6.3

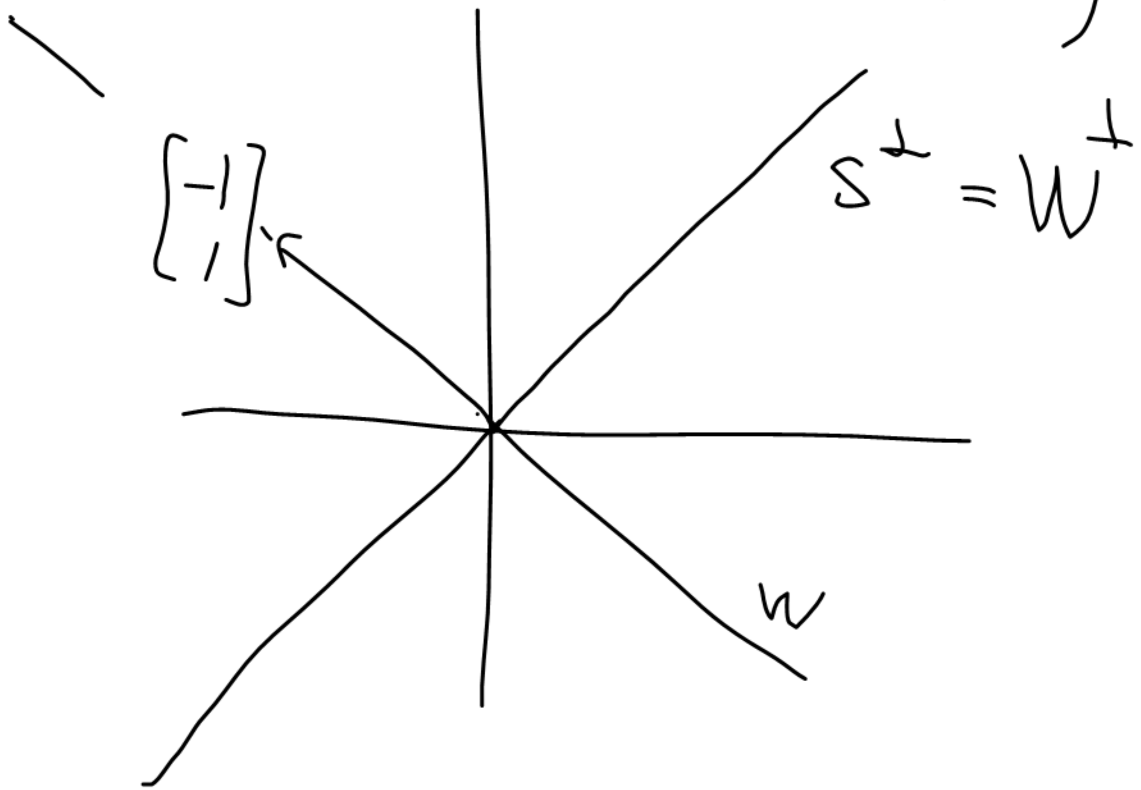
S: mengde af vektorer i $\mathbb{R}^n$

Ortogonal komplement

$$S^{\perp} = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \vec{x} \cdot \vec{v} = 0 \text{ for alle } \vec{v} \in S \right\}$$

Eks $S = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$W = \text{Span } S$



Eks $\vec{n} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$

$\{\vec{n}\}^{\perp}$ er planen med ligning $2x + 4y + 3z = 0$

$2x_1 + 4x_2 + 3x_3 = 0$

Eks i $\mathbb{R}^4$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$S^{\perp} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} 1 \cdot x_1 + 2x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 0 \\ \text{og } -1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 2 \cdot x_4 = 0 \end{array} \right\}$$

