6.3

S: mengde of vektorer i $\mathbb{R}^n$

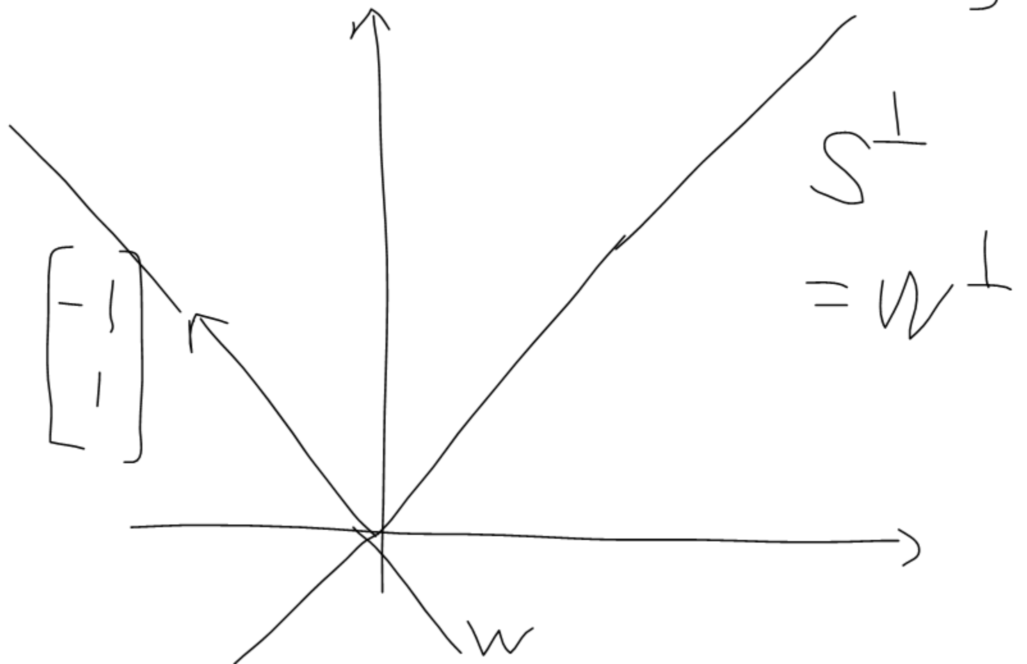
Definer: Det ortogonale komplement

$$S^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \text{ for alle } \vec{v} \in S\}$$

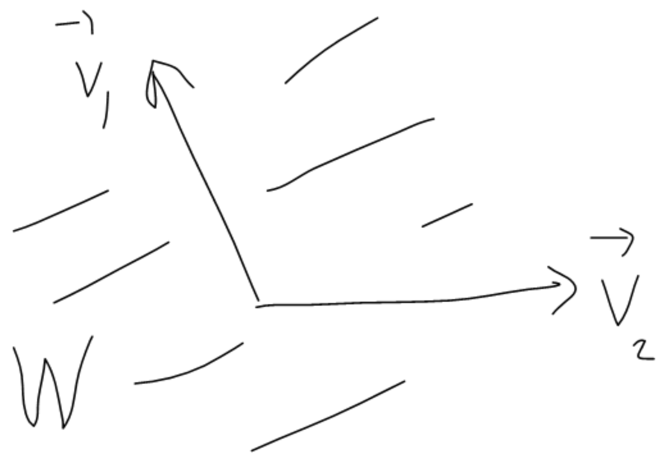
Eks $S = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ i $\mathbb{R}^2$

$W = \text{Span } S$

$W^\perp = S^\perp$



Eks i $\mathbb{R}^3$, $S = \{\vec{v}_1, \vec{v}_2\}$, $W = \operatorname{Span} S$



$$W^\perp = S^\perp$$

Eks ortogonal komplement af

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

$$S^\perp = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} 1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 - 1 \cdot x_4 = 0 \\ \text{og } 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 - 2 \cdot x_4 = 0 \end{array} \right\}$$

= løsningsmængde til ligningssystem med udvidet matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$W = \text{Span}\, S = \text{Col}\, A$$

$$W^{\perp} = S^{\perp} = \text{Null}\, A^T$$ et underrum af $\mathbb{R}^4$

Generelt:

