist, and taking $\frac{0}{0} = 0$). In the examples below we discuss to what extent closed form expressions for $\rho(\xi)$ and $g(\xi, \eta)$ can be derived for GSNCPs as introduced in Examples 1–4 in Section 2.3.

Proposition 1. *The intensity function exists and is given by*

$$\rho(\xi) = \int \gamma k_b(c, \xi)\,\mathrm{d}\zeta(b, c, \gamma) \tag{19}$$

provided that the integral is finite for all $\xi \in \mathbb{R}^d$.

Proof. Follows immediately from (5).

Example 5. By (19), if (7) holds and $k_1(c, \xi) = k_1(\xi - c)$ is invariant under translations, $\rho = \rho(\xi)$ is given by

$$\rho = \int \gamma\,\mathrm{d}\chi(b, \gamma). \tag{20}$$

This reduces as follows for the GSNCPs in Examples 2 and 3: For a generalised Neyman-Scott process, $\rho = \rho_{\mathrm{cent}}\mathrm{E}\gamma$, where the mean is with respect to $(b, \gamma) \sim Q$, cf. (8). For a generalised shot noise G Cox processes, $\rho = \mathrm{E}\big(\kappa\tau^{\alpha-1}\big)$, cf. (11).

However, for the Markov point process setting considered in Example 4, a closed form expression of $\rho(\xi)$ is in general unknown, since $\mathrm{E}\lambda^*(\Phi_{\mathrm{cent}}, \xi)$ in (16) or (18) is not known on closed form.

Proposition 2. *The pair correlation function exists and is given by*

$$g(\xi, \eta) = \frac{\beta_1(\xi, \eta) + \beta_2(\xi, \eta)}{\rho(\xi)\rho(\eta)}$$

provided that $\rho(\xi) < \infty$ *for all* $\xi \in \mathbb{R}^d$ *and the integrals*

$$\beta_1(\xi,\eta) = \int \gamma\gamma' k_b(c,\xi) k_{b'}(c',\eta)\,\mathrm{d}\zeta^{(2)}((c,b,\gamma),(c',b',\gamma'))$$

and

$$\beta_2(\xi,\eta) = \int \gamma^2 k_b(c,\xi) k_b(c,\eta)\,\mathrm{d}\zeta(c,b,\gamma)$$

are finite for all $\xi,\eta \in \mathbb{R}^d$.

Proof. Follows along similar lines as in the proof of Proposition 1 in [29] but using (6).

Example 6. Suppose that $k_1(c,\xi) = k_1(\xi - c)$ is invariant under translations.

If Φ is a Poisson process so that Φ_{cent} is stationary and ρ given by (20) is finite, then by Example 1 and Proposition 2,

$$g(\xi,\eta) = 1 + \beta_2(\xi,\eta)/\rho^2 \tag{21}$$

where

$$\beta_2(\xi,\eta) = \int \gamma^2 \phi_b(\xi - \eta)\,\mathrm{d}\chi(b,\gamma)$$

and

$$\phi_b(\xi) = \int k_b(c) k_b(\xi + c)\,\mathrm{d}c.$$

For the Gaussian kernel (3), this reduces to

$$\phi_b(\xi) = \frac{1}{(4\pi b)^{d/2}} \exp\left(-\frac{\|\xi\|^2}{4b^2}\right).$$

Consider a generalised Neyman-Scott process when Φ_{cent} is a stationary Poisson process with intensity $\rho_{\text{cent}} < \infty$. Then

$$\beta_2(\xi,\eta) = \rho_{\text{cent}} \mathrm{E}\big(\gamma^2 \phi_b(\xi - \eta)\big)$$

where the mean is with respect to $(b,\gamma) \sim Q$. For each of the cases (i)–(iii) in Example 2 we obtain the following from (9) and Proposition 2. In the case (i), $\beta_1(\xi,\eta) = \rho_{\text{cent}}^2 \mathrm{E}\big(\gamma^2\big)$, so

$$g(\xi,\eta) = \frac{\mathrm{E}\big(\gamma^2\big) + \mathrm{E}\big(\gamma^2 \phi_b(\xi - \eta)\big)/\rho_{\text{cent}}}{\big(\mathrm{E}\gamma\big)^2}.$$

In the case (ii), $\beta_1(\xi,\eta)=\rho^2$, so

$$g(\xi,\eta)=1+\frac{\mathrm{E}\big(\gamma^2\phi_b(\xi-\eta)\big)}{\rho_{\mathrm{cent}}\big(\mathrm{E}\gamma\big)^2}. \tag{22}$$

The case (iii) is just the special case of the Poisson case above with $\mathrm{d}\chi(b,\gamma)=\rho_{\mathrm{cent}}\,\mathrm{d}b\,\mathrm{d}\gamma$, and g is again of the form (22).

Consider next the specific example of a generalised shot noise G Cox process in Example 3. By (12) and (13), both when the b_j are identical and when the b_j are i.i.d., it is straightforwardly derived that g is of the form (21) with

$$\beta_2(\xi,\eta)=\mathrm{E}\big((1-\alpha)\kappa\tau^{\alpha-2}\big)\mathrm{E}\phi_b(\xi-\eta)$$

where the means now are with respect to the independent random variables b and $\theta=(\alpha,\kappa,\tau)$.

Note that $g\geq 1$ for the GSNCPs in Example 6. This is in accordance with the usual interpretation that $g\geq 1$ indicates aggregation of the points in X [33, 42]. It seems to be an open problem to what extend $g\geq 1$ for the Markov point process setting considered in Example 4. Most Markov point process models are repulsive, that is, $\varphi(y)\leq 1$ whenever $\mathrm{card}(y)\geq 2$ (see e.g. [33]). In the special case of no interaction, i.e. $\varphi(y)=1$ whenever $\mathrm{card}(y)\geq 2$, we clearly have that $g=1$. So in the repulsive case, since aggregation in X is expected to be more pronounced than if there is no interaction, one may conjecture that $g(\xi,\eta)\geq 1$, at least when ξ and η are sufficiently close. However, we have not succeeded in verify this by combining (16)–(18) and Proposition 2.

3.2. Reduced Palm distributions and J-functions

A simple description of the reduced Palm distribution of an SNCP was established in [29]. This section extends this to GSNCPs and discusses how to use this for deriving certain properties of van Lieshout and Baddeley's J-function [21].

We first need some additional notation and assumptions. Denote the state space of X by N_{lf}, the set of locally finite subsets of $\mathbb{R}^d$. Let N_{lf} be equipped with the σ-field $\mathcal{N}_{\mathrm{lf}}$ generated by the sets $F_{B,n}=\{x\in N_{\mathrm{lf}}:\mathrm{card}(x\cap B)=n\}$ for $n=0,1,\dots$ and bounded Borel sets $B\subset\mathbb{R}^d$. Let the product space $N_{\mathrm{lf}}\times\mathbb{R}^d$ be equipped with the product σ-field $\mathcal{N}_{\mathrm{lf}} \otimes \mathcal{B}^d$, where $\mathcal{B}^d$ is the Borel σ-field on $\mathbb{R}^d$. Similarly, for the point process Φ we define σ-fields by replacing $\mathbb{R}^d$ above with $\Omega = \mathbb{R}^d \times (0,\infty) \times (0,\infty)$.

Assume that the intensity function $\rho(\xi)$ for X exists for Lebesgue almost all $\xi \in \mathbb{R}^d$, cf. Proposition 1. Recall that the reduced Palm distribution $P^!_\xi$ of X at $\xi \in \mathbb{R}^d$ and the reduced Palm distribution $P^!_{(c,b,\gamma)}$ of Φ at $(c,b,\gamma) \in \mathbb{R}^d \times (0,\infty) \times (0,\infty)$ are given by the Campbell-Mecke formula: for Lebesgue almost all $\xi \in \mathbb{R}^d$ with $\rho(\xi) > 0$ and for ζ almost all $(c,b,\gamma) \in \mathbb{R}^d \times (0,\infty) \times (0,\infty)$, $P^!_\xi$ and $P^!_{(c,b,\gamma)}$ are uniquely given by

$$\mathrm{E} \sum_{\xi \in X} f(X \setminus \{\xi\}, \xi) = \int \int f(x,\xi) \rho(\xi) \, \mathrm{d}P^!_\xi(x) \, \mathrm{d}\xi \tag{23}$$

for nonnegative measurable functions f and

$$\mathrm{E} \sum_{(c,b,\gamma) \in \Phi} h(\Phi \setminus \{(c,b,\gamma)\}, (c,b,\gamma)) = \int \int h(\phi, (c,b,\gamma)) \, \mathrm{d}P^!_{(c,b,\gamma)}(\phi) \, \mathrm{d}\zeta(c,b,\gamma) \tag{24}$$

for nonnegative measurable functions h; see [9, 42].

