Variations and Generalizations of Moore Graphs.

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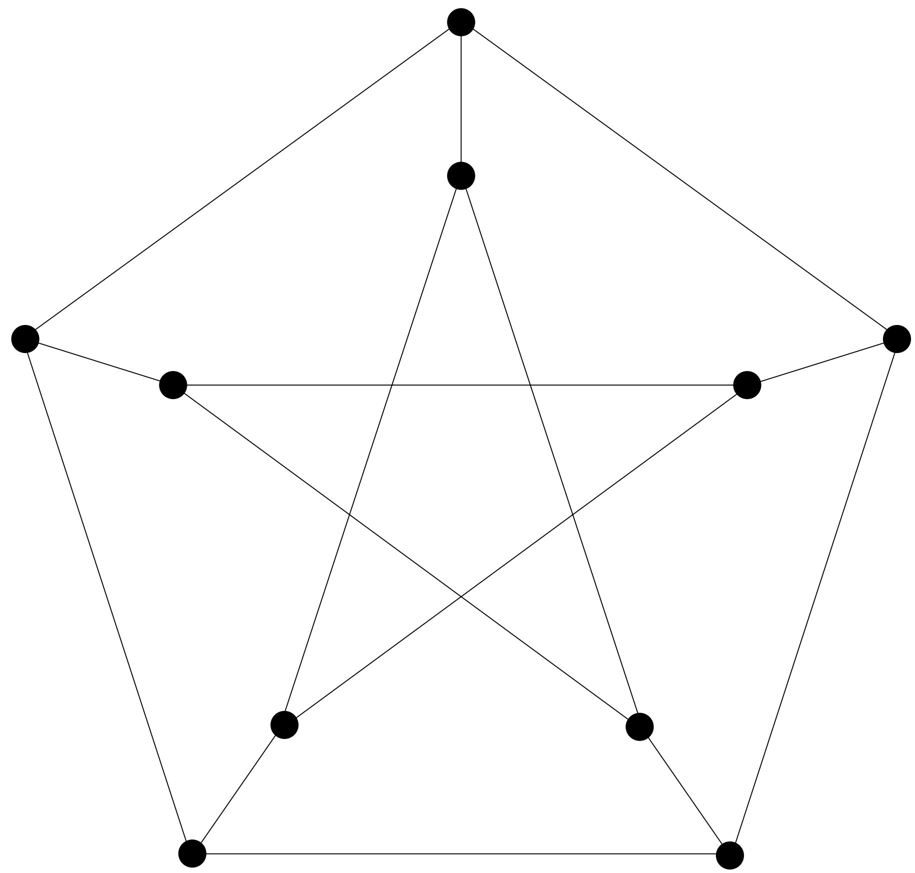


The Petersen graph

Moore Graphs

An (undirected) graph with (maximum) degree Δ and diameter D has order (number of vertices) at most

$$M(\Delta, D) := 1 + \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 + \ldots + \Delta(\Delta - 1)^{D-1}.$$

A graph with with (minimum) degree Δ and girth $2D + 1$ has at least $M(\Delta, D)$ vertices.

$M(\Delta, D)$ is called the Moore bound.

A graph with exactly $M(\Delta, D)$ vertices
has maximum degree Δ and diameter D *if and only if*
it has minimum degree Δ and girth $2D + 1$.

If these properties are satisfied then graph is called a Moore graph.

Theorem Hoffman + Singleton 1960
If there exists a Moore graph with diameter $D = 2$ then the degree is either

- $\Delta = 2$, unique Moore graph: the cycle of length 5,
- $\Delta = 3$, unique Moore graph: the Petersen graph,
- $\Delta = 7$, unique Moore graph: the Hoffman-Singleton graph, or
- $\Delta = 57$, existence of Moore graph is unknown.

Theorem Damerell 1973, Bannai + Ito 1973
A Moore graph with $\Delta \geq 3$ and $D \geq 3$ does not exist.

Does there exist a Moore graph with $D = 2$ and $\Delta = 57$.
Number of vertices is 3250.

Theorem Mačaj + Širáň 2010
A Moore graph of degree 57 has at most 375 automorphisms.