Hvis $W = \text{Col } A = \text{Row } A^T$

underrum af $\mathbb{R}^n$

Så er $W^\perp = \text{Null } A^T$

$\dim W = \dim \text{Col } A = \text{rank } A$

$\dim W^\perp = \dim \text{Null } A^T = \text{nullity } A^T$

$= \text{antal søjler i } A^T - \text{rank } A^T$

$= n - \text{rank } A$

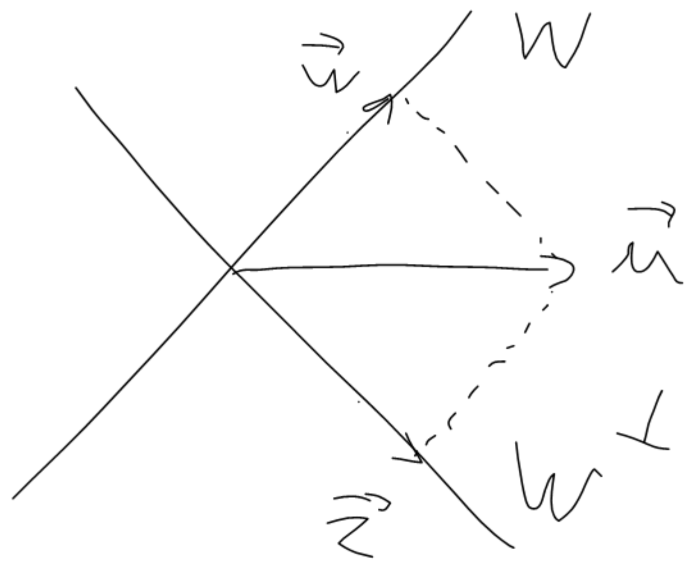
$$\dim W + \dim W^{\perp} = n$$

$$\dim W^{\perp} + \dim (W^{\perp})^{\perp} = n$$

$$\dim W = \dim (W^{\perp})^{\perp}$$

$$W \subseteq (W^{\perp})^{\perp}$$

$$W = (W^{\perp})^{\perp}$$



Sætning 6.7

W: et underrum af $\mathbb{R}^n$, $\vec{u} \in \mathbb{R}^n$

Så findes entydige vektorer $\vec{w} \in W$ og $\vec{z} \in W^\perp$ som opfylder

$$\vec{u} = \vec{w} + \vec{z}$$

$\vec{w}$ kaldes ortogonal projektion af $\vec{u}$ på W

Hvis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ er ortonormal basis for W og $\{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$ er ortonormal basis for $W^\perp$.

Så er $\{\vec{v}_1, \ldots, \vec{v}_n\}$ en ortonormal basis for $\mathbb{R}^n$ og så er (afsnit 6.2)

$$\vec{u} = \underbrace{(\vec{u}\cdot\vec{v}_1)\vec{v}_1 + \ldots + (\vec{u}\cdot\vec{v}_k)\vec{v}_k}_{\vec{w} \in W} + \underbrace{(\vec{u}\cdot\vec{v}_{k+1})\vec{v}_{k+1} + (\vec{u}\cdot\vec{v}_n)\vec{v}_n}_{\vec{z} \in W^\perp}$$

Eks på projektion

$$W = \text{Span}\left\{ \underbrace{\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}}_{\vec{v}_2} \right\}$$

$\{\vec{v}_1, \vec{v}_2\}$ er ortonormal basis W

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Ortogonal projektion af $\vec{u}$ på W

$$(\vec{u} \cdot \vec{v}_1)\vec{v}_1 + (\vec{u} \cdot \vec{v}_2)\vec{v}_2 =$$

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \right) \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} =$$

$$\frac{5}{3}\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 7/9 \\ 11/9 \\ 8/9 \end{bmatrix}$$

Generelt:

Ortogonal projektion af $\vec{u}$ på underrum W skrives $U_W(\vec{u})$

$U_W : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ er en funktion

U_W er lineær (lineær operator)

Standard matrix for U_W: P_W

Hvis $\{\vec{b}_1, \ldots, \vec{b}_k\}$ er basis for W

og $C = [\vec{b}_1 \; \cdots \; \vec{b}_k]$ så er

$$P_W = C(C^TC)^{-1}C^T$$

<u>Eks</u> P_W

$$W = \text{span}\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}, \quad C = \begin{bmatrix}1 & 0\\1 & 0\\0 & 1\\0 & 1\end{bmatrix}$$

$$C^T C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(C^T C)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$P_W = C \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$U_W(\vec{u}) = P_W \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 7/2 \\ 7/2 \end{bmatrix}$$

$U_W(\vec{u})$ er den vektor i W der er

tættest på $\vec{u}$

6.4 (miniprojekt 4)