When $\rho(\xi) > 0$ we define the following. Let

$$Z_\xi(\eta) = \gamma_\xi k_{b_\xi}(c_\xi, \eta), \qquad \eta \in \mathbb{R}^d,$$

where $(c_\xi, b_\xi, \gamma_\xi)$ is a random variable with distribution

$$P((c_\xi, b_\xi, \gamma_\xi) \in D) = \int_D [\gamma k_b(c,\xi)/\rho(\xi)] \, \mathrm{d}\zeta(c,b,\gamma)$$

for Borel sets $D \subseteq \Omega$. Conditional on $(c_\xi, b_\xi, \gamma_\xi)$, let X_ξ and X^ξ denote independent point processes, where X_ξ is a Poisson process on $\mathbb{R}^d$ with intensity function Z_ξ and X^ξ is a GSNCP driven by

$$Z^\xi(\eta) = \sum_{(c,b,\gamma) \in \Phi^{(c_\xi, b_\xi, \gamma_\xi)}} \gamma k_b(c,\eta), \qquad \eta \in \mathbb{R}^d,$$

where $\Phi^{(c,b,\gamma)}$ denotes a point process with law $P^!_{(c,b,\gamma)}$.

Proposition 3. *For Lebesgue almost all $\xi \in \mathbb{R}^d$ with $\rho(\xi) > 0$,*

$$P^!_\xi(F) = P(X^\xi \cup X_\xi \in F), \qquad F \in \mathcal{N}_{\mathrm{lf}}. \tag{25}$$

Proof. By (23) and arguing as in the proof of Proposition 2 in [29], it suffices to show that

$$\mathrm{E} \sum_{\xi \in X} \mathbf{1}[\xi \in A, (X \setminus \{\xi\}) \cap B = \emptyset] = \int_A P((X^\xi \cup X_\xi) \cap B = \emptyset) \rho(\xi) \, \mathrm{d}\xi \tag{26}$$

for bounded $A, B \in \mathcal{B}^d$. It follows also from Proposition 2 in [29] that the left hand side in (26) is given by

$$\int_A \mathrm{E} \sum_{(c,b,\gamma)\in\Phi} \gamma k_b(c,\xi) \exp\left(- \int_B \sum_{(c',b',\gamma')\in\Phi} \gamma' k_{b'}(c',\eta)\,\mathrm{d}\eta \right) \mathrm{d}\xi.$$

By (24),

$$\begin{aligned}
&\mathrm{E} \sum_{(c,b,\gamma)\in\Phi} \gamma k_b(c,\xi) \exp\left(- \int_B \sum_{(c',b',\gamma')\in\Phi} \gamma' k_{b'}(c',\eta)\,\mathrm{d}\eta \right) \\
&= \int\int \gamma k_b(c,\xi) \exp\left(- \int_B \sum_{(c',b',\gamma')\in\phi\cup\{(c,b,\gamma)\}} \gamma' k_{b'}(c',\eta)\,\mathrm{d}\eta \right) \mathrm{d}P^!_{(c,b,\gamma)}(\phi)\,\mathrm{d}\zeta(c,b,\gamma) \\
&= \int \gamma k_b(c,\xi) \exp\left(- \int_B \gamma k_b(c,\eta)\,\mathrm{d}\eta \right) \mathrm{E}\exp\left(- \int_B \sum_{(c',b',\gamma')\in\Phi^{(c,b,\gamma)}} \gamma' k_{b'}(c',\eta)\,\mathrm{d}\eta \right) \\
&\qquad \mathrm{d}\zeta(c,b,\gamma) \\
&= \rho(\xi) P((X^\xi \cup X_\xi)\cap B = \emptyset)
\end{aligned}$$

where we have used the conditional independence of $X^{(\xi)}$ and X_ξ given $(c_\xi, b_\xi, \gamma_\xi)$ to obtain the last equality. Thereby (26) is obtained.

Remark 1. The complication in using Proposition 3 lies in the need of determining $P^!_{(c,b,\gamma)}$. This reduced Palm distribution is particular simple for SNCPs, see Proposition 2 in [29]. Suppose that Φ is a Cox process such that $\Phi\,|\,\theta$ is a Poisson process with intensity function λ_θ, where θ is a random variable. Then

$$\int h(\phi)\,\mathrm{d}P^!_{(c,b,\gamma)}(\phi) = \mathrm{E}[h(\Phi)\lambda_\theta(c,b,\gamma)]/\lambda(c,b,\gamma) \tag{27}$$

for $h \geq 0$ measurable, provided $\lambda(c,b,\gamma) \equiv \mathrm{E}\lambda_\theta(c,b,\gamma) \in (0,\infty)$. This follows from first conditioning on θ in the left hand side of (24) and using the Slivnyak-Mecke theorem, and next by taking expectation and then conditioning on Φ. Suppose instead that Φ_{cent} is a stationary Markov point process (see Example 4) with intensity $\rho_{\mathrm{cent}} > 0$. If we assume for simplicity that the $(b_j, \gamma_j) = (b, \gamma)$ are all equal and fixed, then $P^!_c = P^!_{(c,b,\gamma)}$ is concentrated on N_{lf} and satisfies

$$\int h(\phi)\,\mathrm{d}P^!_c(\phi) = \mathrm{E}[h(\Phi_{\mathrm{cent}})\lambda^*(\Phi_{\mathrm{cent}}, c)]/\rho_{\mathrm{cent}}. \tag{28}$$

This follows from equation (28) in [29] (which, incidentally, should be corrected by replacing $\tilde{\lambda}(x,\xi)$ with $\tilde{\lambda}(x,\xi)/\tilde{\rho}(\xi)$).

We now consider the J-function for a stationary GSNCP with intensity $\rho \in (0,\infty)$, assuming that $k_1(c,\xi) = k_1(\xi - c)$ and that (7) holds.

Corollary 1. *Under the assumptions above, for all* $r \geq 0$*,*

$$J(r) = \frac{1}{\rho I_1(r)} \int\int I_2(c,b,\gamma,r) I_3(c,b,\gamma,r)\,\mathrm{d}c\,\mathrm{d}\chi(b,\gamma) \tag{29}$$

where

$$I_1(r) = \mathrm{E}\exp\left(-\sum_j \int_{b(c_j,r)} \gamma_j k_{b_j}(\xi)\,\mathrm{d}\xi\right),$$

$$I_2(c,b,\gamma,r) = \gamma k_b(-c)\exp\left(-\int_{b(c,r)} \gamma k_b(\eta)\,\mathrm{d}\eta\right),$$

and

$$I_3(c,b,\gamma,r) = \mathrm{E}\exp\left(-\sum_{(c',b',\gamma')\in\Phi^{(c,b,\gamma)}} \int_{b(c',r)} \gamma' k_{b'}(\xi)\,\mathrm{d}\xi\right).$$

Proof. By definition,

$$J(r) = P_0^!(\{x \in N_{\mathrm{lf}} : x \cap b(0,r) = \emptyset\})/P(X \cap b(0,r) = \emptyset).$$

Hence by Proposition 3,

$$\begin{aligned} J(r) &= P((X^0 \cup X_0) \cap b(0,r) = \emptyset)/P(X \cap b(0,r) = \emptyset) \\ &= \frac{1}{\rho P(X \cap b(0,r) = \emptyset)} \int\int P(X^{(0)} \cap b(0,r) = \emptyset | (c_0,b_0,\gamma_0) = (c,b,\gamma)) \\ &\quad \times P(X_0 \cap b(0,r) = \emptyset | (c_0,b_0,\gamma_0) = (c,b,\gamma))\gamma k_b(-c)\,\mathrm{d}c\,\mathrm{d}\chi(b,\gamma). \end{aligned}$$

Since the latter three probabilities are equal to $I_1(r)$, $I_2(c,b,\gamma,r)$, and $I_3(c,b,\gamma,r)$, respectively, we obtain (29).

Example 7. It is well-known that for stationary Poisson cluster processes and SNCPs, $J(r) \leq 1$ and J is non-increasing [21, 29]. Below we show that these properties hold for certain GSNCPs. We let the situation be as in Corollary 1, and recall that $\mathrm{d}\zeta(c,b,\gamma) = \mathrm{d}c\,\mathrm{d}\chi(b,\gamma)$. Notice that if for ζ almost all (c,b,γ) and $r > 0$,

$$I_1(r) \geq I_3(c,b,\gamma,r), \tag{30}$$

then (20) and (29) imply that $J(r) < 1$ for $r > 0$.

If Φ is a Poisson process so that Φ_{cent} is stationary, then $I_3(c, b, \gamma, r) = I_1(r)$, and so by (29),

$$J(r) = \frac{1}{\rho} \int \int \gamma k_b(-c) \exp\left(- \int_{b(c,r)} \gamma k_b(\eta)\, \mathrm{d}\eta \right) \mathrm{d}c\, \mathrm{d}\chi(b, \gamma).$$

This is slightly extending Corollary 1 in [29] (where, incidentally, there is a minor typos: $k(c)$ should read $k(-c)$). It follows that J is non-increasing and $J(r) < 1$ for $r > 0$.

