$$A\vec{x} = \vec{b}$$

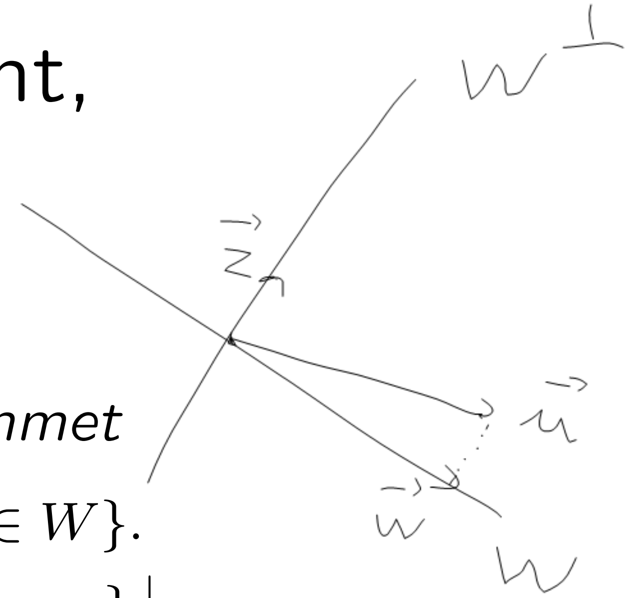
$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

$A\vec{x} = \vec{b}$ er konsistent hvis er linear komb. af søjler i A.

Altså hvis $\vec{b}$ ligger i Col A.

Ortogonalt komplement, ortogonal projektion



W et underrum af $\mathbb{R}^n$.

Det ortogonale komplement af W er *underrummet*

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}\cdot\mathbf{u} = 0 \text{ for alle } \mathbf{u} \in W\}.$$

Hvis $W = \text{span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ så er $W^\perp = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}^\perp$.

For enhver vektor $\mathbf{u} \in \mathbb{R}^n$ findes der entydige vektorer $\mathbf{w} \in W$ og $\mathbf{z} \in W^\perp$ så

$$\mathbf{u} = \mathbf{w} + \mathbf{z}.$$

$\mathbf{w}$ kaldes den ortogonale projektion af $\mathbf{u}$ på W,
og betegnes $U_W(\mathbf{u})$.
U_W er da en *lineær* operator på $\mathbb{R}^n$.

Ortogonal projektion.

Lad W være et underrum af $\mathbb{R}^n$ med $\dim W = k > 0$
og C være en $n \times k$ matrix hvis søjler udgør en basis for W.
Så har ortogonalprojektionsoperatoren U_W standardmatrix

$$P_W = C(C^T C)^{-1} C^T.$$

Den ortogonale projektion af $\mathbf{u}$ på W kan altså beregnes som

$$U_W(\mathbf{u}) = P_W \mathbf{u},$$

eller, hvis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ er en *ortonormal* basis, som

$$U_W(\mathbf{u}) = (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{u} \cdot \mathbf{v}_2)\mathbf{v}_2 + \ldots + (\mathbf{u} \cdot \mathbf{v}_k)\mathbf{v}_k.$$

EKS

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad W = \operatorname{span} S$$

$$C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} \qquad P_W = C(C^TC)^{-1}C^T$$

$$C^TC = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 4 & 3 \end{bmatrix}$$

$$\det C^TC = 7 \cdot 3 - 4 \cdot 4 = 5$$

$$(C^TC)^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ -4 & 7 \end{bmatrix}$$

$$P_W = \frac{1}{5} C \begin{bmatrix} 3 & -4 \\ -4 & 7 \end{bmatrix} C^T = \frac{1}{5} \begin{bmatrix} 3 & 2 & 1 & -1 \\ 2 & 3 & -1 & 1 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Projektion af $\vec{u}$ på W:

$$U_W(\vec{u}) = P_W \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{w}$$

Skriv $\vec{u} = \vec{w} + \vec{z}$, $\vec{z}$ i $W^\perp$

$$\vec{z} = \vec{u} - \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Afstand fra $\vec{u}$ til W: $\|\vec{z}\| = \sqrt{0^2 + 0^2 + 1^2 + 1^2} = \sqrt{2}$

EKS

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \quad \vec{q}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \quad W = \text{Span}\{\vec{q}_1, \vec{q}_2\}$$

$$\vec{q}_1 \cdot \vec{q}_2 = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} - \frac{1}{2}\cdot\left(-\frac{1}{2}\right) = 0$$

$$\|\vec{q}_1\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1$$

$$\|\vec{q}_2\| = 1$$

$\{\vec{q}_1, \vec{q}_2\}$ er ortonormal basis for W

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Projektion af $\vec{u}$ på W:

$$(\vec{u}\cdot\vec{q}_1)\,\vec{q} + (\vec{u}\cdot\vec{q}_2)\,\vec{q}_2 =$$

$$-2\,\vec{q}_1 - 1\cdot\vec{q}_2 = \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

Underrum knyttet til matricer.