Does a Moore graph of degree 57 contain the Petersen graph ?

Bipartite Moore Graphs

A graph with (minimum) degree Δ and girth $2D$ has order at least

$$M_B(\Delta, D) := 2(1 + (\Delta - 1) + (\Delta - 1) + \ldots + (\Delta - 1)^{D-1}).$$

If a graph with (minimum) degree Δ and girth $2D$ has order $M_B(\Delta, D)$ then it is regular and bipartite and it is said to be a bipartite Moore graph.

A bipartite graph with (maximum) degree Δ and diameter D has at most $M_B(\Delta, D)$ vertices. If it has exactly $M_B(\Delta, D)$ vertices then it is a bipartite Moore graph.

A more general class of graphs (Introduced by J. Tits):
(The incidence graph of) a generalized polygon is a biregular bipartite graph with

$$\text{girth} = 2 \cdot \text{diameter}.$$

Biregular = two vertices in the same bipartition class have the same degree.

Theorem Feit + Higman 1964
Every generalized polygon other than a polygon has diameter $2, 3, 4, 6, 8$ or 12.

Feit + Higman and independently Singleton 1966:
A Δ-regular generalized polygon (i.e. a bipartite Moore graph) with $\Delta > 2$ has diameter

- $D = 2$, complete bipartite graph,
- $D = 3$, projective plane,
- $D = 4$, generalized quadrangle, or
- $D = 6$, generalized hexagon.

The theorem does not give restrictions on the degree of a bipartite Moore graph but examples are known (for $D = 3, 4$ and 6) if and only if $\Delta - 1$ is a prime power.
(Benson, Minimal Regular Graphs of Girth 8 and 12, 1966).

Standard construction for $D = 3$ is vertex transitive.

Generalized Moore Graphs

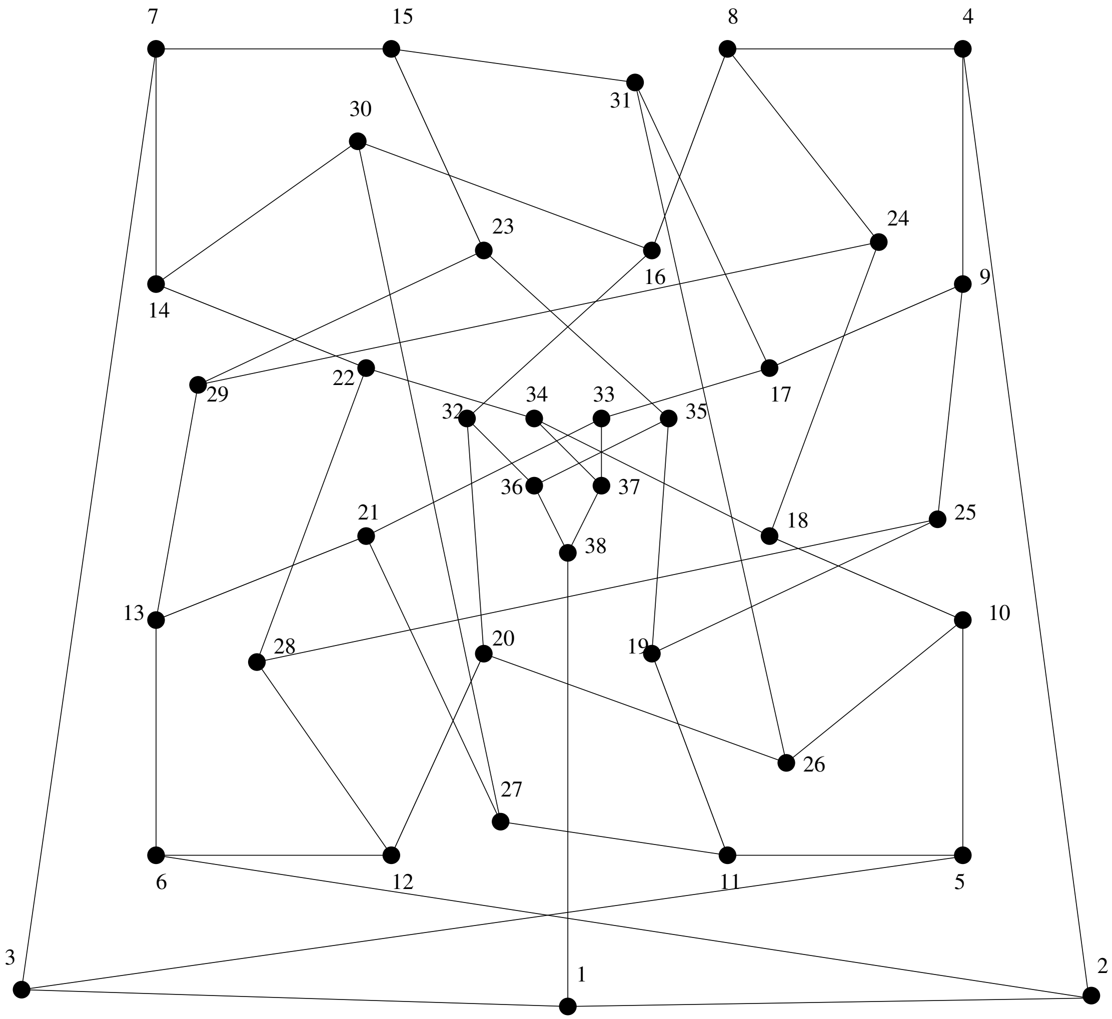
D. Buset (Disc. Appl. Math. 2000): a graph with $\Delta = 3$ and diameter 4 has at most 38 vertices.

