6.6

Hvis A er en symmetrisk $n \times n$ matrix $(A^T = A)$

så er A diagonaliserbar.

Sætning 6.14

A : symmetrisk matrix

Hvis $A\vec{v} = \lambda \vec{v}$ og $A\vec{u} = \mu \vec{u}$

og $\lambda \neq \mu$ Så er $\vec{u}$ og $\vec{v}$ ortogonale

Bevis

$$A\vec{v}\cdot\vec{u} = (A\vec{v})^T\vec{u} = \vec{v}^T A^T \vec{u} = \vec{v}\cdot A^T\vec{u}$$
$$= \vec{v}\cdot A\vec{u}$$

$$\Downarrow$$

$$\lambda\vec{v}\cdot\vec{u} = \vec{v}\cdot\mu\vec{u}$$

$$\Downarrow$$

$$\lambda(\vec{v}\cdot\vec{u}) = \mu(\vec{v}\cdot\vec{u})$$

$$\Downarrow$$

$$\vec{v}\cdot\vec{u} = 0 \qquad \text{da } \lambda \neq \mu$$

Velg en ortonormal basis for hvert egenrom.

Basene samlet:

En basis $\{\vec{P}_1 \ldots \vec{P}_n\}$ for $\mathbb{R}^n$

som ortonormal.

$P = [\vec{P}_1 \ldots \vec{P}_n]$ er en ortogonal matrix.

Setning 6.15

A: $n \times n$ matrix

A symmetrisk

$\Updownarrow$

Der en $n \times n$ ortogonal matrix P
og en $n \times n$ diagonal matrix D
så
$$A = PDP^{-1} = PDP^{T}$$

$(P^{-1} = P^{T}$ da P er ortogonal$)$

Eks diagonalisering af symmetrisk matrix

$$A = \begin{bmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\det(A - t I_3) = \det \begin{bmatrix} 3-t & 4 & 0 \\ 4 & -3-t & 0 \\ 0 & 0 & 5-t \end{bmatrix} \overset{\text{række 3}}{=}$$

$$(5-t) \det \begin{bmatrix} 3-t & 4 \\ 4 & -3-t \end{bmatrix} = (5-t)\left((3-t)(-3-t) - 4 \cdot 4\right) =$$

$$(5-t)\left(-9-3t+3t+t^2-16\right) = (5-t)\left(t^2-25\right)$$

$$t^2 - 25 = 0 \iff t = \pm\sqrt{25} = \pm 5$$

$$\det(A - tI_3) = (5-t)(t-5)(t+5) =$$

$$- (t-5)^2 (t+5)$$

Egenrum $\lambda = 5$

$$A - 5I_3 = \begin{bmatrix} -2 & 4 & 0 \\ 4 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

x_2, x_3 frie

$x_1 - 2x_2 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ orthogonal basis
(Else: Gram-Schmidt)

$$\left\| \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\| = \sqrt{2^2+1^2+0^2} = \sqrt{5} \quad , \quad \left\| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\| = 1$$

$$\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ orthonormal basis

Eigenvalue $\lambda = -5$

$$A + 5I_3 = \begin{bmatrix} 8 & 4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 fri

$x_1 + \frac{1}{2}x_2 = 0$

$x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} = \frac{1}{2}x_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ basis $\qquad \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ ortonormal basis

Sett

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} \text{ og } D = \begin{bmatrix} 5 & & \\ & 5 & \\ & & -5 \end{bmatrix}$$