For ethvert underrum W af $\mathbb{R}^n$ er

$$\dim W + \dim W^{\perp} = n \quad \text{og} \quad (W^{\perp})^{\perp} = W.$$

For enhver matrix A er

$(\text{Col } A)^{\perp} = \text{Null } A^{T}$

$$(\text{Row } A)^{\perp} = \text{Null } A.$$

Ethvert underrum W af $\mathbb{R}^n$ er søjlerum af en matrix og dermed også rækkerum af den transponerede matrix: $W = \text{Row } A$.
Det ortogonale komplement bestemmes altså som $W^{\perp} = \text{Null } A$.

Desuden kan vi nu se at enhvert underrum W af $\mathbb{R}^n$ er nulrum af en matrix:
Til underrummet $W^{\perp}$ findes en matrix A med $\text{Row } A = W^{\perp}$.
Så er $\text{Null } A = (\text{Row } A)^{\perp} = (W^{\perp})^{\perp} = W$.

EKS

$$W = \text{span}\{\vec{v}_1 \ \vec{v}_2 \ \dots \vec{v}_k\} = \text{Col } A$$

$$\text{hvor } A = [\vec{v}_1 \ \vec{v}_2 \ \dots \vec{v}_k]$$

$$W^\perp = \{\vec{v}_1 \dots \vec{v}_k\}^\perp = (\text{Col } A)^\perp = \text{Null } A^T$$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad W = \text{span } S$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}$$

Find basis for $W^{\perp} = S^{\perp} = (\text{Col } A)^{\perp}$

$$A^T = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \to R_1}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

X_3 & X_4 free

$X_1 + X_3 - X_4 = 0, \quad X_2 - X_3 + X_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $W^\perp$: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$W^\perp = \mathrm{Col} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Transpose

$$W = (W^\perp)^\perp = \mathrm{Null} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

7.5

$\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$: en ortonormal basis for $\mathbb{R}^n$

Sæt $Q = [\vec{q}_1 \ \vec{q}_2 \ \cdots \ \vec{q}_n]$

Q siges at være en ortogonal matrix.

EKS

$$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$ ortogonal

$$Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$ ortogonal

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$$Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \dots & \vec{q}_n \end{bmatrix}$$ $n \times n$ matrix

$$A = Q^T Q$$

$$a_{ij} = (\text{række } i \text{ fra } Q^T) \cdot (\text{søjle } j \text{ fra } Q) =$$

$$\vec{q}_i \cdot \vec{q}_j$$

Hvis Q er ortogonal så er $a_{ij} = 0$ hvis $i \neq j$

og $a_{ii} = \vec{q}_i \cdot \vec{q}_i = \|\vec{q}_i\|^2 = 1$

Altså $A = Q^T Q = I_n$

Omvendt hvis $Q^T Q = I_n$ så er Q ortogonal

Hvis Q er ortogonal så er

$$\det(Q^T Q) = \det I_n = 1$$

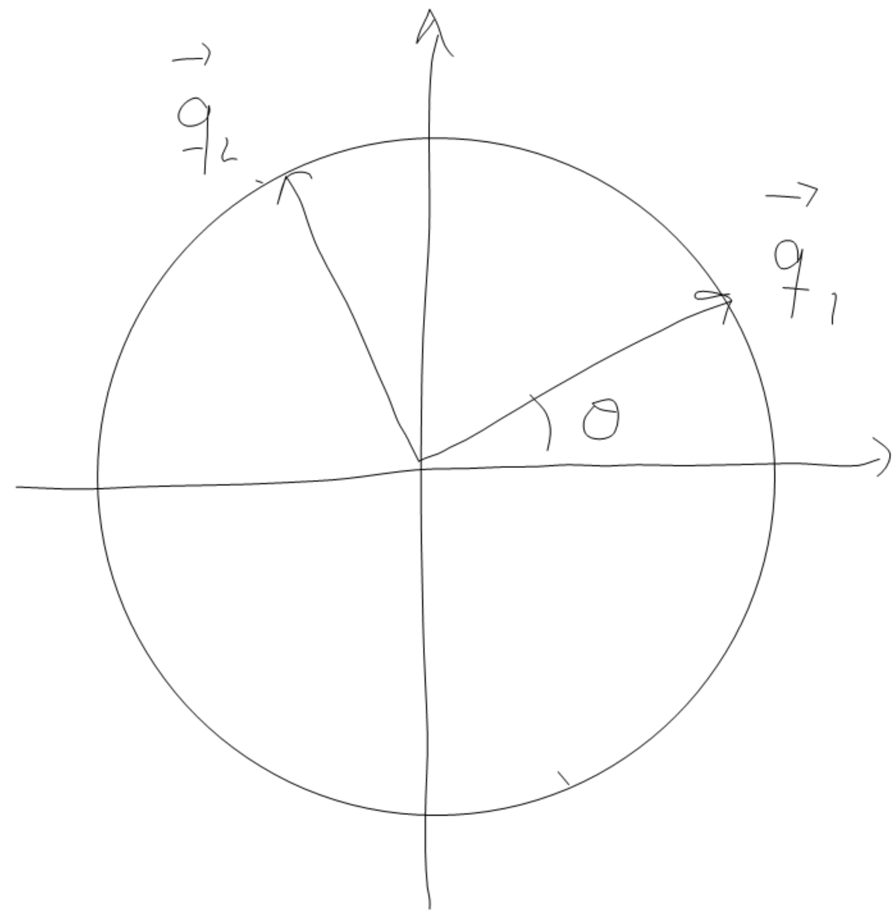
$$\det(Q^T Q) = \det Q^T \cdot \det Q =$$

