Isomorphic switching in tournaments *

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Abstract

For a tournament T with a vertex x, we let T_x be the tournament obtained from T by reversing every edge, except edges joining two out-neighbours or two in-neighbours of x. We consider tournaments, T, with the property that T_x is isomorphic to T for every vertex $x \in T$, and the relation of such tournaments to (vertex-transitivity in) regular digraphs with the property that non-adjacent vertices have no common out-neighbours and adjacent vertices have λ common out-neighbours for some constant λ.

In [4], the author investigates digraphs with the property that every vertex has out-degree k, and the number of common out-neighbours of a pair of vertices x and y is λ if x and y are adjacent and μ if x and y are non-adjacent. For $\mu = 0$ it was shown that if G is a connected digraph then G has this property if and only if either

1. it is a directed cycle ($k = 1$),
2. it is a regular tournament in which the out-neigbours of any vertex is also a regular tournament (this is called a doubly regular tournament), or
3. for some vertex x the out-neighbours of x spans a doubly regular tournament and G has the following structure: G has vertex set $\{x, y, x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and edges $x \to x_i \to y \to y_i \to x$ for $1 \leq i \leq n$ and $x_j \to y_i \to y_j \to x_i$ whenever $x_i \to x_j$. In fact G has this structure for any vertex x.

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The digraph constructed in case 3 is denoted by $\mathcal{D}(\mathcal{T})$ if T is a doubly regular tournament isomorphic the digraph spanned by the out-neighbours of x. $\mathcal{D}(\mathcal{T})$ is uniquely determined from T, but it is possible that $\mathcal{D}(\mathcal{T})$ and $\mathcal{D}(\mathcal{T}')$ are isomorphic even if T and T' are nonisomorphic.

The Paley tournament (or quadratic residue tournament) of a finite field F of order congruent til 3 modulo 4 is the tournament whose vertices are the elements of F and with edges $x \to y$ if $y - x$ is a square in the field F. It is well-known that Paley tournaments are doubly regular.

It seems likely that the Paley tournaments are the only vertex transitive doubly regular tournaments, see Ito [3]. The motivation for this paper is to show that, for a doubly regular tournament T, it is not unusual that $\mathcal{D}(\mathcal{T})$ is vertex transitive and in fact is a Cayley digraph (i. e., has a regular group of automorphisms).

The relation between vertex transitivity of $\mathcal{D}(\mathcal{T})$ and switching in T is investigated in section 2. First we shall give some definitions.

Let x be a vertex in a tournament T. By x^+ and x^-, we denote the set of out-neighbours and the set of in-neighbours of x, respectively.

By switching at x, we mean construction of the tournament T_x obtained from T by reversing those edges that joins two of the three sets $\{x\}$, x^+, and x^-.

We say that T has the weak switching property if for every vertex $x \in T$, T_x is isomorphic to T, and we say that T has the strong switching property if for every vertex $x \in T$ there is an isomorphism from T_x to T fixing x.

The score of a tournament with vertex set $\{x_1, \ldots, x_n\}$ is the multiset $\{d^+(x_1), \ldots, d^+(x_n)\}$, where $d^+(x)$ denotes the out-degree of x. We say that a tournament T has the score switching property if for every vertex $x \in T$, T_x and T has the same score. This property is weaker than the weak switching property.

1 General properties of switching

Lemma 1 *Let T be a tournament on n vertices with the score switching property. If T has a vertex of out-degree d then T has a vertex of out-degree $n - 1 - d$.*

Proof Suppose that x has out-degree d in T. Then x has in-degree d and thus out-degree $n - 1 - d$ in T_x. As T_x has the same score as T, T has a vertex of out-degree $n-1-d$. $\diamond$

It is possible that in every tournament with the score switching property, the number of vertices of out-degree d is equal to the number of vertices of in-degree d.

Although T and the tournaments obtained from T by (repeated application of) switching, in most cases dont all belong to the same isomorphism class, the number of isomorphism classes is always at most the number of vertices plus one. This follows from the next lemma.