Consider instead a generalised Neyman-Scott process: Suppose that Φ_{cent} is a mixed Poisson process driven by a positive random variable θ (i.e. $\Phi_{\text{cent}}|\,\theta$ is a stationary Poisson process with intensity θ), where $(\theta, \Phi_{\text{cent}})$ is independent of the (b_j, γ_j), which are i.i.d. with distribution Q (case (iii) in Example 2). In other words, Φ is a Cox process driven by a random measure Λ given by $\mathrm{d}\Lambda(c, b, \gamma) = \theta\, \mathrm{d}c\, \mathrm{d}Q(b, \gamma)$. Let π denote the distribution of θ and define another positive random variable θ' with distribution

$$\pi'(A) = \frac{1}{\mathrm{E}\theta} \int_A \theta\, \mathrm{d}\pi(\theta)$$

for Borel sets $A \subseteq (0, \infty)$. Suppose that

$$\pi \leq_{\text{st}} \pi' \tag{31}$$

(usual stochastic order), i.e. $\pi((t, \infty)) \leq \pi'((t, \infty))$ for all $t > 0$. For instance, (31) is satisfied if π is a gamma distribution. We claim that (31) implies (30): Since Φ is a Cox process, a slight modification of (27) implies that

$$\begin{aligned} I_3(c, b, \gamma, r) &= \mathrm{E}\left[\exp\left(- \sum_j \int_{b(c_j, r)} \gamma_j k_{b_j}(\xi)\, \mathrm{d}\xi \right) \theta \right] \Big/ \mathrm{E}\theta \\ &= \mathrm{E} \exp\left(- \sum_j \int_{b(c'_j, r)} \gamma_j k_{b_j}(\xi)\, \mathrm{d}\xi \right) \end{aligned}$$

where $\Phi' = \{(c'_j, b_j, \gamma_j)\}$ is a Cox process driven by Λ' given by $\mathrm{d}\Lambda'(c, b, \gamma) = \theta'\, \mathrm{d}c\, \mathrm{d}Q(b, \gamma)$. For locally finite measures μ_1 and μ_2 on Ω, define a partial order $\preceq$ by

$$\mu_1 \preceq \mu_2 \quad \text{whenever } \mu_1(B) \leq \mu_2(B) \text{ for all bounded Borel sets } B \subset \Omega.$$

Further, let $\subseteq_{\rm st}$ denote the usual stochastic order for locally finite random measures Γ_1 and Γ_2 on Ω, i.e.

$$\Gamma_1 \subseteq_{\rm st} \Gamma_2 \;\Leftrightarrow\; \mathrm{E} f(\Gamma_1) \le \mathrm{E} f(\Gamma_2) \qquad \text{whenever } f \text{ is increasing with respect to } \preceq .$$

By (31), $\Lambda \subseteq_{\rm st} \Lambda'$, and so by Theorem 7.4.7 in [34], $\Phi \subseteq_{\rm st} \Phi'$. Now, $I_3(c,b,\gamma,r) = I_3(r)$ depends only on r and it is of the same form as $I_1(r)$ except that $\Phi_{\rm cent}$ is replaced by $\Phi'_{\rm cent}$. Consequently (30) holds, since the function

$$f(\mu) = \exp\left(- \sum_{(c,b,\gamma)\in\mu} \int_{b(c,r)} \gamma k_b(\xi)\,\mathrm{d}\xi \right),$$

where μ is a locally finite measure on Ω, is non-increasing with respect to $\preceq$, and $I_1(r) = \mathrm{E} f(\Phi)$ and $I_3(r) = \mathrm{E} f(\Phi')$.

Consider next a generalised shot noise G Cox process as in Example 3 where we assume that τ is a random variable with distribution π, (α,κ) is fixed, the b_j are i.i.d. with distribution ν, and they are independent of $(\tau, \{(c_j,\gamma_j)\})$. Then the Cox process Φ is driven by Λ given by $\mathrm{d}\Lambda(c,b,\gamma) = \lambda_\tau(\gamma)\,\mathrm{d}c\,\mathrm{d}\nu(b)\,\mathrm{d}\gamma$, where $\lambda_\tau(\gamma) = \lambda_\theta(\gamma)$ is given by (10). We have that

$$I_3(c,b,\gamma,r) = \mathrm{E}\left[\exp\left(- \sum_j \int_{b(c_j,r)} \gamma_j k_{b_j}(\xi)\,\mathrm{d}\xi \right) \lambda_\tau(\gamma) \right] \Big/ \mathrm{E}\lambda_\tau(\gamma).$$

So arguing as above for a generalised Neyman-Scott process, (30) holds if $\pi_\gamma \le_{\rm st} \pi$ for any $\gamma > 0$, where

$$\pi_\gamma(A) = \int_A \lambda_\tau(\gamma)\,\mathrm{d}\pi(\tau) \Big/ \int \lambda_\tau(\gamma)\,\mathrm{d}\pi(\tau)$$

for Borel sets $A \subseteq (0,\infty)$. For instance, if τ is gamma distributed, then $\pi_\gamma \le_{\rm st} \pi$ for any $\gamma > 0$.

Finally, consider a GSNCP where $\Phi_{\rm cent}$ is a stationary Markov point process on $\mathbb{R}^d$ with intensity $\rho_{\rm cent} \in (0,\infty)$, and the $(b_j,\gamma_j) = (b,\gamma)$ are all equal and fixed. Then (29) becomes

$$J(r) = \frac{1}{I_1(r)} \int I_2(c,b,\gamma,r) I_3(c,b,\gamma,r)\,\mathrm{d}c$$

where

$$I_3(c,b,\gamma,r) = \mathrm{E}\left[\exp\left(- \sum_j \int_{b(c_j,r)} \gamma_j k_b(\xi)\,\mathrm{d}\xi \right) \frac{\lambda^*(\Phi_{\rm cent}, c)}{\rho_{\rm cent}} \right],$$

cf. (28). However, we do not know how to get any further with this expression.

4. Simulation of generalised shot noise Cox processes

This section considers various simulation algorithms for the restriction $X_W = X \cap W$ of X to a Borel set $W \subset \mathbb{R}^d$ with volume $|W| \in (0, \infty)$, or for conditional simulation of Φ given X_W.

4.1. Simulation with edge effects and truncation

Clearly, X_W is a Cox process driven by the random field $Z_W(\xi) = \mathbf{1}[\xi \in W]Z(\xi)$. For the simulation of X_W a truncation may be needed to deal with the possibly infinite point process Φ entering Z_W. Assume that $W_{\text{ext}} \subseteq \mathbb{R}^d$, $B \leq \infty$, and $\varepsilon \geq 0$ so that $\zeta(D)$ is finite, where $D = W_{\text{ext}} \times (0, B) \times (\varepsilon, \infty)$. For instance, for the specific example of a generalised shot noise G process in Example 3, the condition that $\varepsilon > 0$ is needed if $\mathrm{P}(\alpha \geq 0) > 0$.

An approximate simulation of X_W is obtained by simulating first the point process $\Phi \cap D$ and next the corresponding independent Poisson processes $X_j \cap W$ with intensity functions $\xi \mapsto \gamma_j k_{b_j}(c_j, \xi)\mathbf{1}[\xi \in W]$, $(c_j, b_j, \gamma_j) \in \Phi \cap D$. As noticed in [29], in applications edge effects may enter, since W_{ext} is typically a bounded window such that $W \subset W_{\text{ext}}$. Below we quantify the error of such approximate simulations by extending Proposition 3 in [29].

Let

$$M_W = \sum_j \mathbf{1}[\{c_j \notin W_{\text{ext}}\} \cup \{b_j \geq B\} \cup \{\gamma_j \leq \varepsilon\}]\,\mathrm{card}(X_j \cap W)$$

denote the number of missing points when we make an approximate simulation of X_W by ignoring clusters X_j with $c_j \notin W_{\text{ext}}$ or $b_j \geq B$ or $\gamma_j \leq \varepsilon$. Further, let q_W be the probability that some cluster X_j with centre $c_j \notin W_{\text{ext}}$ or $b_j \geq B$ or $\gamma_j \leq \epsilon$ has a point in W. Finally, assume there exists a function $k_W^{\text{dom}} : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \mapsto [0, \infty)$ satisfying the following conditions.

Condition 1. If $\xi \in W$ then $k_W^{\text{dom}}(b, c, \xi) \geq k_b(c, \xi)$, and $k_W^{\text{dom}}(b, c, \xi) = 0$ if $\xi \notin W$.

Condition 2. The integral

$$a_W^{\text{dom}}(c, b) = \int_W k_W^{\text{dom}}(b, c, \xi)\,\mathrm{d}\xi, \qquad (c, b) \in \mathbb{R}^d \times (0, \infty),$$

can be easily computed.