$$\det Q \cdot \det Q = (\det Q)^2$$

Altså $(\det Q)^2 = 1$ og $\det Q = \pm 1$

Q : orthogonal 2×2 matrix

$$Q = [\vec{q}_1 \quad \vec{q}_2]$$



$$\vec{q}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

To tilfælde

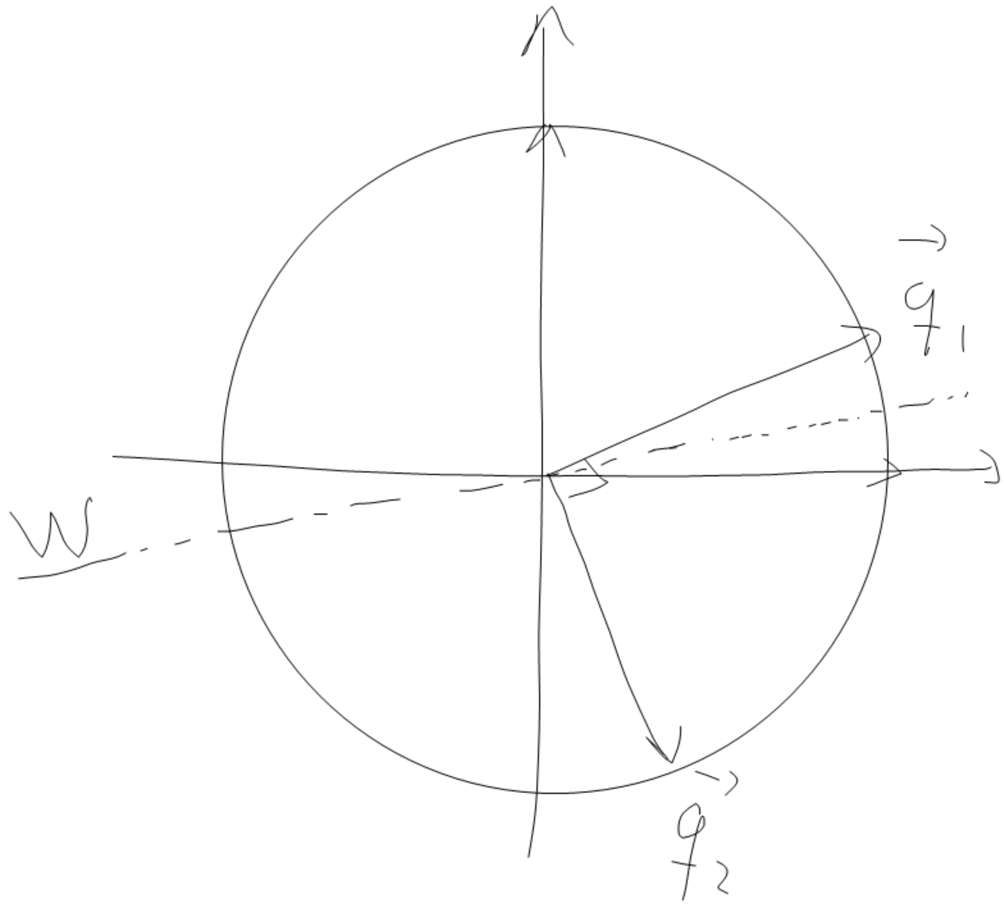
1.

$$\vec{q}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A_\theta$$ rotation med vinkel θ

$$\det Q = \cos\theta \cdot \cos\theta - (-\sin\theta) \cdot \sin\theta = 1$$

2.) $\vec{q}_2 = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$



$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

$$\det Q = \cos\theta \cdot (-\cos\theta) - \sin\theta \cdot \sin\theta$$

$$= -1$$

Spejling om akse W

W er egenrum med egenværdi 1

EKS

$$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$ orthogonal

$$\det Q = \frac{3}{5} \cdot \frac{3}{5} - \left(-\frac{4}{5}\right) \cdot \frac{4}{5} = 1$$

Q er en rotation

$$\cos\theta = \frac{3}{5}, \quad \sin\theta = \frac{4}{5}$$

$$\theta \approx 53^\circ$$