Ligningssystem

$$A\vec{x} = \vec{b}$$

Konsistent hvis $\vec{b} \in \text{Col } A$

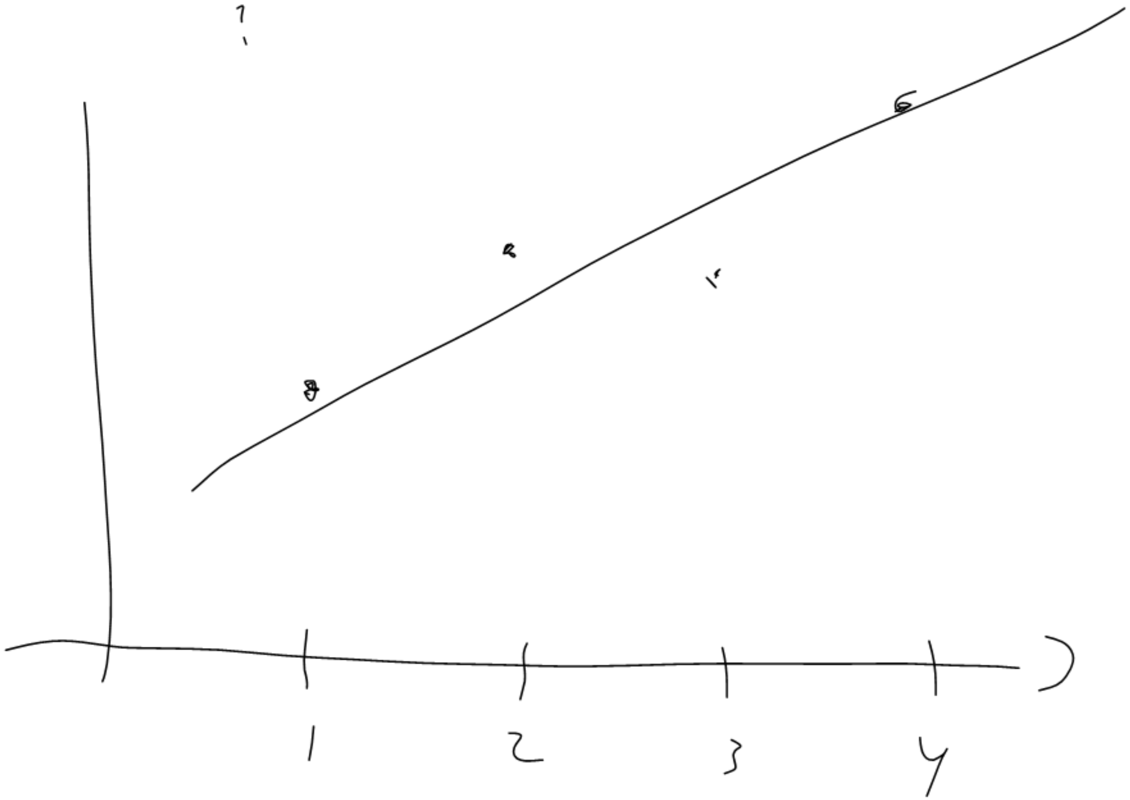
Hvis $\vec{b} \notin \text{Col } A$ så

$$A\vec{x} = \vec{b}'$$

hvor b' er projektion af $\vec{b}$ på $\text{Col } A$

$$-x_1 + x_3 = b_1$$
$$x_2 - x_4 = b_2$$
$$\vdots$$

$y = ax + b$



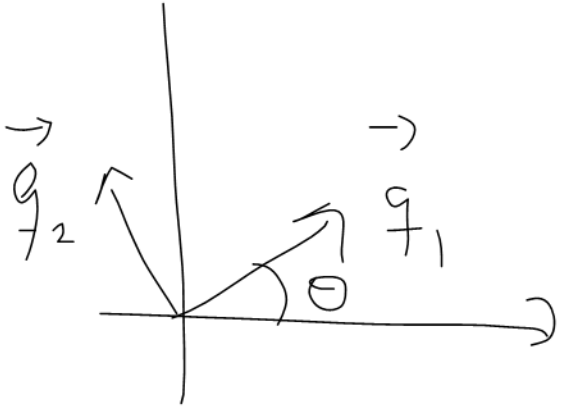
6.5

$$Q = \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix} \quad n \times n$$

Q sies å være ortogonal

hvis $\{\vec{q}_1, \dots, \vec{q}_n\}$ er ortonormal

$$Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \quad \text{er ortogonal}$$



$$\|\vec{q}_1\| = \|\vec{q}_2\| = 1$$

$$Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} = A_\theta$$

ortogonal