Proposition 4. *We have that*

$$\mathrm{E}M_W \leq \int \mathbf{1}[\{c \notin W_{\text{ext}}\} \cup \{b \geq B\} \cup \{\gamma \leq \varepsilon\}]\gamma a_W^{\text{dom}}(c,b)\,\mathrm{d}\zeta(c,b,\gamma) \tag{32}$$

and

$$q_W \leq 1 - \exp\left(-\int \mathbf{1}[\{c \notin W_{\text{ext}}\} \cup \{b \geq B\} \cup \{\gamma \leq \varepsilon\}]\gamma a_W^{\text{dom}}(c,b)\,\mathrm{d}\zeta(c,b,\gamma)\right). \tag{33}$$

Proof. The proof of (32) is similar to that of Proposition 3 in [29]. Conditional on Φ, the clusters X_j are independent Poisson processes with intensity functions $\gamma_j k_{b_j}(c_j,\cdot)$, so

$$q_W = 1 - \mathrm{E}\prod_j \left[\exp\left(-\gamma_j \int_W k_{b_j}(c_j,\xi)\,\mathrm{d}\xi\right)\right]^{\mathbf{1}[\{c_j \notin W_{\text{ext}}\} \cup \{b_j \geq B\} \cup \{\gamma_j \leq \epsilon\}]}. \tag{34}$$

Hence, by Jensen's inequality and (5),

$$\begin{aligned} q_W &\leq 1 - \exp\left(-\mathrm{E}\sum_j \mathbf{1}[\{c_j \notin W_{\text{ext}}\} \cup \{b_j \geq B\} \cup \{\gamma_j \leq \epsilon\}]\gamma_j \int_W k_{b_j}(c_j,\xi)\,\mathrm{d}\xi\right) \\ &= 1 - \exp\left(-\int \mathbf{1}[\{c \notin W_{\text{ext}}\} \cup \{b \geq B\} \cup \{\gamma \leq \epsilon\}]\gamma \int_W k_b(c,\xi)\,\mathrm{d}\xi\,\mathrm{d}\zeta(c,b,\gamma)\right). \end{aligned}$$

Thereby, using Conditions 1 and 2, (33) is obtained.

Remark 2. When Φ is a Poisson process, (33) can be improved, cf. [33]. Indeed, using (34), Conditions 1 and 2, and arguments similar to the derivation of the generating functional of Poisson processes (see [33]), we obtain that

$$\begin{aligned} 1 - q_W = \exp\Bigg[-\int &\mathbf{1}[\{c \notin W_{\text{ext}}\} \cup \{b \geq B\} \cup \{\gamma \leq \epsilon\}] \\ &\times \left(1 - \exp\left(-\gamma \int k_b(c,\xi)\,\mathrm{d}\xi\right)\right)\mathrm{d}\zeta(c,b,\gamma)\Bigg] \\ \geq \exp\Bigg[-\int &\mathbf{1}[\{c \notin W_{\text{ext}}\} \cup \{b \geq B\} \cup \{\gamma \leq \epsilon\}]\left(1 - \exp\left(-\gamma a_W^{\text{dom}}(b,c)\right)\mathrm{d}\zeta(c,b,\gamma)\right)\Bigg]. \end{aligned}$$

Note also that by the coupling inequality (see [23]), the upper bound on q_W in (33) is also an upper bound on the total variation distance between the law of X_W and the law of the truncated process.

Example 8. Suppose that (7) holds. For specific models of χ and k_1, the upper bounds in Proposition 4 can be calculated along similar lines as in [6, 29]. For example, let

$W = b(0,R)$ and $W_{\mathrm{ext}} = b(0,R+r)$, where R and r are two positive constants, and let k_1 be the Gaussian kernel (3). Define $k_W^{\mathrm{dom}}(b,\xi - c) = \sup_{\eta\in W} k_b(\eta - c)$ for $\xi \in W$, and set $\sigma_d = 2\pi^{d/2}/\Gamma(d/2)$. Proceeding as in Example 5 in [29],

$$a(||c||,b) \equiv a_W^{\mathrm{dom}}(c,b) = \left[\frac{\omega_d R^d}{(2\pi b^2)^{d/2}}\right] \exp\left[-\mathbf{1}[\|c\| > R]\frac{(\|c\|-R)^2}{2b^2}\right]$$

and the integral in (32) and (33) is given by

$$\sigma_d \int_0^\infty \int_0^\infty \gamma\,\mathrm{d}\chi(b,\gamma) \int_0^\infty s^{d-1} a(s,b)\,\mathrm{d}s - \sigma_d \int_0^B \int_\varepsilon^\infty \gamma\,\mathrm{d}\chi(b,\gamma)\int_0^{R+r} s^{d-1} a(s,b)\,\mathrm{d}s.$$

This may be determined by numerical methods for specific models of generalised Neyman-Scott and generalised shot noise G Cox processes, cf. [6, 29].

4.2. Simulation without edge effects and truncation

Perfect simulation of X_W can be obtained by independent thinning of the nonempty clusters in a Cox process X_W^{dom} driven by

$$Z_W^{\mathrm{dom}}(\xi) = \sum_j \gamma_j k_W^{\mathrm{dom}}(b_j, c_j, \xi)$$

where k_W^{dom} satisfies Conditions 1 and 2. The details for SNCPs are carefully discussed in [6, 29, 33], so in this section we give only a description of the algorithm and discuss some applications for GSNCPs.

Let X_j^{dom}, $(c_j, b_j, \gamma_j) \in \Phi$, denote the clusters of X_W^{dom}, and let $\Phi_W^{\mathrm{dom}} = \{(c_j, b_j, \gamma_j) \in \Phi : X_j^{\mathrm{dom}} \neq \emptyset\}$. In addition to Conditions 1 and 2, assume that the following condition holds.

Condition 3. It is feasible to simulate Φ_W^{dom} (hence Φ_W^{dom} is almost surely finite).

Example 9 below demonstrates that to check Condition 3 will depend very much on the choice of model for Φ, the kernel k_1, etc.

Perfect simulation algorithm for GSNCPs.

(a) Generate the point process $\Phi_W^{\mathrm{dom}} = \{(c_1, b_1, \gamma_1), \ldots, (c_N, b_N, \gamma_N)\}$.

(b) For each $j = 1, \ldots, N$, generate

- **(i)** X_j^{dom}, which is distributed as a conditional Poisson process with intensity function $\gamma_j k_W^{\mathrm{dom}}(b_j, c_j, \cdot)$ given that it is nonempty;

(ii) X'_j, which is an independent thinning of X_j^{dom} with retention probabilities $k_{b_j}(c_j,\xi)/k_W^{\mathrm{dom}}(b_j,c_j,\xi)$ for $\xi \in X_j^{\mathrm{dom}}$.

(c) Return $\bigcup_j X'_j$.

For the loop in (b), it is implicit that the generation of processes in (i) and (ii) is independent of previous generations. The output in (c) follows the same distribution as X_W, see e.g. Proposition 4 in [29].

Example 9. For the arguments below, it is useful to notice that if Φ is a Cox process, we can obtain Φ_W^{dom} by independent thinning of Φ with retention probabilities

$$p_W^{\mathrm{dom}}(c,b,\gamma) = 1 - \exp\big(-\gamma a_W^{\mathrm{dom}}(c,b)\big), \qquad (c,b,\gamma) \in \Omega.$$

Suppose that Φ is a Poisson process. Then clearly Φ_W^{dom} is a Poisson process on Ω, with intensity measure

$$\zeta_W^{\mathrm{dom}}(D) = \int_D p_W^{\mathrm{dom}}(c,b,\gamma)\,\zeta(c,b,\gamma).$$

Condition 3 means that $\zeta_W^{\mathrm{dom}}(\Omega) < \infty$ and we are able to generate Φ_W^{dom}. Example 6 in [29] gives a detailed discussion of this condition for SNCP, and this discussion easily extends to the present case.

Suppose instead that Φ is a generalised Neyman-Scott process, where the (b_j,γ_j) are i.i.d. with distribution Q, and Φ_{cent} is a Cox process driven by a random field $Y(\cdot)$, which is bounded by a constant M and independent of the (b_j,γ_j). Further, let k_1, W, W_{ext}, k_W^{dom}, and $a(\|c\|,b) = a_W^{\mathrm{dom}}(c,b)$ be defined as in Example 8. Then Φ_W^{dom} is a Cox process on Ω driven by the random measure $\mathrm{d}m(c,b,\gamma) = p_W^{\mathrm{dom}}(c,b,\gamma)Y(c)\,\mathrm{d}c\,\mathrm{d}Q(b,\gamma)$. So

$$\mathrm{E}\Phi_W^{\mathrm{dom}} = \mathrm{E}m(\Omega) \le M\sigma_d \int_0^\infty \int_{(0,\infty)^2} s^{d-1}(1-\exp(-\gamma a(s,b)))\,\mathrm{d}s\,\mathrm{d}Q(b,\gamma) \equiv \beta_W^{\mathrm{dom}}$$

where we assume that β_W^{dom} is finite and can be determined by numerical integration (see e.g. page 628 in [29]). At least in principle simulation of Φ_W^{dom} is straightforward by thinning: First, simulate a Poisson variate N with mean β_W^{dom}. Second, generate independent points $(s_j u_j, b_j, \gamma_j)$ for $j = 1,\dots,N$, where u_j is a uniformly distributed unit vector in $\mathbb{R}^d$, (s_j,b_j,γ_j) has distribution

$$\mathrm{P}(A) \propto \int\int \mathbf{1}[(s,b,\gamma)\in A] s^{d-1}\,[1-\exp(-\gamma a(s,b))]\,\mathrm{d}s\,\mathrm{d}Q(b,\gamma), \qquad A \subseteq (0,\infty)^3,$$

and u_j is independent of (s_j, b_j, γ_j) (here rejection sampling may be useful). Third, simulate $Y(s_j u_j)$ for $j = 1, \ldots, N$, and assign each $(s_j u_j, b_j, \gamma_j)$ to Φ_W^{dom} with probability $Y(s_j u_j)/M$ (where these assignments are independent).

