4.4 Koordinatsystem

$\mathbb{R}^2$, Eksempel

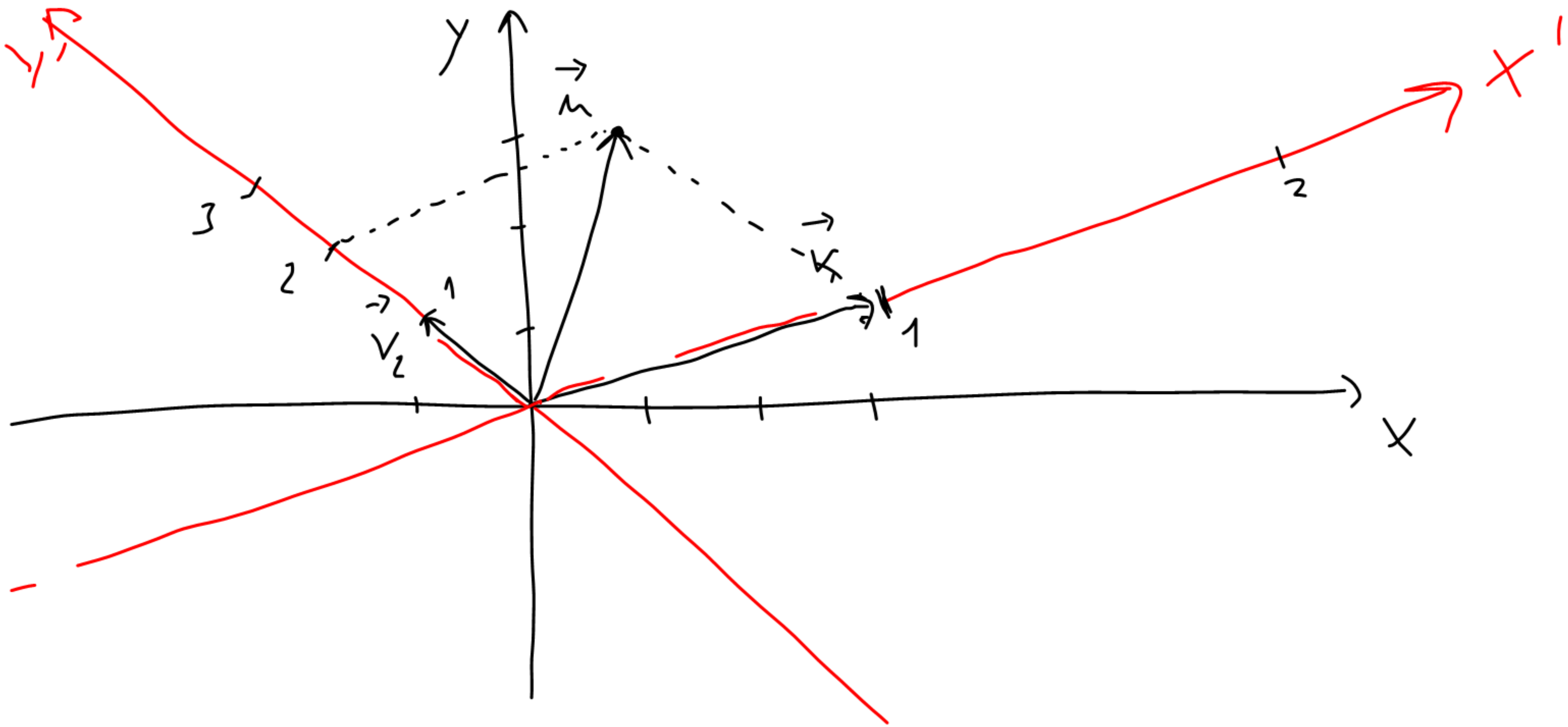
$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\{\vec{v}_1, \vec{v}_2\}$ er basis for $\mathbb{R}^2$

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \in \mathbb{R}^2 = \text{span}\{\vec{v}_1, \vec{v}_2\}$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{u} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\vec{u} = \vec{v}_1 + 2\vec{v}_2$$



Nytt koordinatsystem med x'-akse og y'-akse

$\vec{u}$ har koordinater $x' = 1$ og $y' = 2$

Sætning 4.10

V: underrum af $\mathbb{R}^n$

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_k\}$

Så kan enhver vektor $\vec{v} \in V$ skrives entydigt som

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + \dots + a_k\vec{b}_k$$

Bevis

Da $\operatorname{span} \mathcal{B} = V$ kan vi skrive

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2 + \dots + a_k \vec{b}_k$$

Hvis også

$$\vec{v} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \dots + c_k \vec{b}_k$$

så er (1. ligning − 2. ligning)

$$\vec{0} = \vec{v} - \vec{v} = (a_1 - c_1)\vec{b}_1 + (a_2 - c_2)\vec{b}_2 + \dots + (a_k - c_k)\vec{b}_k$$

Da $\mathcal{B}$ er lineært uafhængig er

$$a_1 - c_1 = 0 \quad , \quad a_2 - c_2 = 0 \quad , \quad \dots \quad a_k - c_k = 0$$

Altså $a_1 = c_1 \quad , \quad a_2 = c_2 \quad \dots \quad a_k = c_k$.