Så er $A = PDP^T$, P ortogonal

6.5 Ortogonal matrix Q

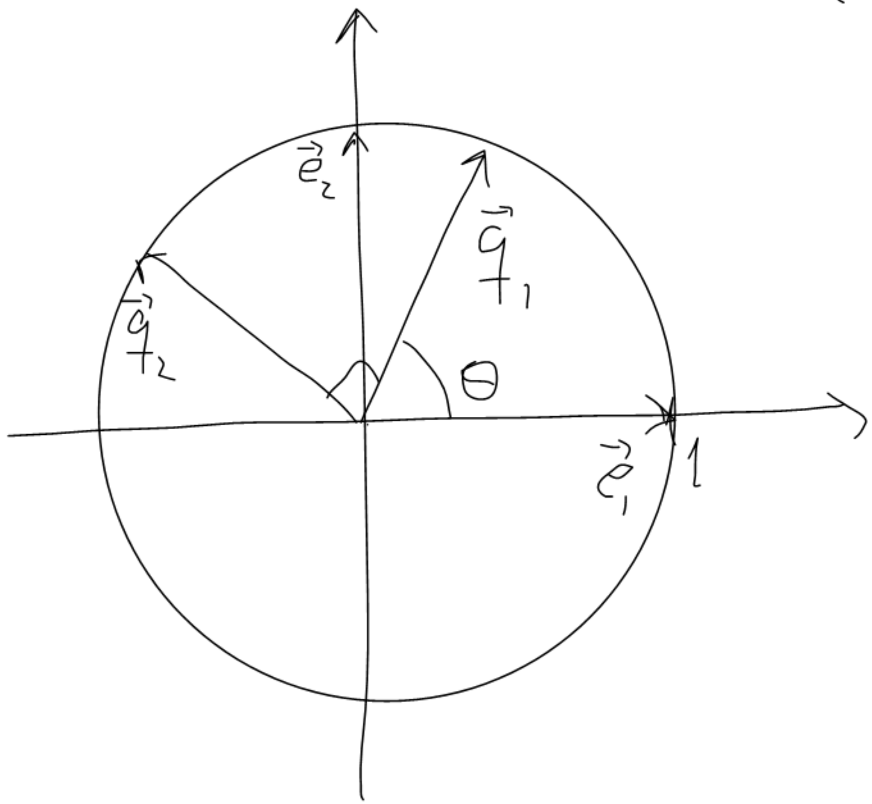
$$Q^T Q = I_n$$

$$1 = \det I_n = \det Q^T Q =$$

$$\det Q^T \cdot \det Q = \det Q \cdot \det Q$$

$$(\det Q)^2 = 1 \Rightarrow \det Q = \pm 1$$

Ortogonal 2×2 matrix $\quad Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$



$$\vec{q}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\vec{q}_2 = \begin{bmatrix} \cos(\theta + 90^\circ) \\ \sin(\theta + 90^\circ) \end{bmatrix}$$

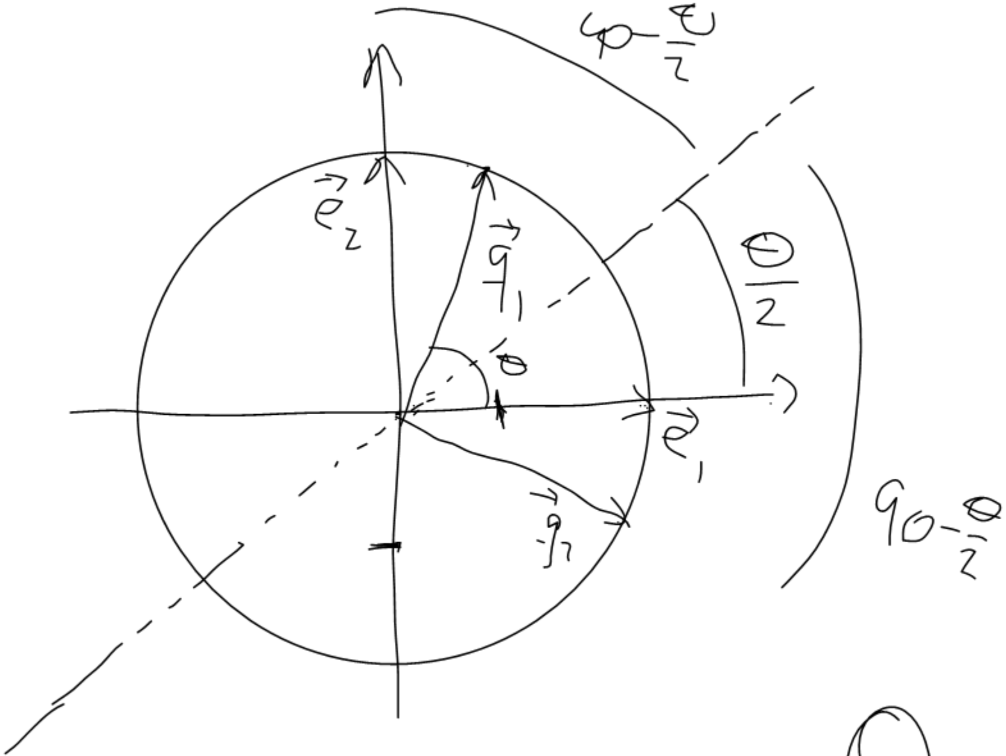
$$= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation

Vinkel θ

$$\det Q = (\cos\theta)^2 + (\sin\theta)^2 = 1$$



$$\vec{q}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\vec{q}_2 = \begin{bmatrix} \cos(\theta - 90^\circ) \\ \sin(\theta - 90^\circ) \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

$$\det Q = -(\cos\theta)^2 - (\sin\theta)^2 = -1$$

Spejling om akse, som er egenrum hørende til egenværdi 1

<u>Eks</u> rotation

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$ en ortogonal matrix

$$\det Q = \frac{2}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} - \left(-\frac{1}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{4}{5} + \frac{1}{5} = 1$$

En rotation med vinkel Θ,
hvor $\cos\Theta = \frac{2}{\sqrt{5}}$, $\sin\Theta = \frac{1}{\sqrt{5}}$

$$\Theta \approx 26{,}565^\circ$$

Sætning 6.9

Q er ortogonal matrix

$$\Updownarrow$$

$Q\vec{u} \cdot Q\vec{v} = \vec{u} \cdot \vec{v}$ for alle vektor $\vec{u}$ og $\vec{v}$

$$\Updownarrow$$

$\|Q\vec{u}\| = \|\vec{u}\|$ for alle $\vec{u} \in \mathbb{R}^n$

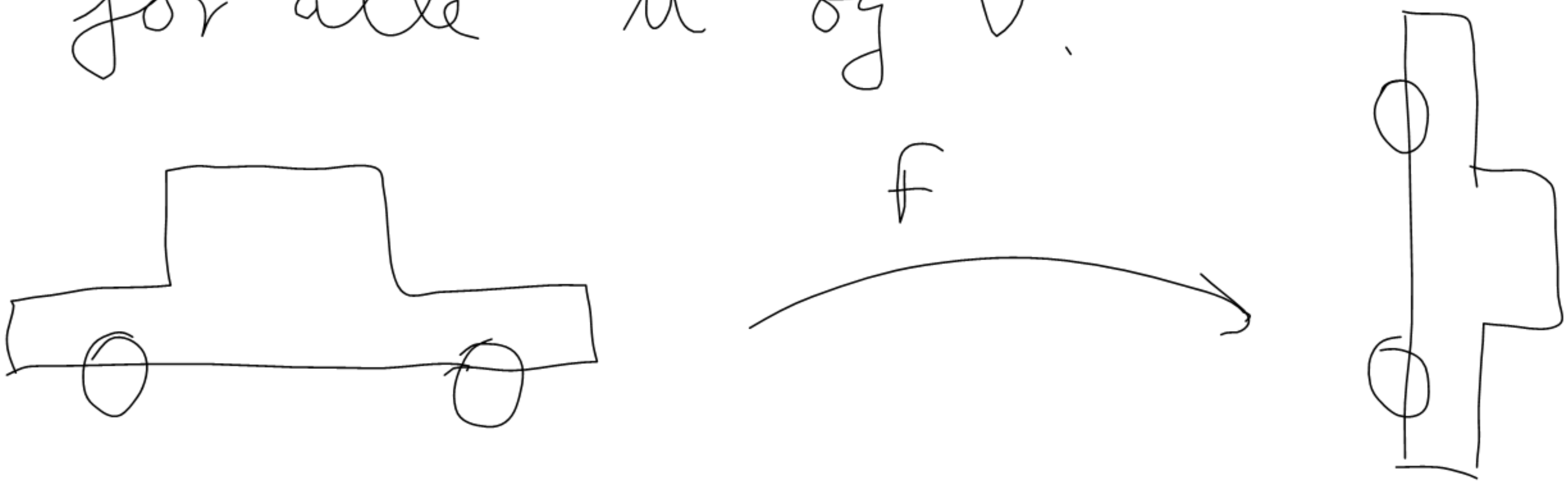
Rigid motion = flytning

$F : \mathbb{R}^n \to \mathbb{R}^n$ funktion, måske ikke lineær

F kaldes en flytning hvis

$$\|F(\vec{u}) - F(\vec{v})\| = \|\vec{u} - \vec{v}\|$$

for alle $\vec{u}$ og $\vec{v}$.



Måske er $F(\vec{0}) \neq \vec{0}$

Definer $T : \mathbb{R}^n \to \mathbb{R}^n$

ved $T(\vec{v}) = F(\vec{v}) - F(\vec{0})$

Så er

- T lineær
- T bevarer afstand
- T's standardmatrix Q er ortogonal

$T(\vec{x}) = Q\vec{x}$ og $F(\vec{x}) = T(\vec{x}) + F(\vec{0}) = Q\vec{x} + F(\vec{0})$