Similar ideas apply for a generalised shot noise G Cox process: Consider the case in Example 3 where the b_j are i.i.d. with distribution ν, and let again k_1, W, etc. be as in Example 8. Then Φ_W^{dom} is a Cox process driven by the random measure $\mathrm{d}m_\theta(c, b, \gamma) = p_W^{\mathrm{dom}}(c, b, \gamma)\lambda_\theta(\gamma)\,\mathrm{d}c\,\mathrm{d}\nu(b)\,\mathrm{d}\gamma$, where λ_θ is given by (10). So Φ_W^{dom} is almost surely finite if

$$\mathrm{E}\Phi_W^{\mathrm{dom}} = \sigma_d \int_0^\infty \int_0^\infty \int_0^\infty s^{d-1}(1 - \exp(-\gamma a(s, b)))\mathrm{E}\lambda_\theta(\gamma)\,\mathrm{d}s\,\mathrm{d}\nu(b)\,\mathrm{d}\gamma$$

is finite, and we can then at least in principle simulate Φ_W^{dom} as follows. Generate first θ and next Φ_W^{dom} as a Poisson process with intensity measure m_θ.

Finally, consider the Markov point process setting in Example 4, where for simplicity we assume that Φ_{cent} is a finite Markov point process contained in W_{ext} which is bounded. We cannot exploit the same ideas as above, since Φ_{cent} is not easily viewed as a Cox process. However, dominated coupling from the past [18] may be used for making a perfect simulation of Φ_{cent}. Then, since Φ_{cent} is independent of the (b_j, γ_j), which are either i.i.d. or identical (cf. Example 4), it may be straightforward to simulate Φ_W^{dom}.

4.3. Conditional simulation

Assume that $X_W = x = \{x_1, \ldots, x_m\} \neq \emptyset$ is observed. Simulation from the conditional distribution of Φ given $X_W = x$ is needed for predicting Φ and also for performing likelihood and Bayesian inference based on MCMC methods, cf. Section 4.3 in [29] and the references therein. In this section firstly, following [14] (see also [27, 33]), we describe the birth-death-move Metropolis-Hastings algorithm for conditional simulation of the process $\Phi | X_W = x$. Secondly, we give sufficient conditions which guarantee geometric ergodicity of the algorithm (or in fact V-uniform; the reader is referred to [26] or Section 7.2 in [33] for background material on Markov chains).

Throughout this section we assume the following. The process Φ is almost surely finite and contained in $D = W_{\mathrm{ext}} \times B \times (\varepsilon, \infty)$, with $W \subseteq W_{\mathrm{ext}} \subseteq \mathbb{R}^d$, $B \subseteq (0, \infty)$, and $\varepsilon \geq 0$. We will usually have that W_{ext} is bounded, while depending on the context it may be natural to consider cases where B is a bounded or unbounded interval. The case where ε is positive will first be needed for technical reasons when establishing geometric ergodicity. Further, ν denotes a Poisson process on D with a diffuse intensity measure μ such that $0 < \mu(D) < \infty$. Finally, Φ has a density p with respect to ν.

Then $X_W|\Phi$ is a Poisson process and it has a density with respect to a unit rate Poisson process on W,

$$f(x|\Phi) = \exp\left(|W| - \int_W Z(\xi)\,\mathrm{d}\xi\right)\prod_{\xi\in x} Z(\xi).$$

Hence, an unnormalised density for $\Phi|X_W = x$ with respect to ν is given by

$$\pi(\phi|x) = f(x|\phi)p(\phi)$$

where the normalising constant is unknown but depends only on the data x.

We turn now to the description of the birth-death-move Metropolis-Hastings algorithm. It generates a Markov chain defined on

$$\Omega_x = \{\phi \subset D : \mathrm{card}(\phi) < \infty,\ \pi(\phi|x) > 0\}.$$

Let $0 < q_1 < 1$ and $0 < q_2 \le 1$ be given numbers, and for $\phi \in \Omega_x$ and $(c, b, \gamma) \in D$, define

$$r[\phi, (c, b, \gamma)] = \frac{(1 - q_1)\pi(\phi \cup \{(c, b, \gamma)\}|x)\mu(D)}{q_1\pi(\phi|x)(\mathrm{card}(\phi) + 1)}. \tag{35}$$

If $\phi = \{(c_1, b_1, \gamma_1), \ldots, (c_n, b_n, \gamma_n)\} \in \Omega_x$ is the current state of the chain, the next state is generated as follows:

- with probability q_1q_2 make a birth step:
  - generate (c, b, γ) with law $\mu/\mu(D)$
  - with probability $\min\{1, r[\phi, (c, b, \gamma)]\}$ return $\phi \cup \{(c, b, \gamma)\}$ as the next state
- if $n > 0$, with probability $(1 - q_1)q_2$ make a death step:
  - generate i according to the uniform law on $\{1, \ldots, n\}$
  - with probability $\min\{1, r[\phi\backslash\{(c_i, b_i, \gamma_i)\}, (c_i, b_i, \gamma_i)]^{-1}\}$ return $\phi\backslash\{(c_i, b_i, \gamma_i)\}$ as the next state
- if $n > 0$, with probability $1 - q_2$ make a move step:

- generate i according to the uniform law on $\{1, \ldots, n\}$
- generate (c, b, γ) with law $\mu/\mu(D)$
- with probability $\min\{1, \pi((\phi \setminus \{(c_i, b_i, \gamma_i)\}) \cup \{(c, b, \gamma)\}|x)/\pi(\phi|x)\}$ return $(\phi \setminus \{(c_i, b_i, \gamma_i)\}) \cup \{(c, b, \gamma)\}$ as the next state

- else return ϕ as the next state.

Assuming that the initial state is in Ω_x, the chain stays in Ω_x. Note that the empty point configuration is not contained in Ω_x, since $x \neq \emptyset$. Further, the chain is reversible with invariant (unnormalised) density $\pi(\cdot|x)$. This follows along similar lines as in the proofs of Propositions 7.11, 7.12, and 7.15 in [33].

We consider two situations where irreducibility of the chain is satisfied. If for any $\phi \in \Omega_x$, $\phi' \in \Omega_x$, $\psi \subset \phi$, and $\psi' \subset \phi'$, we have that $\phi \cup \psi' \in \Omega_x$ and $(\phi \setminus \psi) \cup \phi' \in \Omega_x$, then irreducibility holds. Briefly this follows because the chain can then move up from ϕ to $\phi \cup \phi'$ and then down to ϕ'. Note that $f(x|\phi) > 0$ if $f(x|\phi') > 0$ and $\phi' \subset \phi$, and that $\pi(\phi|x) > 0$ if and only if $f(x|\phi) > 0$ and $p(\phi) > 0$. Thus irreducibility holds if for all $\phi \in \Omega_x$ and $(c, b, \gamma) \in D$,

$$p(\phi) > 0 \;\Rightarrow\; p(\phi \cup \{(c, b, \gamma)\}) > 0. \tag{36}$$

This condition is satisfied for SNCPs (since $p = 1$ in [29]). However, (36) is not always satisfied for other models of interests. For example, if a hard core condition is imposed so that $p(\phi \cup \{(c, b, \gamma)\}) = 0$ if a cluster centre from ϕ is sufficiently close to c, then (36) is violated. In fact, the opposite will often hold: p is said to be hereditary on Ω_x if for all $\phi \in \Omega_x$ and $(c, b, \gamma) \in D$,

$$p(\phi \cup \{(c, b, \gamma)\}) > 0 \;\Rightarrow\; p(\phi) > 0. \tag{37}$$

In that case we need to impose further conditions: If $q_2 < 1$ (i.e. moves are possible),

$$k_1(c, x_i) > 0 \qquad \text{for all } c \in W_{\text{ext}} \text{ and } i = 1, \ldots, m, \tag{38}$$

(i.e. $f(x|\phi) > 0$ for all finite non-empty $\phi \subset D$), and (37) hold, then we have irreducibility. Briefly, this follows since $\pi(\cdot|x)$ is now hereditary on Ω_x, so if $\phi \in \Omega_x$, $\phi' \in \Omega_x$, $(c, b, \gamma) \in \phi$, and $(c', b', \gamma') \in \phi'$, the chain can first move down from ϕ to (c, b, γ), then move to (c', b', γ'), and finally move up to ϕ'. Note that (38) is clearly satisfied for a positive kernel such as the Gaussian kernel (3), but (38) may easily be violated for the uniform kernel (4).