Definition $V = \mathbb{R}^n$

Lad $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ være basis for $\mathbb{R}^n$.

Hvis $\vec{v} \in \mathbb{R}^n$ kan skrives

$$\vec{v} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \ldots + c_n \vec{b}_n$$

Vektoren

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

kaldes så koordinat-vektoren for $\vec{v}$ m.h.t. basen $\mathcal{B}$.

Eks

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$ basis for $\mathbb{R}^2$

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eks

$$\mathcal{E} = \{ \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n \}$$

standard basen for $\mathbb{R}^n$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_n\vec{e}_n$$

$$[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \vec{v}$$

Teori

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$$

basis for $\mathbb{R}^n$

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \end{bmatrix} \quad n \times n \text{ matrix}$$

Da $\mathcal{B}$ er basis har B pivot i alle rækker / søjler.

B har altså en invers B^{-1}

Hvis $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ så er

$$(*) \quad \vec{v} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \dots + c_n \vec{b}_n$$

$$= \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = B [\vec{v}]_{\mathcal{B}}$$

Altså (Setning 4.11)

$$\vec{v} = B\,[\vec{v}]_{\mathcal{B}}$$

$$[\vec{v}]_{\mathcal{B}} = B^{-1}\,\vec{v}$$

Eller bestem $[\vec{v}]_{\mathcal{B}}$ ved å løse (*)
(se første eksempel)

Eks

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Er $\mathcal{B}$ en basis for $\mathbb{R}^3$

Hvis ja så omregn mellom $\vec{v}$ og $[\vec{v}]_{\mathcal{B}}$.

$$[B \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

Da $\mathrm{ref}(B) = I_3$ så er $\mathcal{B}$ en basis

og $B^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

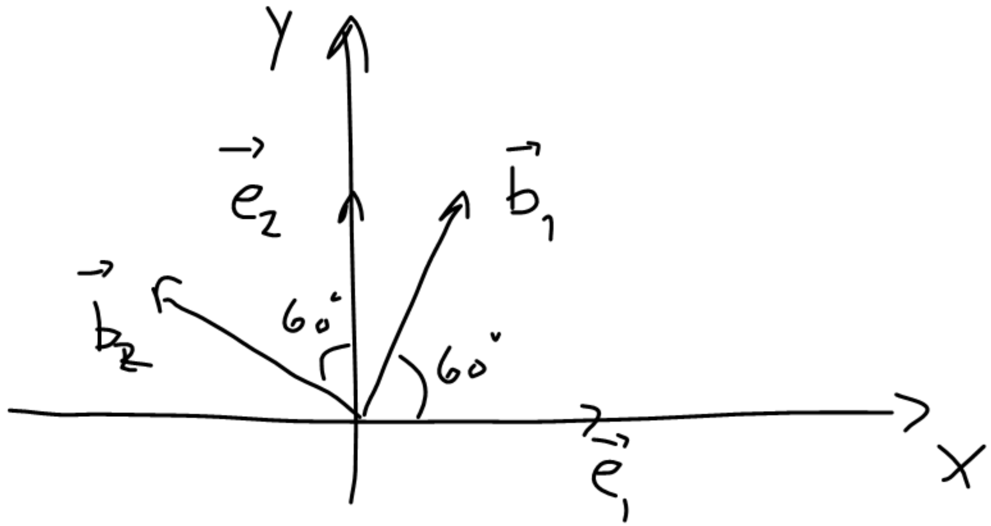
Lad $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $[\vec{v}]_{\mathcal{B}}$.