Lemma 2 *Let x and y be distinct vertices in a tournament T. Then $(T_x)_y$ is isomorphic to T_y.*

Proof There is an isomorphism from $(T_x)_y$ to T_y, that interchanges x and y and fixes every other vertex. To show this, for each pair of sets among

$$\{x\}, \{y\}, x^+ \cap y^+, x^+ \cap y^-, x^- \cap y^+, x^- \cap y^-$$

count how many times an edge between these two sets is reversed in $(T_x)_y$ and in T_y (this depends on whether $x \to y$ or $y \to x$). $\diamond$

For any natural number n the transitive tournament on n vertices, i. e. the tournament with vertices $1, \ldots, n$ and edges $i \to j$ when $i < j$, has the weak switching property. Another important example is the Paley tournaments.

Theorem 3 *Let T be a Paley tournament. Then T has the strong switching property.*

Proof As T is vertex-transitive it suffices to show that the tournament obtained by switching at the vertex 0 is isomorphic to T. Let $\phi : T \mapsto T_0$ be defined by $\phi(0) = 0$ and $\phi(x) = -x^{-1}$, for every vertex $x \neq 0$. Let $Q = \{x^2 \mid x \neq 0\}$. Then Q is the set of out-neighbours and $-Q$ is the set of in-neighbours of 0.

Suppose that $x \to y$ in T, i. e., $y - x \in Q$. If $x, y \in Q$ then $\phi(y) - \phi(x) = x^{-1} - y^{-1} = x^{-1}y^{-1}(y-x) \in Q$, as Q is closed under multiplication. Thus $\phi(x) \to \phi(y)$ in T and in T_0. If $x \in Q$ and $y \in -Q$ then $\phi(x) - \phi(y) = -x^{-1} + y^{-1} = x^{-1}(-y^{-1})(y-x) \in Q$. Thus $\phi(y) \to \phi(x)$ in T and so $\phi(x) \to \phi(y)$ in T_0. The remaining cases are similar. It follows that ϕ is an isomorphism. $\diamond$

From the Paley tournament we can construct new examples of tournaments with the weak switching property.

Theorem 4 *Let T be a Paley tournament, and let x be a vertex of T. Then $T - x$ has the weak switching property.*

Proof Let $y \neq x$ be a vertex of T. Then there is an isomorphism from T_y to T that maps x to, say, x'. As T is vertex transitive, there is an automorphism of T mapping x' to x. Thus there is an isomorphism $T_y \mapsto T$ fixing x, and so $(T - x)_y = T_y - x \simeq T - x$. $\diamond$

In this theorem T may be replaced by any other vertex transitive tournament with the weak switching property. But any such tournament would be a doubly regular vertex transitive tournament (see the proof of Theorem 6), and as mentioned above the Paley tournaments are probably the only examples.

Theorem 5 *Let T be a Paley tournament, and let x and y be distinct vertices of T. Then $T - \{x, y\}$ has the weak switching property.*

Proof Let $z \neq x, y$ be a vertex in T. Then there is an isomorphism from T_z to T that maps $\{x, y\}$ to, say, $\{x', y'\}$. As T is edge transitive there is automorphism mapping $\{x', y'\}$ to $\{x, y\}$ (note that the edge joining x and y may have been reversed by switching at z). Thus there is an isomorphism from T_z to T fixing the set $\{x, y\}$, and so $T - \{x, y\} \simeq (T - \{x, y\})_z$. $\diamond$

Note that the Paley tournaments are the only tournaments that are edge transitive, see Berggren [1].

n	1	2	3	4	5	6	7	8	9
Strong switch. prop.	1		1				1		
Weak switch. prop.	1	1	2	1	3	2	4	3	7
Score switch. prop.	1	1	2	1	3	2	8	6	19

Table 1 The number of tournaments with the strong, weak, and score switching property, respectively on n vertices for $n \leq 9$.

