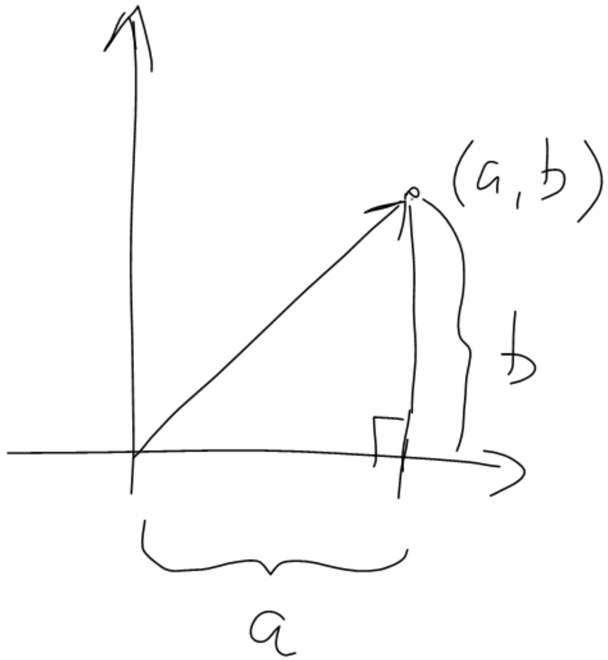
6.1

$\mathbb{R}^2$



Længde af $\begin{bmatrix} a \\ b \end{bmatrix}$:

$$\sqrt{a^2+b^2}$$

I $\mathbb{R}^n$: $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{bmatrix}$

Længden elle norm af $\vec{v}$:

$$\|\vec{v}\| = \sqrt{v_1^2+v_2^2+\dots+v_n^2}$$

Hvis $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ så er $\|\vec{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$

$$\|c\vec{v}\| = |c|\,\|\vec{v}\|$$

Normalisering af $\vec{v}$, $\vec{v} \neq \vec{0}$

$\frac{1}{\|\vec{v}\|}\vec{v}$ har samme retning som $\vec{v}$

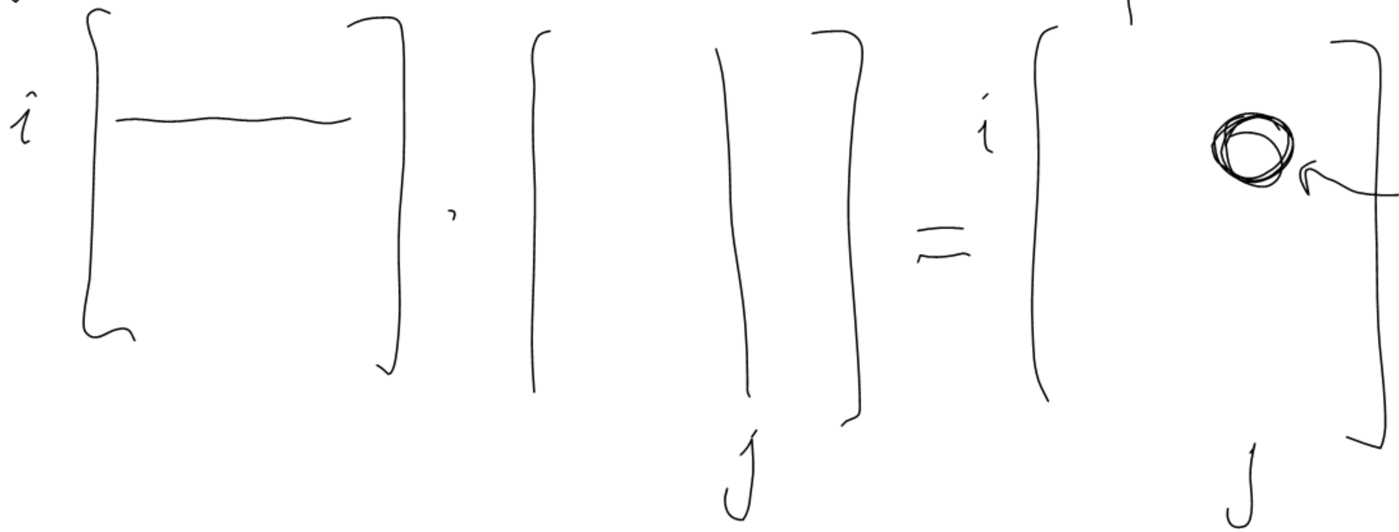
og lengde $\left\| \frac{1}{\|\vec{v}\|}\vec{v} \right\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1$