In the sequel irreducibility is assumed. Note that the chain is clearly aperiodic, since it can be staying in the same state for one or more transitions. Below stronger conditions than those above will be assumed to establish V-uniform ergodicity. We consider the cases with and without moves separately (i.e. the two cases $q_2 < 1$ and $q_2 = 1$).

The following Proposition 5 concerns the case $q_2 = 1$, in which case we refer to our Metropolis-Hastings algorithm as the birth-death algorithm. The proposition encompasses Proposition 5 in [29] (where the b_j are equal and fixed, Φ is a Poisson process, and $p = 1$). The following conditions are assumed to hold.

(a) p is locally stable, i.e. there is a finite constant Λ so that

$$p(\phi \cup \{(c, b, \gamma)\}) \leq \Lambda p(\phi)$$

for all finite $\phi \subset D$ and $(c, b, \gamma) \in D$.

(b) For any positive integer K there exists a positive function h_K such that

$$h_K(c, b, \gamma) p(\phi) \leq p(\phi \cup \{(c, b, \gamma)\})$$

for all $(c, b, \gamma) \in D$ and $\phi \in \Omega_x$ for which $\mathrm{card}(\phi) \leq K$.

(c) There exists a positive constant $\delta > 0$ such that

$$\int_W k_b(c, \xi) \,\mathrm{d}\xi \geq \delta$$

for any $(c, b) \in W_{\mathrm{ext}} \times B$.

(d) For $i = 1, \ldots, m$,

$$\int \mathbf{1}[k_b(c, x_i) > 0] \,\mathrm{d}\mu(c, b, \gamma) > 0.$$

(e) There exist positive constants $\delta'_1, \ldots, \delta'_m, \delta''_1, \ldots, \delta''_m$ such that

$$\delta'_i \leq k_b(c, x_i) \leq \delta''_i \qquad \text{whenever } k_b(c, x_i) > 0 \text{ and } (c, b) \in W_{\mathrm{ext}} \times B.$$

Conditions (a) and (b) are automatically satisfied for SNCPs (as $p = 1$). For GSNCPs, condition (b) implies (36), and so irreducibility is ensured. Conditions (a)–(e) are further discussed in Example 10 below.

Proposition 5. *Assume that $q_2 = 1$, $\varepsilon > 0$, and conditions (a)–(e) are satisfied. Then for any function $V(\phi) = \beta^{\mathrm{card}(\phi)}$, $\phi \in \Omega_x$, with $\beta > 1$, the birth-death algorithm is V-uniformly ergodic.*

Proof. We show first that for any positive integer N, the set $S_N = \{\phi \in \Omega_x : \mathrm{card}(\phi) \le N\}$ is a small set. This means that $P^t(\phi, F) \ge \varepsilon' Q(F)$ for any $\phi \in S_N$ and any measurable set $F \subseteq \Omega_x$, where $\varepsilon' > 0$ is some positive constant, Q is some non-zero measure on Ω_x, and $P^t(\phi, \cdot)$ denotes the t-step transition probability of the Metropolis-Hastings chain when it starts in ϕ.

We shall use the following bounds. Let $\phi \in \Omega_x$ and $(c, b, \gamma) \in D$. Arguing as at the beginning of the proof of Proposition 5 in [29], using (a), (c), and (e), it is seen that

$$\exp\left(-\gamma/b^d\right) p(\phi \cup \{(c, b, \gamma)\})/p(\phi) \le \pi(\phi \cup \{(c, b, \gamma)\}|x)/\pi(\phi|x) \le M \tag{39}$$

where $M > 0$ is a constant. Further, $0 < \mu(D) < \infty$ implies that for any positive integer K, there exist a constant $L_K > \varepsilon$ and a non-void interval $B_K \subseteq B$ such that $\rho_K = \inf B_K \in (0, \infty)$, $((1-q_1)/q_1) \exp\left(-L_K/\rho_K^d\right) \mu(D)\Lambda \le 1$ and $\mu(W_{\mathrm{ext}} \times B \times (\varepsilon, L_K)) > 0$. Furthermore, by (d) and since h_K is a positive function,

$$\int \mathbf{1}[k_b(c, x_i) > 0,\ b \in B_K,\ \gamma < L_K] h_K(c, b, \gamma)\, \mathrm{d}\mu(c, b, \gamma) > 0, \qquad i = 1, \ldots, m. \tag{40}$$

Let $m' > \max\{m, ((1-q_1)/q_1) M \mu(D) - 1\}$ be an integer and set $a = \min\{q_1, 1-q_1\}$. For any $\phi = \{(c_1, b_1, \gamma_1), (c_2, b_2, \gamma_2), \ldots (c_k, b_k, \gamma_k)\} \in S_N$ and measurable $F \subseteq \Omega_x$,

$$P^{m'+N}(\phi, F) \ge \int_D \cdots \int_D \int_D \cdots \int_D$$

$$\prod_{l=1}^{m'} a \min\{1, r[\phi \cup \{(v_1, u_1, \sigma_1), \ldots, (v_{l-1}, u_{l-1}, \sigma_{l-1})\}, (v_l, u_l, \sigma_l)]\} \tag{41}$$

$$\times \prod_{j=1}^{k} \frac{a}{k + m' - j + 1} \min\{1, r[(\phi \setminus \{(c_1, b_1, \gamma_1), \ldots, (c_j, b_j, \gamma_j)\}) \cup$$

$$\{(v_1, u_1, \sigma_1), \ldots, (v_{m'}, u_{m'}, \sigma_{m'})\}, (c_j, b_j, \gamma_j)]^{-1}\} \tag{42}$$

$$\times \prod_{i=1}^{N-k} a(1 - \min\{1, r[\{(v_1, u_1, \sigma_1), \ldots, (v_{m'}, u_{m'}, \sigma_{m'})\}, (v_i', u_i', \sigma_i')]\}) \tag{43}$$

$$\times \mathbf{1}[\{(v_1, u_1, \sigma_1), \ldots, (v_{m'}, u_{m'}, \sigma_{m'})\} \in F] \frac{\mathrm{d}\mu(v_1, u_1, \sigma_1)}{\mu(D)} \cdots \frac{\mathrm{d}\mu(v_{m'}, u_{m'}, \sigma_{m'})}{\mu(D)}$$

$$\times \frac{\mathrm{d}\mu(v_1', u_1', \sigma_1')}{\mu(D)} \cdots \frac{\mathrm{d}\mu(v_{N-k}', u_{N-k}', \sigma_{N-k}')}{\mu(D)}$$

corresponding to first adding $(v_1, u_1, \sigma_1), \ldots, (v_{m'}, u_{m'}, \sigma_{m'})$ to ϕ, next deleting the points in ϕ, and finally making no changes when $N-k$ births are proposed. From (35) and the last inequality in (39), we obtain for each of the terms in (42) that $r[\ldots]^{-1} \geq (q_1/(1-q_1))(m'+1)/(M\mu(D)) \geq 1$, i.e. $\min\{1, r[\ldots]^{-1}\} = 1,\ j = 1, \ldots, k$, and for each of the terms in (43) that $r[\ldots] \leq ((1-q_1)/q_1)M\mu(D)/(m'+1) \leq 1$, i.e. $1 - \min\{1, r[\ldots]\} = 1 - ((1-q_1)/q_1)M\mu(D)/(m'+1),\ i = 1, \ldots, N-k$. Further, combining (35), (b), and the first inequality in (39), we obtain for each of the terms in (41) that

$$r[\ldots] \geq \frac{(1-q_1)\exp\left(-L_{m'+N}/\rho^d_{m'+N}\right)\mu(D)}{q_1(k+l)} h_{m'+N}(v_l, u_l, \sigma_l)$$

if $\sigma_l < L_{m'+N},\ u_l \in B_{m'+N},\ l = 1, \ldots, m'$. Therefore,

$$P^{m'+N}(\phi, F) \geq a^{m'+N}\left(\frac{1-q_1}{q_1}\right)^{m'} \frac{\left[\exp\left(-L_{m'+N}/\rho^d_{m'+N}\right)\mu(D)\right]^{m'}}{\left[\prod_{l=1}^{m'}(k+l)\right]\left[\prod_{j=1}^{k}(m'+j)\right]} \times \left[1 - \frac{(1-q_1)M\mu(D)}{q_1(m'+1)}\right]^{N-k} Q(F)$$

where

$$Q(F) = \int_{(v_1,u_1,\sigma_1)\in D:\, u_1 \in B_{m'+N},\, \sigma_1 < L_{m'+N}} \cdots \int_{(v_{m'},u_{m'},\sigma_{m'})\in D:\, u_{m'} \in B_{m'+N},\, \sigma_{m'} < L_{m'+N}} \mathbf{1}[\{(v_1,u_1,\sigma_1), \ldots, (v_{m'},u_{m'},\sigma_{m'})\} \in F] \times h_{m'+N}(v_1,u_1,\sigma_1) \ldots h_{m'+N}(v_{m'},u_{m'},\sigma_{m'}) \frac{\mathrm{d}\mu(v_1,u_1,\sigma_1)}{\mu(D)} \cdots \frac{\mathrm{d}\mu(v_{m'},u_{m'},\sigma_{m'})}{\mu(D)}.$$

