6.1

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$= \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{u}^T \vec{v}$$

Hvis $A : n \times n$

$$A\vec{u} \cdot \vec{v} = (A\vec{u})^T \vec{v} = (\vec{u}^T A^T) \vec{v} =$$

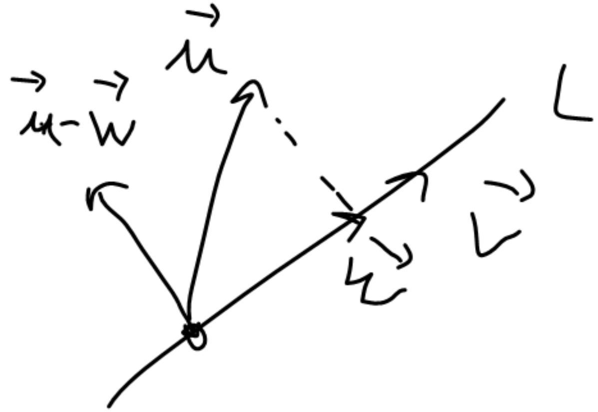
$$\vec{u}^T (A^T \vec{v}) = \vec{u} \cdot A^T \vec{v}$$

Ortogonal projektion

$\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, $L = \operatorname{span}\{\vec{v}\}$ linie

$\vec{u} \in \mathbb{R}^n$

$\vec{w}$: ortogonal projektion af $\vec{u}$ på L



Så er

$$\vec{w} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\,\vec{v}$$

$\vec{u}-\vec{w}$ er ortogonal på $\vec{v}$. Dvs $(\vec{u}-\vec{w})\cdot\vec{v} = 0$

6.2

$S = \{ \vec{v}_1, \ldots, \vec{v}_k \}$ vektorer i $\mathbb{R}^n$

S er ortogonal hvis $\vec{v}_i \cdot \vec{v}_j = 0$ når $i \neq j$

S er ortonormal hvis S er ortogonal og $\| \vec{v}_i \| = 1$ for alle i

Eks

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\vec{v}_1 \cdot \vec{v}_2 = 1\cdot 1 + 0\cdot 1 + 1\cdot(-1) = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = 1 + 0 - 1 = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = 1 - 2 + 1 = 0$$

$\left\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \right\}$ er ortogonal

$\left\{ \frac{1}{\|\vec{v}_1\|} \vec{v}_1, \frac{1}{\|\vec{v}_2\|} \cdot \vec{v}_2, \frac{1}{\|\vec{v}_3\|} \vec{v}_3 \right\}$ er ortonormal.

Setn. 6.5

Hvis $S = \{ \vec{v}_1, \ldots, \vec{v}_k \}$ er ortogonal og $\vec{v}_i \neq \vec{0}$ for alle i

Så er S lineært uafhængig.

Antag $\{\vec{v}_1, \dots, \vec{v}_k\}$ er en ortogonal basis for V.

Hvis $\vec{u} \in V$ så er $\vec{u}$ linear kombination

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

Find c_i:

$$\underline{\vec{u} \cdot \vec{v}_i} = (c_1 \vec{v}_1 + \dots + c_k \vec{v}_k) \cdot \vec{v}_i =$$

$$c_1 \underbrace{\vec{V}_1 \cdot \vec{V}_i}_{0,\ \text{hvis } i \neq 1} + \ldots + c_i \vec{V}_i \cdot \vec{V}_i + \ldots + c_k \vec{V}_k \cdot \vec{V}_i =$$

$$\underline{c_i \, \vec{V}_i \cdot \vec{V}_i}$$

Altså $\quad c_i = \dfrac{\vec{u} \cdot \vec{V}_i}{\vec{V}_i \cdot \vec{V}_i}$

> Hvis $\{\vec{v}_1 \ldots \vec{v}_k\}$ er ortonormal så er
>
> $$c_i = \vec{u} \cdot \vec{V}_i$$

<u>Eks</u>

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

en ortonormal basis for $\mathbb{R}^3$

$\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$, $\vec{u} \cdot \vec{e}_1 = 3$, $\vec{u} \cdot \vec{e}_2 = 2$, $\vec{u} \cdot \vec{e}_3 = 4$