2 Transitivity of $\mathcal{D}(\mathcal{T})$

In this section we consider the connection between switching in T and vertex transitivity of $\mathcal{D}(\mathcal{T})$.

In a tournament with the strong switching property, any vertex has equal in- and out-degree, and so it is regular. And in fact,

Theorem 6 *Any tournament with the strong switching property is doubly regular.*

Proof We prove that a regular tournament T with the score switching property is doubly regular. Let x be any vertex of T. We have to show that x^+ spans a regular tournament. Let $y \in x^+$. After switching at x out-neighbours of y outside x^+ becomes in-neighbours of y, and vice versa. As both T and T_x are regular, the number of out-neighbours of y outside x^+ is equal to the number of in-neighbours of y outside x^+. And so in x^+ the in-degree of y is equal to the out-degree of y. $\diamond$

Although not every doubly regular tournament has the strong switching property we can say something.

Theorem 7 *Every doubly regular tournament has the score switching property.*

In order to prove this, for a doubly regular tournament T, we consider the digraph $\mathcal{D}(\mathcal{T})$ in which the out-neighbours of any vertex spans a doubly regular tournament. The next lemma shows how one of those subtournaments is constructed from another. The theorem follows immediately from this.

Lemma 8 *Let x be a vertex of the digraph $D = \mathcal{D}(\mathcal{T})$, for some doubly regular tournament T. Let z be a vertex in x^+. Then the tournaments $(x^+)_z$ and z^+ are isomorphic.*

Proof Let y be the unique vertex of D not adjacent to x, and let the out-neighbours of x (and in-neighbours of y) be $x_0, \ldots, x_{2\lambda}$, the in-neighbours of x (and out-neighbours of y) be $y_0, \ldots, y_{2\lambda}$, such that if $x_i \to x_j$ then also $x_j \to y_i \to y_j \to x_i$. We may assume that $z = x_0$ and $z^+ \cap x^+ = \{x_1, \ldots, x_\lambda\}$. Then $z^+ = \{x_1, \ldots, x_\lambda, y_{\lambda+1}, \ldots, y_{2\lambda}, y\}$. Now it is easy to check that $\phi : (x^+)_z \mapsto z^+$ given by $\phi(z) = y$, $\phi(x_i) = x_i$, $\phi(x_{\lambda+i}) = y_{\lambda+i}$, for $1 \leq i \leq \lambda$, is an isomorphism. $\diamond$

As the the digraph $\mathcal{D}(\mathcal{T})$ is uniquely determined up to isomorphism by the tournament spanned by the out-neighbours of any vertex, we get another consequence of this lemma.

Corollary 9 *For a doubly regular tournament T, $\mathcal{D}(\mathcal{T})$ is vertex transitive if and only if T has the weak switching property.*

We have seen that the Paley tournaments have the strong switching property. We will now show that this property is shared with two other families of doubly regular tournaments.

Let q be a prime power congruent to 5 modulo 8, and let F be a field of order q. Let Q be the unique subgroup of the multiplicative group F^* of index 4. Let $Q, -Q, R,$ and $-R$ be the cosets. Then Szekeres [5] constructed a tournament on $2q+1$ vertices: v and x_i, where $x \in F$ and $i = 1, 2$. The edges are: $v \to x_1 \to x_2 \to v$, for $x \in F$, and $x_1 \to y_1$ if $y - x \in Q \cup R$, $x_2 \to y_2$ if $y - x \in -Q \cup -R$, and $x_i \to y_{3-i}$, $i = 1, 2$ if $y - x \in Q \cup -R$. We denote this tournament by S_q.

Theorem 10 *The Szekeres tournament S_q has the strong switching property, for every prime power $q \equiv 5 \bmod 8$.*

Proof For every vertex $z \in S_q$ we must show that there is an isomorphism from S_q to $(S_q)_z$ fixing z. For $z = v$, we define $\mu : S_q \mapsto (S_q)_v$ by $\mu(v) = v$ and $\mu(x_i) = (-x)_{3-i}$, for $x \in F$, $i = 1, 2$. An easy case examination shows that μ is an isomorphism.