Two 3-regular graphs of order 38 has diameter 4:

von Conta - graph (IEEE Transactions on computers 1983)
has girth 7.
Number of vertices at distance $0, 1, 2, 3, 4$ from a vertex: $1, 3, 6, 12, 16$.

Doty - graph (IEEE 1982)
has two cycles of length 6.
Number of vertices at distance $0, 1, 2, 3, 4$ from a vertex on a 6-cycle: $1, 3, 6, 11, 17$.

von Conta - graph has the smallest average distance.



von Conta graph

A Generalized Moore Graph is a regular graph with diameter D and girth at least $2D-1$.
Cerf, Cowan, Mullin and Stanton, 1973.

A Δ-regular generalized Moore graph with n vertices has the least possible average distance.

Furthermore

$$M(\Delta, D-1) < n \leq M(\Delta, D).$$

Every Moore graph is a generalized Moore graph.

Every bipartite Moore graph is a generalized Moore graph.

Constructions/enumerations of cubic generalized Moore graphs:

von Conta, IEEE Transactions on computers 1983

$(n, \Delta) = (38, 3), (60, 3), (70, 3),$

McKay and Stanton, Lecture Notes in Math 1979:

cubic generalized Moore graphs with ≤ 30 vertices.

Sampels, Discr. Appl. Math. 2004:

$3-$ and $4-$regular gen. Moore graphs with $n \leq 150$ as Cayley graphs.

$(n, \Delta) = (60, 3).$

(and some with degree 4)

D	Order n	# GM graphs	vertex-transive	
2	10	1	1	Petersen
3	12	2	0	
3	14	7	1	Heawood
3	16	6	0	
3	18	1	0	
3	20	1	0	
3	22	0	0	(Moore-bound)
4	24	1	0	
4	26	2	1	
4	28	9	1	Coxeter graph
4	30	40	1	Tutte 8-cage
4	32	56	0	
4	34	3	0	
4	36	0	0	
4	38	1	0	
4	40	0	0	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	46	0	0	(Moore-bound)

D	Order n	# GM graphs	vertex-transive	
5	48	0	0	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
5	58	0*	0	
5	60	≥ 2	≥ 2	von Conta
5	62	?	?	$M_B(3,5)$
5	$\vdots$	?	?	
5	70	≥ 1	?	von Conta
5	$\vdots$	?	?	
5	94	0	0	(Moore-bound)
6	$\vdots$	?	?	
6	126	≥ 1	?	12-cage

* Brinkmann, McKay and Saager 1995: girth $9 \Rightarrow \geq 58$ vertices.