$$[\vec{v}]_{\mathcal{B}} = B^{-1}\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Hvis $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ så er

$$\vec{u} = B\,[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

<u>Eks</u> kurve i $\mathbb{R}^2$



$$A_{60^\circ} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

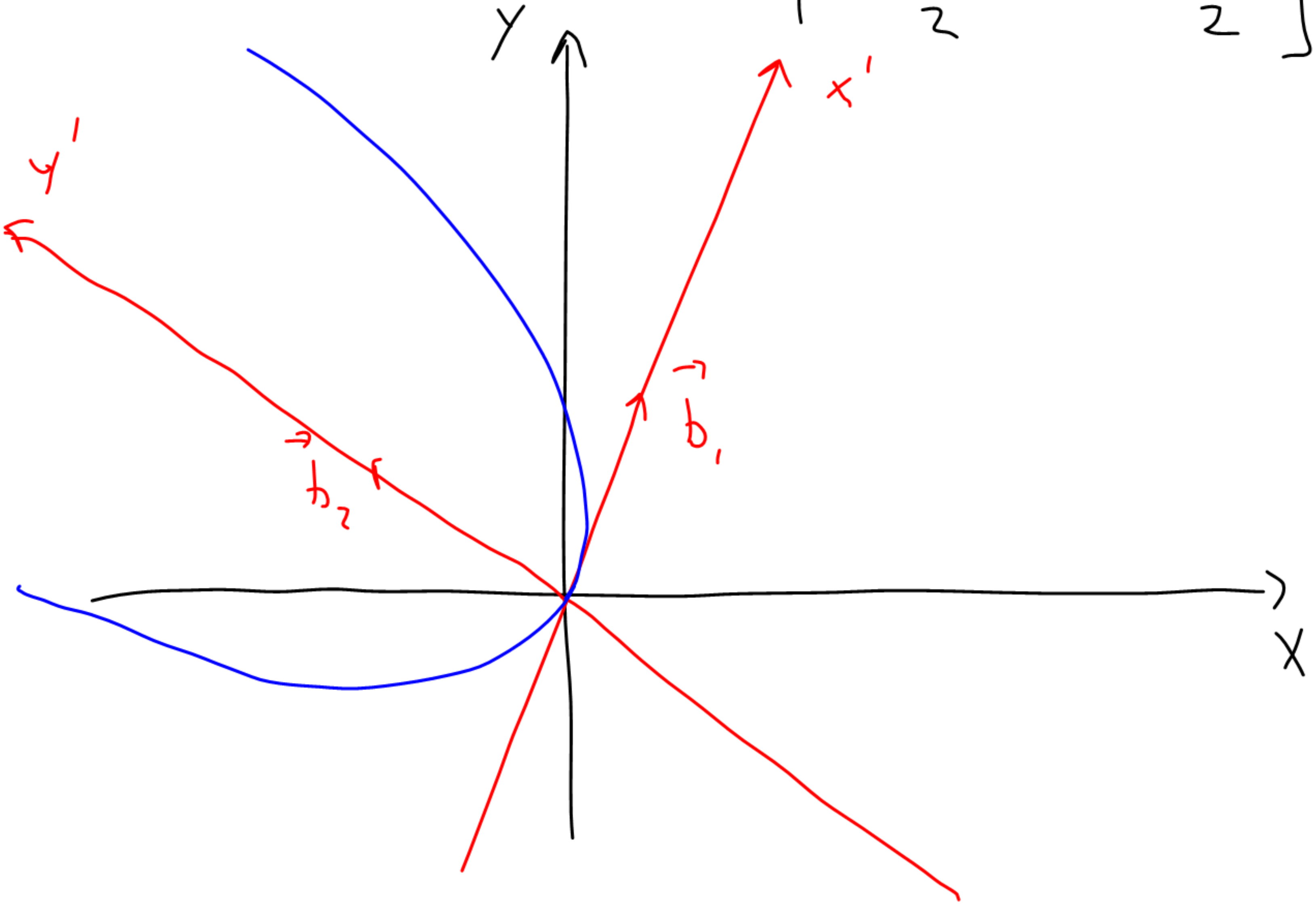
$$\vec{b}_1 = A_{60^\circ} \vec{e}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \quad \vec{b}_2 = A_{60^\circ} \vec{e}_2 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ basis for $\mathbb{R}^2$

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = A_{60^\circ}$$

$$B^{-1} = A_{-60^\circ} = \begin{bmatrix} \cos -60^\circ & -\sin -60^\circ \\ \sin -60^\circ & \cos -60^\circ \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$



Parabel med ligning $y' = (x')^2$

Find ligning for parablen i xy-koordinater.

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix} = B^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$$

$$y' = -\frac{\sqrt{3}}{2}x + \frac{1}{2}y$$

Parablen har ligning

$$y' = (x')^2$$

$$\Updownarrow$$

$$-\frac{\sqrt{3}}{2}x + \frac{1}{2}y = \left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^2$$