Suppose that $z = x_1$, for some $x \in F$. Since there is an isomorphism of S_q mapping x_1 to 0_1, we may assume that $x = 0$. Define $\phi(0_1) = 0_1$, $\phi(0_2) = v$, $\phi(v) = 0_2$, and $\phi(x_i) = (-x^{-1})_i$, for $x \in Q \cup -Q$, $i = 1, 2$, and $\phi(x_i) = (-x^{-1})_{3-i}$, for $x \in R \cup -R$, $i = 1, 2$. Then $\phi : S_q \mapsto (S_q)_{0_1}$ is an isomorphism.

Suppose that $z = x_2$. Then we may assume that $x = 0$. Define $\nu : S_q \mapsto (S_q)_{0_2}$ as follows: $\nu(0_2) = 0_2$, $\nu(0_1) = v$, $\nu(v) = 0_1$, and $\nu(x_i) = (x^{-1})_{3-i}$, for $\in Q \cup -Q$, $i = 1, 2$, and $\nu(x_i) = (x^{-1})_i$, for $x \in R \cup -R$, $i = 1, 2$. Then ν is an isomorphism. $\diamond$

Theorem 11 *Let T be a tournament with the strong switching property. Suppose that for every vertex z of T there is an isomorphism from T to the tournament obtained from T by reversing every edge and suppose that this isomorphism fixes z. Let the vertices of T be $v_1, \ldots, v_n$. Let S be the tournament with vertex-set $\{v, x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and edges $v \to x_i \to y_i \to v$, for $1 \le i \le n$, and $y_j \to x_i \to x_j \to y_i, y_j \to y_i$, when $v_i \to v_j$ in T.*

Then S has the strong switching property.

Note that the Paley tournaments have the property of the tournament T in the theorem.

Proof We may identify T with the subtournament of S spanned by $x_1, \ldots, x_n$. Let T' denote the subtournament of S spanned by $y_1, \ldots, y_n$. Then T' is isomorphic to the tournament obtained by reversing every edge of T. Let $\mu : T \mapsto T'$ be an isomorphism, and define $\mu(v) = v$ and $\mu(y_i) = x_j$ where $\mu(x_i) = y_j$. Then $\mu : S \mapsto S_v$ can be shown to be an isomorphism.

Let $z \in T$ and let $\phi : T \mapsto T_z$ be an isomorphism, $\phi(z) = z$. For $z = x_k$, let $z' = y_k$. Then let $\phi(v) = z'$, $\phi(z') = v$ and let $\phi(y_i) = y_j$ where $\phi(x_i) = x_j$, for $i \neq k$. Then $\phi : S \mapsto S_z$ is an isomorphism.

Let $z \in T'$, $z = y_k$ and let $z' = x_k$. By composition of the isomorphisms which we have assumed exist, we get an isomorphism $\nu : T \mapsto T'_z$ with $\nu(z') = z$. Define $\nu(y_i) = x_j$ where $\nu(x_i) = y_j$, redefine $\nu(z) = z$, $\nu(z') = v$ and let $\nu(v) = z'$. Then $\nu : S \mapsto S_z$ is an isomorphism. $\diamond$

We shall now investigate when the digraphs $\mathcal{D}(\mathcal{T})$ are Cayley digraphs. Recall that a Cayley digraph of the group G generated by $S \subset G$ is the digraph whose vertices are the elements of G and with edges $x \to y$ if $x^{-1}y \in S$. It is well-known that D is a Cayley digraph of G if and only if G acts regulary as a group of automorphisms on the vertices of D, see [6].