Does there exist an upper bound on the diameter of a generalized Moore graph
(maybe depending on the degree) ?

When restricted to Cayley graphs ?

Conjecture
Every generalized Moore graph has diameter $D \leq 6$.

Directed Moore Graphs

The number of vertices in a directed graph with (maximum) out-degree Δ and diameter D is at most

$$\overrightarrow{M}(\Delta, D) = 1 + \Delta + \Delta^2 + \ldots + \Delta^D.$$

A directed Moore graph with out-degree Δ, diameter D and with $\overrightarrow{M}(\Delta, D)$ vertices.

Theorem Plesník and Znám 1974, Bridges and Toueg 1980.
Directed Moore graphs do not exist except in the trivial cases:

- $D = 1$, complete directed graphs $K_{\Delta+1}$, and

- $\Delta = 1$, directed cycles C_{D+1}.

Mixed Moore Graphs

Mixed graph: directed and undirected edges.

A mixed Moore graph with degree $\Delta = t + z$ and diameter D is a mixed graph if

- every vertex is incident to t undirected edges,
- for every vertex there are z edges directed into and z edges directed out from this vertex, and
- for any ordered pair (x, y) of vertices there is a unique path from x to y of length at most D.

Nguyen, Miller, Gimbert 2007: mixed Moore graphs of diameter at least 3 do not exist.

Duval 1988: a directed strongly regular graph with parameters (n, k, t, λ, μ) is a mixed graph with n vertices so that

- every vertex is incident to t undirected edges, and
  for every vertex there are $k - t$ edges directed into and $k - t$ edges directed out from this vertex,

- if there is an edge from x to y then there are exactly λ paths of length 2 from x to y and
  if there is no edge from x to y then there are exactly μ paths of length 2 from x to y.

The adjacency matrix A satisfies

$$A^2 = tI + \lambda A + \mu(J - A - I), \quad JA = AJ = kJ.$$

A mixed Moore graph (of diameter 2) is a directed strongly regular graph with $\lambda = 0$ and $\mu = 1$.

Bosák 1979: If the graph is not a directed triangle or a 5-cycle then the eigenvalues are $\Delta = t + z$ with multiplicity 1 and (integers) $\frac{-1 \pm c}{2}$ where $c = \sqrt{4t - 3}$.

c is an odd positive integer $t = \frac{c^2+3}{4}$ and $c \mid (4z - 3)(4z + 5)$.

Possible values of t: $1, 3, 7, 13, 21, \ldots$.
For each value of t: infinitely many values of z.