$\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ har lengde 1 og samme retning som $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Prik produkt af $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ og $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

er $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$

Bruges ved matrix multiplikation



(række i)$\cdot$(søjle j)

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{u}^T \vec{v}$$

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = (\vec{u}^T \vec{v})^T = \vec{v}^T (\vec{u}^T)^T =$$

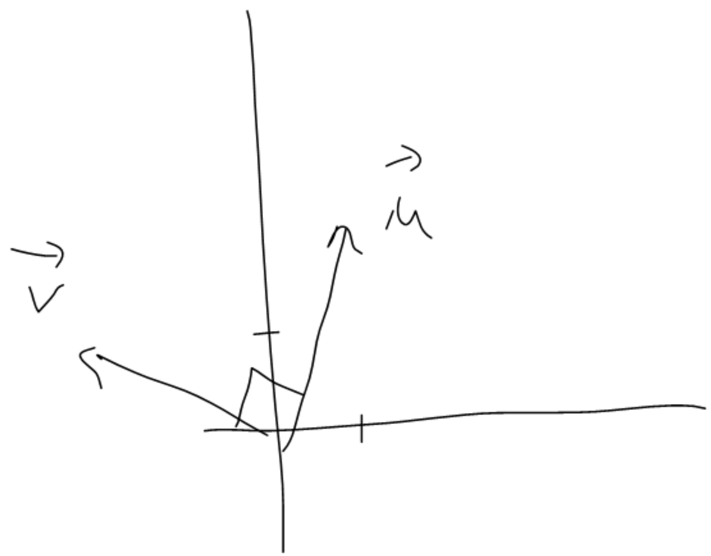
$$\vec{v}^T \vec{u} = \vec{v} \cdot \vec{u}$$

$$\| \vec{v} \| = \sqrt{v_1^2 + \dots + v_n^2} = \sqrt{\vec{v} \cdot \vec{v}}$$

$$\| \vec{v} \|^2 = \vec{v} \cdot \vec{v}$$

$\vec{u}$ og $\vec{v}$ siges at være ortogonale (vinkelret) hvis $\vec{u} \cdot \vec{v} = 0$

$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



$\vec{u} \cdot \vec{v} = 1 \cdot (-2) + 2 \cdot 1 = 0$

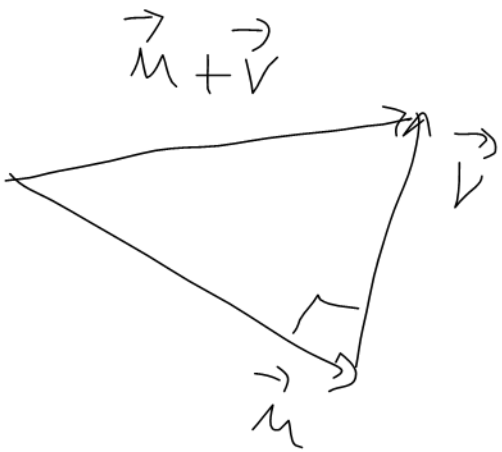
$\vec{u}$ og $\vec{v}$ ortogonale

<u>Pythagoras:</u>

$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

$\Updownarrow$

$\vec{u} \cdot \vec{v} = 0$



Bevis

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) =$$

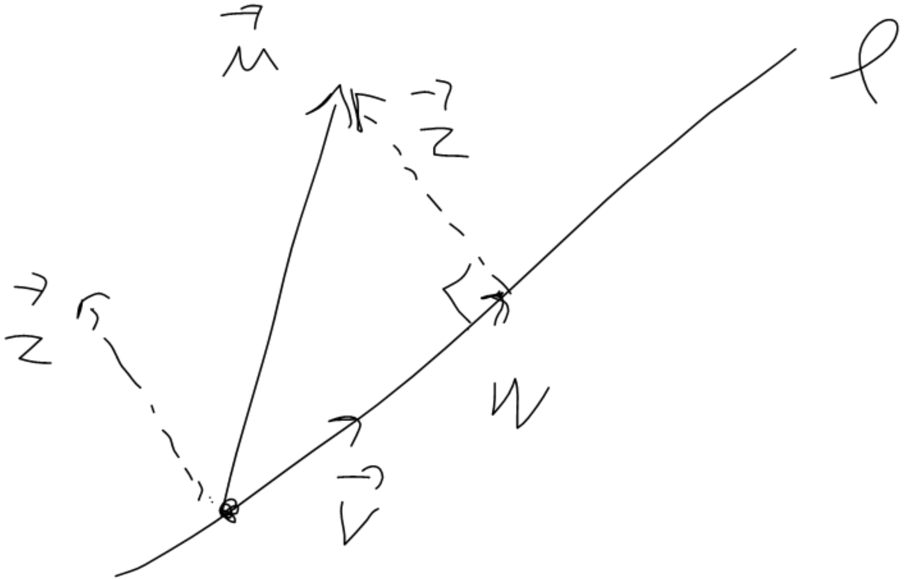
$$\vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} =$$