Lemma 12 *For a doubly regular tournament T, $\mathcal{D}(\mathcal{T})$ cannot be a Cayley digraph of an abelian group.*

Proof Suppose that $\mathcal{D}(\mathcal{T})$ is a Cayley digraph of an abelian group G. Consider the set $H = \{x + x \mid x \in G\}$. As $|G|$ is even, $|H| \leq \frac{1}{2}|G|$. Suppose that $x \to y$ in the tournament spanned by the out-neighbours of the group identity, i. e., $x, y \in S$ and there exist $z \in S$ such that $-x + y = z$. But then $z \to y$, as $-z + y = x \in S$. Thus the in-neighbours of y in S comes in pairs, except possibly when $y \in H$. As S spans a doubly regular tournament, every vertex in S has odd in-degree in S and so $S \subseteq H$. Also the in-neighbours of the group identity, $-S \subseteq H$, and so $|H| > \frac{1}{2}|G|$, a contradiction. $\diamond$

Lemma 13 *Let T be a doubly regular tournament, and let ψ be the permutation of the vertices of $\mathcal{D}(\mathcal{T})$ mapping each vertex x to the unique vertex not adjacent to x. Then ψ is the unique involutory automorphism of $\mathcal{D}(\mathcal{T})$.*

Proof It is easy to check that ψ is an automorphism. Let ϕ be an involutory automorphism. If $x \to \phi(x)$ then $\phi(x) \to \phi^2(x) = x$, a contradiction. Also

$\phi(x) \to x$ leads to a contradiction. Thus, for every vertex x, either $\phi(x) = x$ or $\phi(x)$ is the vertex nonadjacent to x. Suppose that for some vertex x, $\phi(x) = x$. Then ϕ fixes every vertex in x^+ and thus every vertex in $\mathcal{D}(\mathcal{T})$. It follows that $\phi = \psi$. $\diamond$

Theorem 14 *Let T be a doubly regular tournament with the weak switching property, and suppose that T has $2^n - 1$ vertices for some number n. Then $\mathcal{D}(\mathcal{T})$ is a Cayley digraph of the generalized quarternion group of order 2^{n+1}.*

Proof From Corollary 9 we see that the automorphism group acts transitively on the vertices. Thus $|\mathcal{D}(\mathcal{T})| = \in^{\backslash+\infty}$ divides the order of the automorphism group. Let G be a Sylow 2-subgroup. Let $\phi \in G$ be an automorphism fixing some vertex x. Then the restriction of ϕ to x^+ is an automorphism of a tournament, and so it has odd order (a well-known fact with a proof similar to that of Lemma 13). It follows that ϕ is the identity map. Since the order of G is at least $|\mathcal{D}(\mathcal{T})|$ and the stabilizer of a vertex is trivial, G acts regulary on the vertices of $\mathcal{D}(\mathcal{T})$, and so $\mathcal{D}(\mathcal{T})$ is a Cayley digraph of G. Since G has a unique involution (Lemma 13), G is either a cyclic group or a generalized quarternion group, see [2]. By Lemma 12 it cannot be cyclic. $\diamond$

Computerexperiments shows that small digraphs $\mathcal{D}(\mathcal{T})$ often appear as Cayley digraphs (especially when T is a Paley tournament) of the groups

$$\langle x, y \mid x^{2n} = y^2, y^4 = 1, xyx = y \rangle$$

which are semi direct products of a cyclic group and a generalized quarternion group.

References

[1] J. L. Berggren, An algebraic characterization of finite symmetric tournaments, *Bull. Austral. Math. Soc.* **6** (1972) 53–59.

[2] D. Gorenstein, *Finite Groups*, Harper and Row, New York (1968).

[3] N. Ito, Hadamard tournaments with transitive automorphism groups, *Europ. J. Combin.* **5** (1984) 37–42.

[4] L. K. Jørgensen, On normally regular digraphs.

[5] G. Szekeres, Tournaments and Hadamard matrices, *Enseignment Math.* **15** (1969) 269–278.

[6] H. P. Yap, *Some topics in graph theory*, London Math. Soc. Lecture Note Series 108, Cambridge Univ. Press (1986).