$n = (t + z)^2 + z + 1$

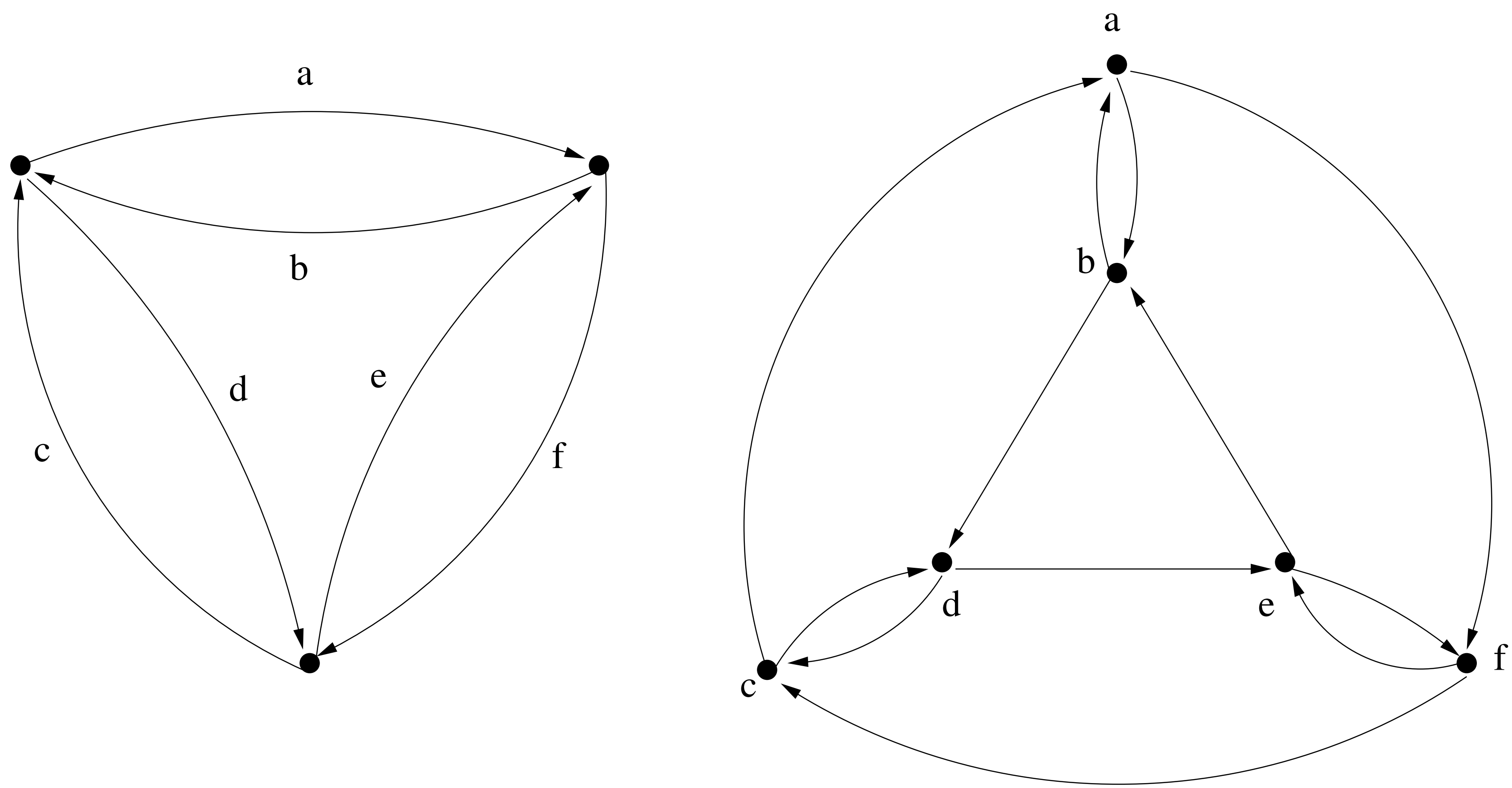
$t = 1$

$z = 1, 2, 3, 4, \ldots$

The Kautz digraph on $(z+2)(z+1)$ vertices is the line-digraph of a the complete directed graph on $z+2$ vertices.

This is a mixed Moore graph for every z.

Gimbert 2001: Kautz digraphs are unique mixed Moore graph with $t = 1$.



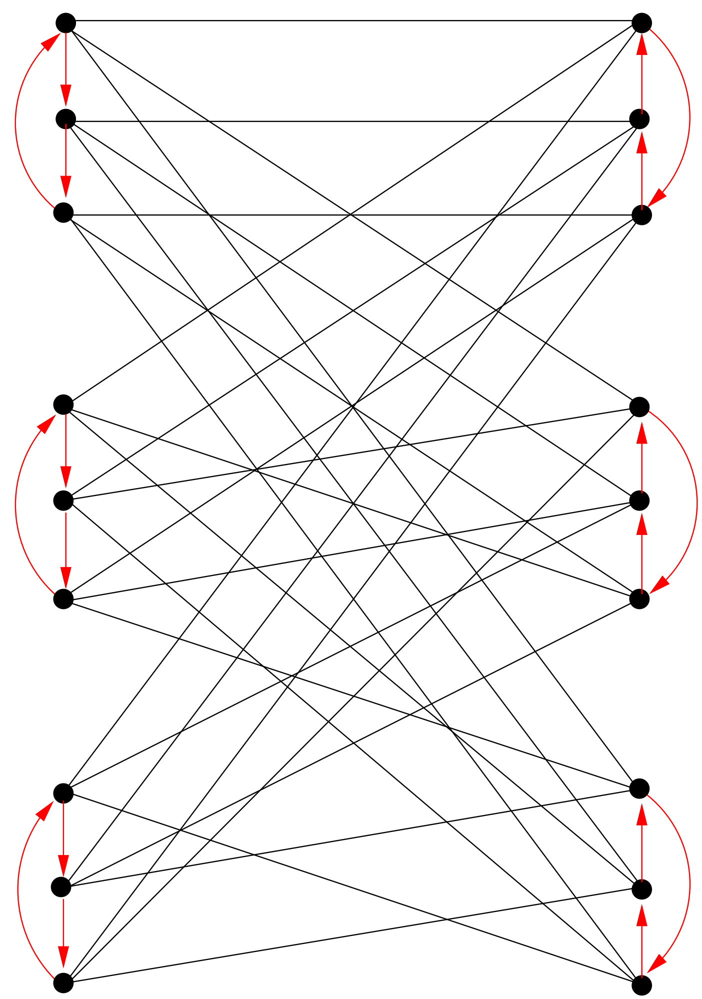
The complete digraph K_3 and its linedigraph: the Kautz digraph with 6 vertices