Consequently, $P^{m'+N}(\phi, F) \geq \varepsilon' Q(F)$, where

$$\varepsilon' = a^{m'+N}\left(\frac{1-q_1}{q_1}\right)^{m'} \frac{\left[\exp\left(-L_{m'+N}/\rho^d_{m'+N}\right)\mu(D)\right]^{m'}}{(N+1)\cdots(N+m')} \times \frac{m'!}{(N+m')!}\left[1 - \frac{(1-q_1)M\mu(D)}{q_1(m'+1)}\right]^{N}.$$

Notice that Q is a non-zero measure, because if we set $x_i = x_m$ for $i \geq m$, then

$$Q(\Omega_x) \geq \int_D \cdots \int_D \mathbf{1}[k_{u_i}(v_i, x_i) > 0,\ u_i \in B_{m'+N},\ \sigma_i < L_{m'+N},\ i = 1, \ldots, m'] \times h_{m'+N}(v_1,u_1,\sigma_1) \ldots h_{m'+N}(v_{m'},u_{m'},\sigma_{m'}) \frac{\mathrm{d}\mu(v_1,u_1,\sigma_1)}{\mu(D)} \cdots \frac{\mathrm{d}\mu(v_{m'},u_{m'},\sigma_{m'})}{\mu(D)}$$

which is positive, cf. (40). Thus S_N is a small set.

We can next verify the following geometric drift condition which implies V-uniform ergodicity: for each $\beta > 1$ there exists constants $b < \infty$ and $b' < 1$ such that for any $\phi \in \Omega_x$,

$$\mathrm{E}[\beta^{\mathrm{card}(X_1)}|X_0 = \phi] \leq b'\beta^{\mathrm{card}(\phi)} + b\mathbf{1}[\phi \in S_N] \tag{44}$$

where $\{X_n\}_{n\geq 0}$ denotes the Metropolis-Hastings chain, and where $N > \beta M\mu(D)$. The proof follows along lines similar to those in the proof of Proposition 3.3 in [13] or Proposition 7.14 in [33] or Proposition 6 below. Finally, (44) implies V-uniform ergodicity, cf. Proposition 7.9 in [33].

Example 10. In applications W_{ext} is bounded, and it makes not much sense to consider arbitrary large bandwidths, so let us suppose that W_{ext} and B are bounded, and let $\varepsilon > 0$. Further, suppose that $\{c_j\}$ has density $\tilde{p}$ with respect to the unit rate Poisson process on W_{ext}, and $\{c_j\}$ is independent of the (b_j, γ_j) which are i.i.d. with density q (this setting covers e.g. the finite Markov point process considered in Example 4). Let the Poisson process ν be specified by the intensity measure $\mathrm{d}\mu(c, b, \gamma) = q(b, \gamma)\,\mathrm{d}c\mathrm{d}b\mathrm{d}\gamma$. Then Φ has a density with respect to ν,

$$p(\{(c_1, b_1, \gamma_1), \ldots, (c_n, b_n, \gamma_n)\}) = \tilde{p}(\{c_1, \ldots, c_n\}).$$

Clearly, conditions (c)–(e) are then satisfied if k_1 is positive (e.g. the Gaussian kernel (3)). If e.g. k_1 is the uniform kernel (4), then (c) and (e) still hold, and (d) is satisfied provided

$$\int_{W_{\mathrm{ext}}} \int_B \int_\varepsilon^\infty \mathbf{1}[\|c - x_i\| < b] q(b, \gamma)\,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}\gamma > 0, \qquad i = 1, \ldots, m.$$

The latter condition may be expected to hold for almost any natural choice of $W_{\mathrm{ext}}, B, \varepsilon$, and q.

Suppose also that $\tilde{p}$ is a Markov density, i.e.

$$\tilde{p}(\mathbf{c}) = \prod_{y \subseteq \mathbf{c}} \varphi(y)$$

for all finite $\mathbf{c} \subset W_{\mathrm{ext}}$, where φ is an interaction function, cf. (14). Then (a) is satisfied for most choices of φ used in practice (including repulsive models and the examples considered below), cf. Chapter 6 in [33]. However, (b) is usually not satisfied. Two exceptions are the Strauss process [17, 43] and the area-interaction process [2, 45]: For a Strauss process,

$$\tilde{p}(\{c_1, \dots, c_n\}) \propto \beta^n \theta^{\sum_{i<j}^{1,n} \mathbf{1}[\|c_i - c_j\| \le R]}$$

where $\beta > 0$, $0 < \theta \le 1$, and $R > 0$ are parameters. Then (a) is satisfied with $\Lambda = \beta$, and (b) holds with $h_K(c, b, \gamma) = \beta\theta^{K(K+1)/2}$. For an area-interaction process,

$$\tilde{p}(\{c_1, \dots, c_n\}) \propto \beta^n \theta^{-|\cup_{i=1}^n b(c_i, R)|}$$

where $\beta > 0$, $\theta > 0$, and $R > 0$ are parameters, and $b(c_i, R)$ is the ball in $\mathbb{R}^d$ with centre c_i and radius R. Then (a) and (b) are satisfied with $\Lambda = \beta$ and $h_K(c, b, \gamma) = \beta\theta^{-|b(0,R)|}$ if $\theta \ge 1$, and $\Lambda = \beta\theta^{-|b(0,R)|}$ and $h_K(c, b, \gamma) = \beta$ if $\theta \le 1$.

The following Proposition 6 concerns the case $q_2 < 1$ when certain conditions including the following are satisfied.

(f) $p(\phi)$ is constant and positive for all $\phi \in \Omega_x$ with $\text{card}(\phi) = 1$.

(g) $D = W_{\text{ext}} \times B \times (\varepsilon, L)$ with $W_{\text{ext}}, B, \varepsilon$ as before and where $L > \varepsilon$ is finite.

Note that the conditions in Proposition 6 imply irreducibility, and we do not need to assume (b) which, as noticed in Example 10, is violated for most Markov models. (Moreover, (d) is implied by (38) since $\mu(D) > 0$, but we shall not use (d) in the proof).

Proposition 6. *Assume that $q_2 < 1$, $\varepsilon > 0$, and conditions (38), (a), (c), (e), (f), and (g) are satisfied. Then for any function $V(\phi) = \beta^{\text{card}(\phi)}$, $\phi \in \Omega_x$, with $\beta > 1$, the birth-death-move algorithm is V-uniformly ergodic.*

Proof. As in the proof for Proposition 5 the first step is to show that S_N is a small set for any positive integer N. Let $m' > N$ be an integer, $a = \min\{q_1 q_2, (1-q_1)q_2, 1-q_2\}$, $\phi = \{(c_1, b_1, \gamma_1), (c_2, b_2, \gamma_2), \dots (c_k, b_k, \gamma_k)\} \in S_N$, $(c_0', b_0', \gamma_0') = (c_k, b_k, \gamma_k)$, and $F \subseteq$

Ω_x. Then

$$
\begin{aligned}
P^{m'}(\phi, F) \geq a^{m'} \frac{1}{k!} \prod_{j=1}^{k-1} & \min\{1, r[\phi \setminus \{(c_1, b_1, \gamma_1), \dots, (c_j, b_j, \gamma_j)\}, (c_j, b_j, \gamma_j)]^{-1}\} \\
& \times \int_{D^{m'-k+1}} \prod_{j=1}^{m'-k+1} \min\{1, \pi(\{(c'_j, b'_j, \gamma'_j)\}|x)/\pi(\{(c'_{j-1}, b'_{j-1}, \gamma'_{j-1})\}|x)\} \\
& \times \mathbf{1}[\{(c'_{m'-k+1}, b'_{m'-k+1}, \gamma'_{m'-k+1})\} \in F] \\
& \times \frac{\mathrm{d}\mu(c'_1, b'_1, \gamma'_1)}{\mu(D)} \cdots \frac{\mathrm{d}\mu(c'_{m'-k+1}, b'_{m'-k+1}, \gamma'_{m'-k+1})}{\mu(D)}
\end{aligned}
$$

corresponding to first deleting all except one point of ϕ and then applying the movement step $m' - k + 1$ times. (Note that condition (b) was used in the proof for Proposition 5 when we considered the m' additions to ϕ; in the present proof (b) is not needed because we consider no births above). A straightforward computation shows that for any $j = 1, \dots, k-1$,

$$
\begin{aligned}
& \min\{1, r[\phi \setminus \{(c_1, b_1, \gamma_1), \dots, (c_j, b_j, \gamma_j)\}, (c_j, b_j, \gamma_j)]^{-1}\} \\
& \geq \min\left\{1, \frac{q_1}{(1-q_1)M\mu(D)}\right\} > 0
\end{aligned}
$$