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\,\vec{u}\cdot\vec{v}$$

Projektion

$l = \operatorname{span}\{\vec{v}\}$ $\quad$ $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$

$\vec{u} \in \mathbb{R}^n$



Find ortogonal projektion $\vec{w}$ af $\vec{u}$ på ℓ

$\vec{u} = \vec{w} + \vec{z}$ og $\vec{z} \cdot \vec{v} = 0$

$\vec{w} \in \operatorname{span}\{\vec{v}\}$ Dvs der findes skalar c

så $\vec{w} = c\,\vec{v}$

$$0 = \vec{z} \cdot \vec{v} = (\vec{u} - \vec{w}) \cdot \vec{v} = (\vec{u} - c\vec{v}) \cdot \vec{v} =$$

$$\vec{u} \cdot \vec{v} - c\,\vec{v} \cdot \vec{v}$$

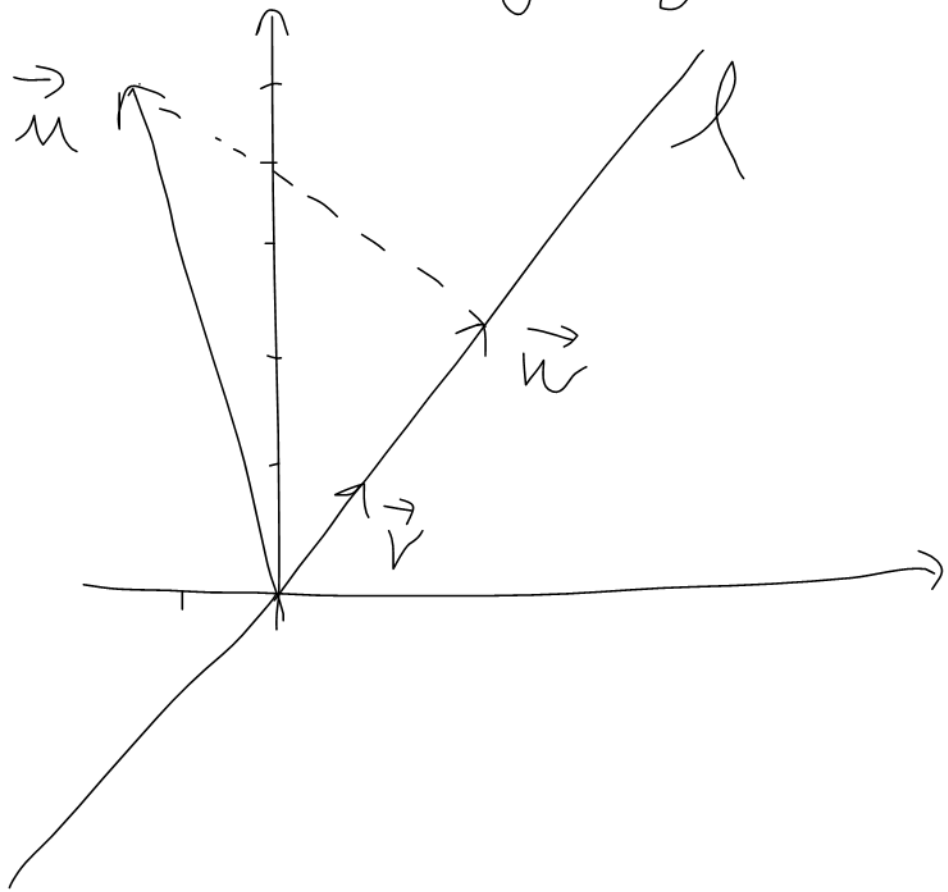
$$c\,\vec{v} \cdot \vec{v} = \vec{u} \cdot \vec{v} \Rightarrow c = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$$