$z = 1$

The vertices are partitioned in directed triangles.

$t = 1, 3, 21$

- $t = 1$: Kautz digraph
- $t = 3$: Bosák graph is unique.
- $t = 21$:
  Undirected graph is an antipodal distance regular graph with diameter 4.
  A 3-cover of a strongly regular graph with
  $n = 162$, $\lambda = 0, \mu = 3$ and degree t



Pappus graph and Bosák graph

Feasible cases with $t > 1$ and $n \leq 200$:

n	t	z	Graphs
18	3	1	Bosák (unique)
40	3	3	
54	3	4	
84	7	2	
88	3	6	
108	3	7	G_{108} and transpose
150	7	5	
154	3	9	
180	3	10	

Mixed Moore graph as Cayley graph

From theory of directed strongly regular: the group must be non-Abelian.

Bosák graph is a Cayley graph:

$$\text{Cay}(S_3 \times \mathbb{Z}_3, \{(1,2), (1,2,3), (2,3)(4,5,6), (2,3)(4,6,5)\}),$$

where $\mathbb{Z}_3 = \langle (4,5,6) \rangle$.

And a Cayley graph of another group:

$\mathsf{Cay}((S_3 \times S_3) \cap A_6, \{(1,2,3)(4,5,6), (1,2)(4,5), (1,2)(4,6), (1,3)(4,5)\})$.

Bosák graph is a subgraph of

$$G_{36} = \mathsf{Cay}(S_3 \times S_3, \{(1,2,3)(4,5,6), (1,2)(4,5), (1,2)(4,6), (1,3)(4,5),$$
$$(1,3), (1,2)(4,5,6), (1,3)(4,5,6), (4,6), (1,2,3)(4,5), (1,2,3)(4,6)\}).$$

G_{36} is directed strongly regular graph with $n = 36$, $k = 10$, $t = 5$, $\lambda = 2$, $\mu = 3$.

Theorem

There exist at least two mixed Moore graphs with $n = 108, t = 3, z = 7$.

We denote these graph by G_{108} and G_{108}^T.

One is transposed of the other.

They are both Cayley graphs of groups number 15 and 17 is the GAP catalogue of groups of order 108.

Group number 17 is isomorphic to automorphism group of the Bosák graph.

The automorphism group of G_{108} has a (unique) normal subgroup of order 3.

This partitions the vertices of G_{108} in 36 orbits of length 3.

The quotient graph is G_{36}.

G_{108} has a partition in four sets of 27 vertices. Each of these sets induce a subgraph consisting nine directed triangles.

Suppose that G is mixed Moore graph with $n = 40, t = z = 3$.

Then G does not have an automorphism of order 5.

Subgraphs that are mixed Moore:

G does not contain an undirected 5-cycle.

G does not contain the Bosák graph.

G does not contain the Kautz digraph on 12 vertices.

The Kautz digraph on 6 vertices may be a subgraph of G.