where $M > 0$ is the upper bound in (39) (which was obtained without using (b)). By (38), (e), (f), and (g), for all $(c, b, \gamma), (c', b', \gamma') \in D$,

$$
\begin{aligned}
\frac{\pi(\{(c, b, \gamma)\}|x)}{\pi(\{(c', b', \gamma')\}|x)} &= \frac{\exp\left(-\int_W \gamma k_b(c, \xi)\,\mathrm{d}\xi\right)}{\exp\left(-\int_W \gamma' k_{b'}(c', \xi)\,\mathrm{d}\xi\right)} \left(\frac{\gamma}{\gamma'}\right)^m \prod_{j=1}^m \left[\frac{k_b(c, x_j)}{k_{b'}(c', x_j)}\right] \\
&\geq \exp\left(-\gamma/b^d\right) \left(\frac{\gamma\delta'}{\gamma'\delta''}\right)^m \geq \exp\left(-\gamma/b^d\right) \left(\frac{\varepsilon\delta'}{L\delta''}\right)^m
\end{aligned}
$$

where $\delta' = \min\{\delta'_1, \dots, \delta'_m\}$ and $\delta'' = \max\{\delta''_1, \dots, \delta''_m\}$. Therefore,

$$
\begin{aligned}
P^{m'}(\phi, F) \geq \frac{a^{m'}}{k!} & \left[\min\left\{1, \frac{q_1}{(1-q_1)M\mu(D)}\right\}\right]^{k-1} \\
& \times \int_D \cdots \int_D \prod_{j=1}^{m'-k+1} \min\left\{1, \exp\left(-\gamma'_j/b'^d_j\right) \left(\frac{\varepsilon\delta'}{L\delta''}\right)^m\right\} \\
& \qquad \times \mathbf{1}[\{(c'_{m'-k+1}, b'_{m'-k+1}, \gamma'_{m'-k+1})\} \in F] \\
& \qquad \frac{\mathrm{d}\mu(c'_1, b'_1, \gamma'_1)}{\mu(D)} \cdots \frac{\mathrm{d}\mu(c'_{m'-k+1}, b'_{m'-k+1}, \gamma'_{m'-k+1})}{\mu(D)}.
\end{aligned}
$$

Since $\mu(D) > 0$, there exists $B' \subseteq B$ such that $\rho \equiv \inf B' \in (0, \infty)$ and $\mu(W_{\text{ext}} \times B' \times$

$(\epsilon, L)) > 0$. Thus

$$
\begin{aligned}
P^{m'}(\phi, F) \geq{}& \frac{a^{m'}}{k!}\left[\min\left\{1, \frac{q_1}{(1-q_1)M\mu(D)}\right\}\right]^{k-1} \\
&\times \int_D \cdots \int_D \prod_{j=1}^{m'-k+1} \min\left\{1, \exp\left(-\gamma_j'/b_j'^{d}\right)\left(\frac{\varepsilon\delta'}{L\delta''}\right)^m\right\} \\
&\qquad \times \mathbf{1}[b_1' \in B', \ldots, b_{m'-k+1}' \in B',\ \{(c_{m'-k+1}', b_{m'-k+1}', \gamma_{m'-k+1}')\} \in F] \\
&\qquad\quad \frac{\mathrm{d}\mu(c_1', b_1', \gamma_1')}{\mu(D)} \cdots \frac{\mathrm{d}\mu(c_{m'-k+1}', b_{m'-k+1}', \gamma_{m'-k+1}')}{\mu(D)} \\
\geq{}& \frac{a^{m'}}{k!}\left[\min\left\{1, \frac{q_1}{(1-q_1)M\mu(D)}\right\}\right]^{k-1} \\
&\times \left[\min\left\{1, \exp\left(-L/\rho_j^d\right)\left(\frac{\varepsilon\delta'}{L\delta''}\right)^m\right\}\right]^{m'-k+1}\left(\frac{\mu(D')}{\mu(D)}\right)^{m'-k} Q(F)
\end{aligned}
$$

where $D' = W_{\text{ext}} \times B' \times (\varepsilon, L)$ and

$$
Q(F) = \int_{D'} \mathbf{1}[\{(v, u, \sigma)\} \in F]\frac{\mathrm{d}\mu(v, u, \sigma)}{\mu(D)}.
$$

Notice that Q is a non-zero measure, since by (38) and (f), $Q(\Omega_x) = \mu(D')/\mu(D) > 0$. Furthermore, $P^{m'}(\phi, F) \geq \varepsilon' Q(F)$ where

$$
\begin{aligned}
\varepsilon' ={}& \frac{a^{m'}}{N!}\left[\min\left\{1, \frac{q_1}{(1-q_1)M\mu(D)}\right\}\right]^{N-1} \\
&\times \left[\min\left\{1, \exp\left(-L/\rho^d\right)\left(\frac{\varepsilon\delta'}{L\delta''}\right)^m\right\}\right]^{m'+1}\left(\frac{\mu(D')}{\mu(D)}\right)^{m'} > 0.
\end{aligned}
$$

Thus S_N is a small set.

Next we establish V-uniform ergodicity by checking the geometric drift condition (44) with $N > \max\{\beta, (1-q_1)/q_1\}M\mu(D)$. Assume that the current state ϕ of the Metropolis-Hastings chain is such that $\phi \notin S_N$, and set $k = \text{card}(\phi)$. As for the birth-death Metropolis-Hastings algorithm, here again the probability of accepting a death is 1. Now, letting

$$
B(\phi) = \int_D \min\{1, r[\phi, (c, b, \gamma)]\}\frac{d\mu(c, b, \gamma)}{\mu(D)}
$$

denote the probability of accepting a birth, we have that

$$
B(\phi) \leq \frac{(1-q_1)M\mu(D)}{q_1(k+1)} \leq \frac{(1-q_1)M\mu(D)}{q_1(N+1)},
$$

since $\phi \notin S_N$. So

$$\begin{aligned}
&\mathrm{E}[\beta^{\mathrm{card}(X_1)}|X_0=\phi] \\
&= q_1q_2\beta^{k+1}B(\phi) + q_1q_2\beta^k(1-B(\phi)) + (1-q_1)q_2\beta^{k-1} + (1-q_2)\beta^k \\
&\le \beta^k\left[(1-q_1)q_2(\beta-1)\frac{M\mu(D)}{N+1} + (1-q_1)q_2\left(\frac{1}{\beta}-1\right)+1\right] \qquad (45)
\end{aligned}$$

where we have used that $\beta > 1$. Therefore, when $\phi \notin S_N$, (44) follows setting b' equal to $[\cdots]$ from (45). Indeed, since $N > \max\{\beta, (1-q_1)/q_1\}M\mu(D)$, it is easily seen that $b' < 1$. Finally, when $\phi \in S_N$,

$$\mathrm{E}[\beta^{\mathrm{card}(X_1)}|X_0=\phi] \le \beta^{\mathrm{card}(\phi)+1} \le \beta^{N+1}$$

whereby (44) is verified.

Example 11. Conditions (38) and (c) are satisfied if k_1 is positive, but as mentioned (38) may easily be violated for the uniform kernel. If the situation is as in Example 10, then (f) means that $\varphi(y)$ is a constant $\beta > 0$ whenever $\mathrm{card}(y) = 1$. If this is not the case, (a) can still be obtained by redefining the intensity measure of ν as $\mathrm{d}\mu(c, b, \gamma) = \varphi(\{c\})q(b, \gamma)\,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}\gamma$. Finally, (g) is needed for technical reasons, cf. the proof above, and the upper bound L on the γ_j may in many applications be a less serious assumption.

5. Concluding remarks

Although we have demonstrated that GSNCPs to some extend possess many appealing properties, it remains to investigate the statistical aspects and practical use of GSNCPs. The results in Section 3.1 for the intensity and pair correlation function may be useful for model checking and parameter estimation, particularly minimum contrast estimation, see [33]. The results in Section 3.2 on reduced Palm distributions and J-functions may to some extend be useful for statistical applications [20, 21, 33]. The most important part of this paper from a statistical viewpoint is probably the simulation algorithms in Section 4, since they provide ways of performing simulation-based inference, cf. Section 2.1 and [33]. The algorithms in Sections 4.1 and 4.2 for simulation of GSNCPs may be useful in connection to model checking, while the conditional simulation algorithm in Sections 4.3 will be needed in connection to both Bayesian and likelihood inference, cf. [33].

In this paper we have concentrated on the probabilistic aspects of GSNCPs. Future research should address the following: 1) Markov properties of GSNCPs, cf. the discussion in [29] for SNCPs. 2) Implementation and experimentation with the simulation algorithms treated in Section 4. 3) Convergence properties of MCMC hybrid (or Gibbs within Metropolis, see e.g. [10]) algorithms for conditional simulation, when we extend the setting in Section 4.3 by imposing a prior distribution on hyperparameters for the distribution of Φ. 4) Exploit our current understanding of GSNCPs for related models, e.g. spatial Hawkes processes (see point (e) in Section 2.1).

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