Så er

$$\vec{u} = 3\vec{e}_1 + 2\vec{e}_2 + 4\vec{e}_3$$

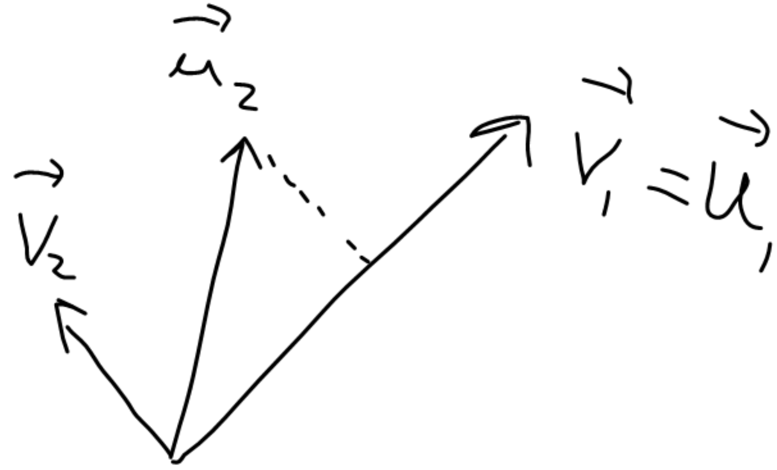
Gram - Schmidt

$\{\vec{u}_1, \ldots \vec{u}_k\}$ er basis for et underrum W

Find en ortogonal basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$

$$\vec{v}_1 = \vec{u}_1$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$



$$\vec{v}_3 = \underbrace{\vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1}_{\text{ortogonal på } \vec{v}_1} - \underbrace{\frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2}$$

$$\vec{v}_4 = \ldots$$

Eks

$$\vec{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix}1\\2\\2\end{bmatrix}$$

Basis for $\mathbb{R}^3$

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \vec{v}_1 \cdot \vec{v}_1 = 1$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} =$$

$$\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \vec{v}_2 \cdot \vec{v}_2 = 1$$

$$\vec{V}_3 = \begin{bmatrix}1\\2\\2\end{bmatrix} - \frac{\begin{bmatrix}1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}}{1} \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}{1} \begin{bmatrix}0\\1\\0\end{bmatrix} =$$

$$\begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

<u>Eks</u>

$$\vec{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix}1\\2\\2\end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\vec{V}_1 = \vec{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{V}_1 \cdot \vec{V}_1 = 1$$

$$\vec{V}_2 = \begin{bmatrix}1\\2\\2\end{bmatrix} - \frac{\begin{bmatrix}1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}}{1} \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\2\\2\end{bmatrix}$$

$$\vec{V}_2 \cdot \vec{V}_2 = 0 + 2^2 + 2^2 = 8$$

$$\vec{V}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}}{8} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{8} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{V}_3 \cdot \vec{V}_3 = 0 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{2}$$

$\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ er ortogonal

$$\|\vec{V}_1\| = 1, \quad \|\vec{V}_2\| = \sqrt{8} = 2\sqrt{2}, \quad \|\vec{V}_3\| = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

$$\vec{w}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{2\sqrt{2}}\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{w}_3 = \frac{1}{\|\vec{v}_3\|}\vec{v}_3 = \frac{2}{\sqrt{2}}\begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ er ortonormal.

Sol $Q = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix}$

$A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$

Skriv $A = QR$

$$\vec{w}_1 = \vec{v}_1 = \vec{u}_1 \quad , \quad \vec{u}_1 = 1 \cdot \vec{w}_1 + 0 \cdot \vec{w}_2 + 0 \cdot \vec{w}_3$$

$$\vec{u}_2 \cdot \vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$$

$$\vec{u}_2 \cdot \vec{w}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot 4 = 2\sqrt{2}$$

$$\vec{u}_2 \cdot \vec{w}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0$$

$$\vec{u}_2 = 1 \cdot \vec{w}_1 + 2\sqrt{2}\, \vec{w}_2 + 0 \cdot \vec{w}_3$$

$$\vec{u}_3 = \vec{w}_1 + \frac{1}{\sqrt{2}} \vec{w}_2 + \frac{1}{\sqrt{2}} \vec{w}_3$$

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2\sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$QR = [\vec{w}_1 \ \vec{w}_2 \ \vec{w}_3] R = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3] = A$$

$A = QR$ hvor R er övre triangulær og søjlerne i Q er ortonormale kaldes en QR faktorisering.