Moore Bipartite Digraphs

Fiol and Yebra, 1990

A directed, bipartite graph with (maximum) out-degree Δ and diameter D has order at most

$$\overrightarrow{M}_B(\Delta, D) := \begin{cases} 2(1+\Delta^2+\Delta^4+\ldots+\Delta^{2m}) & \text{if } D = 2m+1, \\ 2(\Delta+\Delta^3+\ldots+\Delta^{2m-1}) & \text{if } D = 2m. \end{cases}$$

A directed, bipartite graph with out-degree Δ and diameter D and with exactly $\overrightarrow{M}_B(\Delta, D)$ vertices is called a Moore bipartite digraph.

Theorem Fiol+Yebra 1990
If a Moore bipartite digraph with out-degree $\Delta > 1$ and diameter D exists then

$$2 \le D \le 4.$$

Adjacency matrix of a Moore bipartite digraph

$$A = \begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}.$$

Then

$$I + A + A^2 + A^3 = \begin{bmatrix} I + A_1A_2 & A_1(I + A_2A_1) \\ A_2(I + A_1A_2) & I + A_2A_1 \end{bmatrix} = \begin{bmatrix} J & \Delta J \\ \Delta J & J \end{bmatrix},$$

and

$$\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix} \begin{bmatrix} J & \Delta J \\ \Delta J & J \end{bmatrix} = A(I + A + A^2 + A^3) =$$

$$(I + A + A^2 + A^3)A = \begin{bmatrix} J & \Delta J \\ \Delta J & J \end{bmatrix} \begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}.$$

Thus a Moore bipartite digraph is in- and out- regular.

A Moore bipartite digraph with diameter $D = 2$ is a complete bipartite digraph (exists for every Δ).

A Moore bipartite digraph with diameter $D = 3$ exists for every Δ.

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The line digraph of a Moore bipartite digraph with diameter $D = 3$ is a Moore bipartite digraph with diameter $D = 4$.

Theorem Fiol, Gimbert, Gómez and Wu, 2003
Every Moore bipartite digraph with diameter 4 is the line digraph of Moore bipartite digraph with diameter 3.

Proof:

A: adjacency matrix of a Moore bipartite digraph with with diameter 4.
Eigenvalues and their multiplicities can be computed:

$$\text{Rank } A = 2(1 + \Delta^2).$$

$n = 2(\Delta + \Delta^3)$. Each row has Δ 1's. There is a 1 in each row.
Thus A has at least $\frac{n}{\Delta} = 2(1 + \Delta^2)$ different rows.

Then there are exactly $\frac{n}{\Delta}$ different rows.

But this is satisfied if and only if the digraph is a linedigraph, by a theorem of Heuchenne, 1964.

Multipartite Moore Digraphs

Fiol, Gimbert and Miller, 2006.

An r-partite digraph with partition of vertex set

$$V_1 \cup \ldots \cup V_r$$

is said to be δ-equioutregular if every vertex in V_i has exactly δ outneighbours in V_j, for all $i \neq j$.

The number of vertices in a δ-equioutregular r-partite digraph with outdegree $\Delta = (r-1)\delta > 1$ and diameter D is at most

$$\overrightarrow{M}_r(\Delta, D) := \begin{cases} \frac{\Delta^{D+1}-1}{\Delta-1} - (r-1)\frac{\delta^{D+1}-1}{\delta-1} & \text{if } D = 2m+1 \\ \\ \frac{\Delta^{D+1}-1}{\Delta-1} - \frac{\delta^{D+1}+1}{\delta+1} & \text{if } D = 2m. \end{cases}$$

A δ-equioutregular r-partite digraph with diameter D and with $\overrightarrow{M}_r((r-1)\delta, D)$ vertices is called a multipartite Moore digraph.

