Exam in Discrete Mathematics

First Year at The TEK-NAT Faculty

June 11th, 2014, 9.00–13.00

ANSWERS

Part I (”regular exercises”)

Exercise 1 (6%).

Find the expansion of $(2x - y)^4$ using The Binomial Theorem.
Answer: $16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$

Exercise 2 (8%).

Find witnesses proving that $f(x) = 2x^3 + x^2 + 5$ is $O(x^3)$.

Exercise 3 (12%).

1. Use the Euclidean algorithm to find the greatest common divisor of 46 and 21.

   Answer: 1

2. Find integers s and t satisfying that $\gcd(46, 21) = s \cdot 46 + t \cdot 21$.

   Answer: $s = -5, t = 11$

3. Determine all integers x such that

$$x \equiv 2 \pmod{46} \quad \text{and} \quad x \equiv 1 \pmod{21}.$$

   Answer: $x \equiv 232 \pmod{966}$

Exercise 4 (9%).

Prove by induction that

$$\sum_{i=1}^{n}(4i+1) = 2n^2 + 3n,$$

for every positive integer n.

Exercise 5 (6%).

1. Construct a truth table for the compound proposition $(p \wedge \neg q) \rightarrow (r \vee q)$.

   Answer:

p	q	r	$(p \wedge \neg q) \rightarrow (r \vee q)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

2. Is the compound proposition in question 1 a tautology?

   Answer: No.

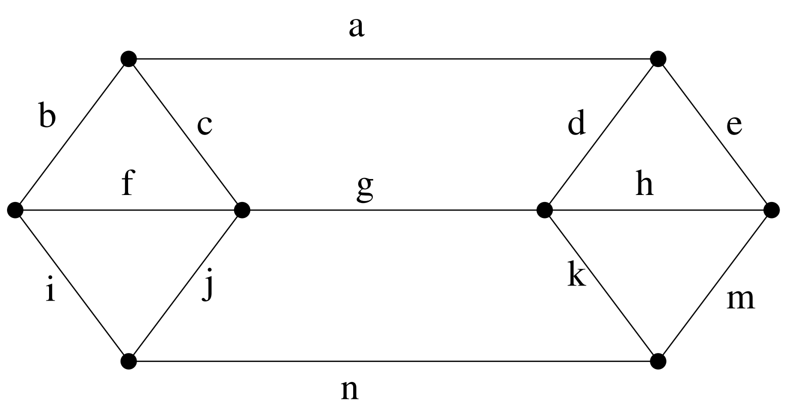


Figure 1: A graph G considered in Exercise 6.

Exercise 6 (10%).

A graph G with 13 edges is shown in Figure 1. The edges of G have weights given by the following table

Edge	a	b	c	d	e	f	g	h	i	j	k	m	n
Weight	1	1	3	3	6	4	5	6	2	4	2	7	2

1. Use Prim's algorithm to find a minimum spanning tree S in G. Write the edges of S in the order in which they are added to S by Prim's algorithm. (If there is more than one possible solution then write only one of them.)

   One possible solution: a, b, i, n, k, c, e

2. Use Kruskal's algorithm to find a minimum spanning tree T in G. Write the edges of T in the order in which they are added to T by Kruskal's algorithm. (If there is more than one possible solution then write only one of them.)

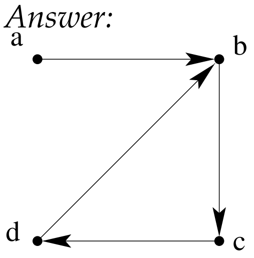
   One possible solution: a, b, i, k, n, c, e

Exercise 7 (9%).

Let $A = \{a, b, c, d\}$ and let $R = \{(a, b), (b, c), (c, d), (d, b)\}$ be a relation on A.

1. Draw the directed graph representing R.

   Answer:



2. Determine the transitive closure R^* of R.

   Answer:
   $R^* = \{(a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, b), (c, c), (c, d), (d, b), (d, c), (d, d)\}$

3. Determine a matrix $\mathbf{M}_{R^*}$ representing R^*.

   Answer:
   $$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Exercise 8 (10%).

A set S is defined recursively by

Basis step: $0 \in S$

Recursive step: if $a \in S$ then $a + 3 \in S$ and $a + 5 \in S$.

1. Determine the set $S \cap \{a \in \mathbb{Z} \mid 0 < a < 12\}$.

   Answer: $\{3, 5, 6, 8, 9, 10, 11\}$
2. Prove that every integer $a \geq 8$ is contained in S.

Part II ("multiple choice" exercises)

Exercise 9 (10%).

Let $f(x) = (x^2 + 5x + 3)(x + 2\log x)$, for $x > 0$. Answer the following 5 true/false exercises

1. $f(x)$ is $O(x^4)$.

   ☒ True ☐ False

2. $f(x)$ is $O(x^3)$.

   ☒ True ☐ False

3. $f(x)$ is $O(x^2)$.

   ☐ True ☒ False

4. $f(x)$ is $O(x^3 \log x)$.

   ☒ True ☐ False

5. $f(x)$ is $O(x^2 \log x)$.

   ☐ True ☒ False

Exercise 10 (6%).

Let $A = \{1,3,5\}$ and $B = \{3,4,5\}$ be sets.

1. What is the cardinality of the power set $\mathcal{P}(A \cup B)$

   ☐ 4 ☐ 8 ☒ 16 ☐ 32 ☐ 64

2. Which of the following are elements of $A \times B$?

   ☐ $\{1,3\}$ ☒ $(1,3)$ ☐ $(4,5)$ ☒ $(5,5)$

Exercise 11 (8%).

Consider the following algorithm:

Procedure sum(n: positive integer)
$s := 0$
for $i := 1$ **to** n
  for $j := 1$ **to** i
    $s := s + j$
return s

1. Suppose that procedure sum is started with input $n = 4$. Then what number is returned by the algorithm?

   ☐ 10 ☒ 20 ☐ 40 ☐ 45

2. The worst-case time complexity of procedure sum is:

   ☐ $O(n)$ ☐ $O(n \log n)$ ☐ $O(n^{3/2})$ ☒ $O(n^2)$

Exercise 12 (6%).

Let

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

be a matrix representing a relation R on a set A. Answer the following 3 true/false exercises

1. R is reflexive.

   ☒ True ☐ False

2. R is symmetric.

   ☐ True ☒ False

3. R is antisymmetric.

   ☐ True ☒ False