MCG - 11

$\mathbf{R}$ is a 3×3 rotation matrix.

Determine axis-angle representation of this rotation,
i.e., a vector $\hat{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and an angle θ so that $\mathbf{R} = \mathbf{R}_{\hat{\mathbf{r}}\theta}$.

Compute $\text{trace}(\mathbf{R}) = R_{00} + R_{11} + R_{22}$.

Then $\theta = \cos^{-1}(\frac{\text{trace}(\mathbf{R})-1}{2})$. This gives $0^\circ \leq \theta \leq 180^\circ$.
$\cos^{-1}$ is also written as arccos.

If $\theta = 0^\circ$: no rotation, $\hat{\mathbf{r}}$ is arbitrary (and $R = I$).

If $\theta \neq 0^\circ$ and $\theta \neq 180^\circ$:

$$\mathbf{r} = (R_{21} - R_{12}, R_{02} - R_{20}, R_{10} - R_{01}), \quad \hat{\mathbf{r}} = \frac{1}{||\mathbf{r}||}\mathbf{r}.$$

If $\theta = 180^\circ$: Determine the largest of the numbers R_{00}, R_{11}, R_{22}.

R_{00} largest: $\quad x = \frac{1}{2}\sqrt{R_{00} - R_{11} - R_{22} + 1}, \quad y = \frac{R_{01}}{2x}, \quad z = \frac{R_{02}}{2x}.$

R_{11} largest: $\quad y = \frac{1}{2}\sqrt{R_{11} - R_{00} - R_{22} + 1}, \quad x = \frac{R_{01}}{2y}, \quad z = \frac{R_{12}}{2y}.$

R_{22} largest: $\quad z = \frac{1}{2}\sqrt{R_{22} - R_{00} - R_{11} + 1}, \quad x = \frac{R_{02}}{2z}, \quad y = \frac{R_{12}}{2z}.$

A **quaternion** q is written as

$$q = (w, x, y, z),$$

or

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

If we let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then we also write

$$q = (w, \mathbf{v}),$$

or

$$q = w + \mathbf{v}.$$

Addition of quaternions:

$$(w_1, x_1, y_1, z_1)+(w_2, x_2, y_2, z_2) = (w_1+w_2, x_1+x_2, y_1+y_2, z_1+z_2).$$

Scalar multiplication:

$$a(w, x, y, z) = (aw, ax, ay, az).$$

Magnitude of a quaternion $q = (w, x, y, z)$:

$$||q|| = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

If $q \neq (0, 0, 0, 0)$ then the quaternion

$$\frac{1}{||q||}q$$

has magnitude 1 and is said to be normalized.

Rotation around the axis $\hat{\mathbf{r}}$ with angle θ is represented by the quaternion

$$q = (\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}).$$

Or by

$$(\cos\left(\frac{360^\circ - \theta}{2}\right), \sin\left(\frac{360^\circ - \theta}{2}\right)(-\hat{\mathbf{r}})) = (-\cos\left(\frac{\theta}{2}\right), -\sin\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}) = -q.$$

The matrix for the rotation, represented by the normalized quaternion $q = (w, x, y, z)$:

$$\begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & 1 - 2x^2 - 2z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & 1 - 2x^2 - 2y^2 \end{bmatrix}$$