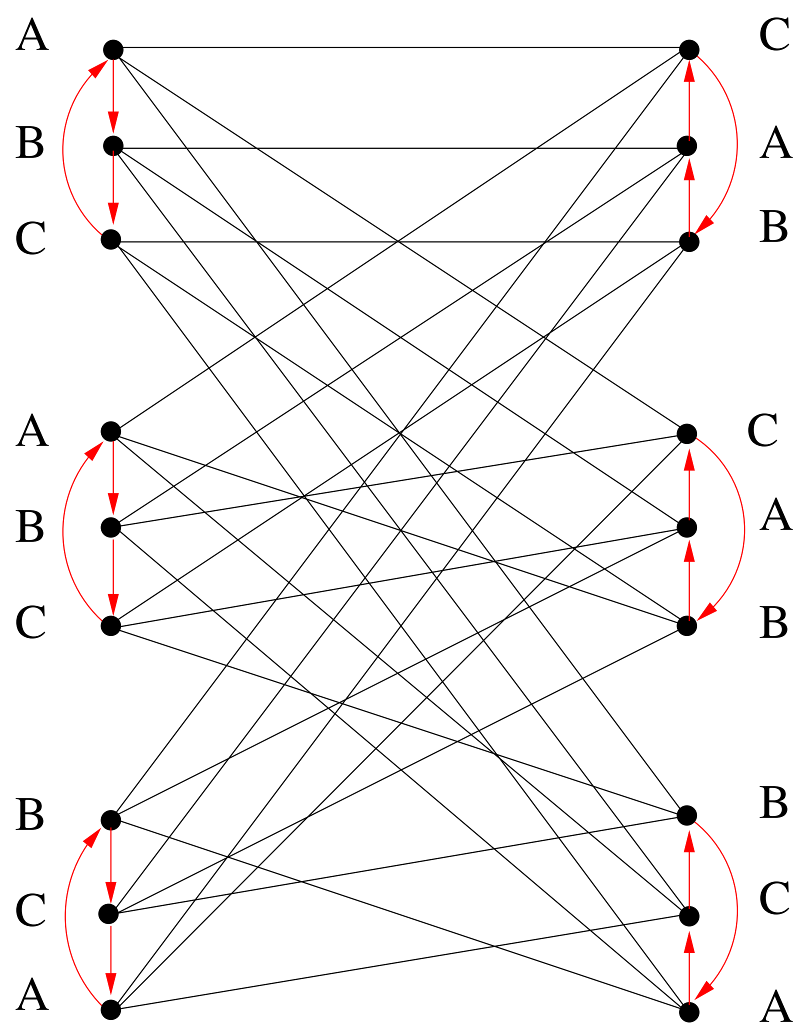
Examples:

- A directed complete graph K_r is an r-partite Moore digraph with $\delta = 1$ and $D = 1$.
  One vertex in each class of the partition.

- A Kautz digraph $L(K_r)$ is an r-partite Moore digraph with $\delta = 1$ and $D = 2$.[†]
  The set of edges of K_r directed into a vertex correspond to the vertices of a class of the partition.

- The Bosák graph is a 3-partite Moore digraph with $\delta = 2$ and $D = 2$.

[†]It is possible that an almost Moore digraph is a multipartite digraph with $\delta = 1$ when D is even.


A
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If $D > 1$ and $\Delta > 1$ then $\overrightarrow{M}_r(\Delta, D) < \overrightarrow{M}(\Delta, D)$.
I.e., for some pairs (x, y) there is more than one path of length $\leq D$ from x to y.
If D is even then x and y must be in the same partite set.
If D is odd then x and y must be in different partite sets.

Suppose that D is even and $x = y$ for all of the above pairs.
Then the graph is weakly distance regular and then eigenvalues and their multiplicities can be computed.

The most interesting case is $D = 2$. Then the graph is a mixed Moore graph.

Theorem Fiol, Gimbert, Fiol
If mixed Moore graph is a multipartite Moore graph with $\delta > 1$ then

$$r > \delta \quad \text{and} \quad 2\delta - 1 \text{ divides } \quad r(r\delta(\delta - 1) + 1).$$

$\Delta = (r-1)\delta$, $t = \delta^2 - \delta + 1$, $z = \Delta - t$ and $n = r\delta((r-2)\delta + 1)$.

Feasible parameters:

n	t	z	δ	r	
18	3	1	2	3	Bosák graph
40	3	3	2	4	??
54	3	4			
84	7	2	3	4	??
88	3	6			
108	3	7	2	6	G_{108}
150	7	5	3	5	??
154	3	9	2	6	??
180	3	10			