$$\vec{w} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

<u>Eks.</u> projektion

l : linie i $\mathbb{R}^2$ med ligning $Y = X$

$$\vec{u} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$



$$\vec{v} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$l = \text{span}\{\vec{v}\}$$

$$\vec{w} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{\begin{bmatrix} -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 5 \end{bmatrix}}{\begin{bmatrix} 5 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 5 \end{bmatrix}} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{-5+25}{25+25} \begin{bmatrix} 5 \\ 5 \end{bmatrix} =$$

$$\frac{20}{50} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

6.2

$S = \{\vec{v}_1, \dots, \vec{v}_k\}$ vektorer i $\mathbb{R}^n$

S er ortogonal hvis $\vec{v}_i \cdot \vec{v}_j = 0$ når $i \neq j$

S er ortonormal hvis S er ortogonal og $\|\vec{v}_i\|$ for alle i.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_1 \cdot \vec{v}_2 = 1 + 0 - 1 = 0, \quad \vec{v}_1 \cdot \vec{v}_3 = -1 + 0 + 1 = 0$$

$\vec{v}_2 \cdot \vec{v}_3 = -1 + 2 - 1 = 0$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ er ortogonal

$\left\{\frac{1}{\|\vec{v}_1\|}\vec{v}_1, \frac{1}{\|\vec{v}_2\|}\vec{v}_2, \frac{1}{\|\vec{v}_3\|}\vec{v}_3\right\}$ ortonormal.

Hvis $\{\vec{v}_1 \ldots \vec{v}_k\}$ er ortonormal basis for underrom V og $\vec{u} \in V$

så er $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$

hvis $\quad c_i = \dfrac{\vec{u} \cdot \vec{v}_i}{\|\vec{v}_i\|^2} = \vec{u} \cdot \vec{v}_i$

ortogonal basis $\qquad\qquad$ ortonormal

Bevis

$$\vec{u} \cdot \vec{v}_i = \left(c_1 \vec{v}_1 + \dots + c_k \vec{v}_k\right) \cdot \vec{v}_i = c_i \, \vec{v}_i \cdot \vec{v}_i$$

$$c_i = \frac{\vec{u} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

<u>Eks</u>

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

ortonormal basis for $\mathbb{R}^3$

$$\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad , \quad \vec{u} \cdot \vec{v}_1 = 3 \ , \ \vec{u} \cdot \vec{v}_2 = 4 \ , \ \vec{u} \cdot \vec{v}_3 = 5$$

$$\vec{u} = 3\vec{v}_1 + 4\vec{v}_2 + 5\vec{v}_3$$

Gram - Schmidt

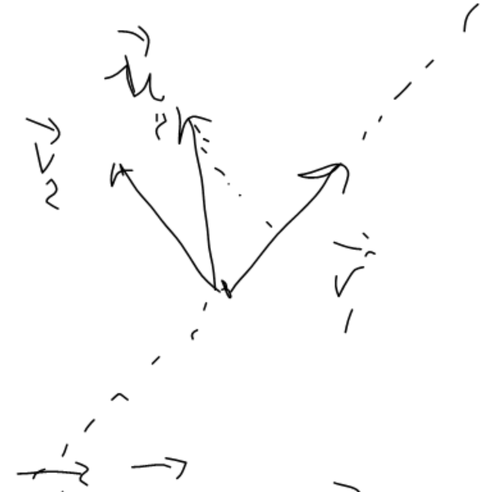
$\{\vec{u}_1 \dots \vec{u}_k\}$ basis for underrum W

Find ortogonal basis for W

$\{\vec{v}_1 \dots \vec{v}_k\}$

$$\vec{v}_1 = \vec{u}_1$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1$$



$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$\vdots$$

<u>Eks</u> på Gram-Schmidt

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Basis for $\mathbb{R}^3$

$$\vec{v}_1 = \vec{u}_1 \qquad \|\vec{v}_1\|^2 = 1^2 + 0^2 + 0^2 = 1$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} =$$

$$\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\|\vec{v}_2\|^2 = 1$$

$$\vec{v}_3 = \begin{bmatrix}1\\2\\2\end{bmatrix} - \frac{\begin{bmatrix}1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} =$$

$$\begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix}$$