$$= \text{nulrum af } \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}$$

Sæt $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$ søjler fra S

Sæt $W = \text{Span } S = \text{Col } A$

Så er $W^{\perp} = S^{\perp} = \text{Null } A^T$

Generelt $A: n \times m$

Hvis $W = \text{Col } A = \text{Row } A^T$

Så er $W^\perp = \text{Null } A^T$

W og $W^\perp$ er underrum af $\mathbb{R}^n$

$\dim W = \dim \text{Col } A = \text{rank } A$

$\dim W^\perp = \dim \text{Null } A^T = \text{nullity } A^T$

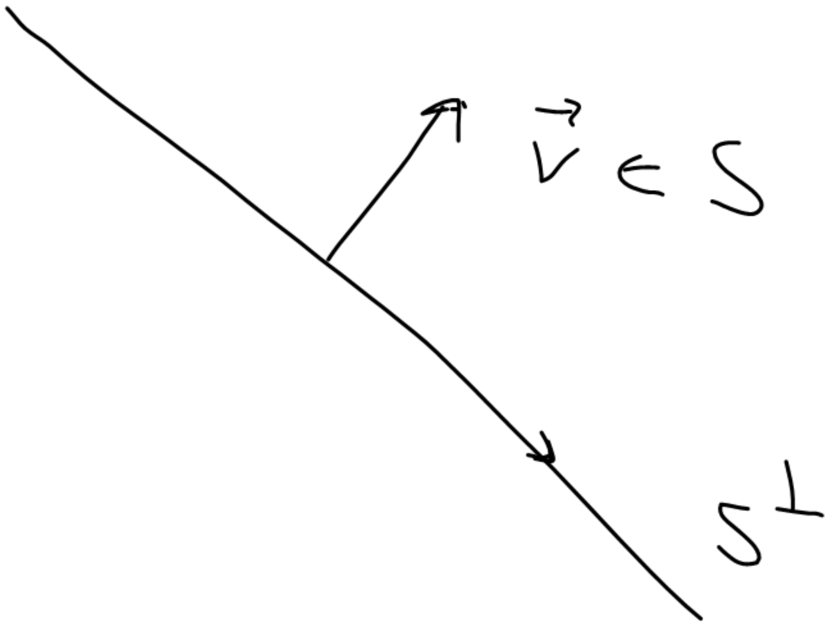
$= \text{antal søjler i } A^T - \text{rank } A^T =$

$$n - \operatorname{rank} A$$

$$\dim W + \dim W^\perp = \operatorname{rank} A + n - \operatorname{rank} A$$
$$= n$$

S: mængde af vektorer

$$S \subseteq (S^\perp)^\perp$$



W: underrum

$$W \subseteq (W^\perp)^\perp$$

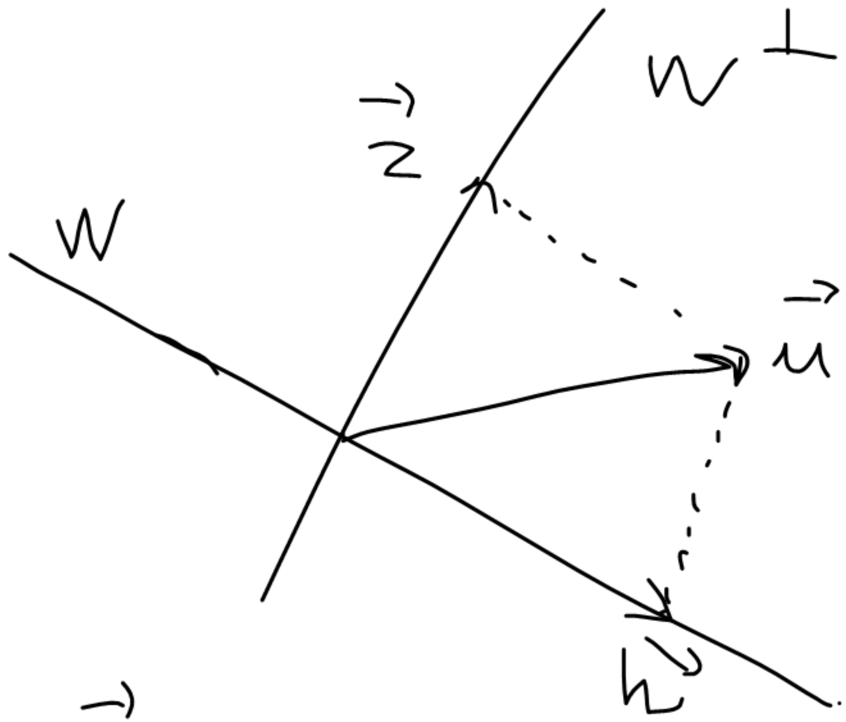
$$\dim W + \dim W^\perp = n \quad , \quad \dim W^\perp + \dim (W^\perp)^\perp = n$$

Altså $\dim W = \dim (W^\perp)^\perp$ og $W = (W^\perp)^\perp$

Sætning 6.7

W: underrum af $\mathbb{R}^n$

$\vec{u} \in \cancel{W}\ \mathbb{R}^n$



Der findes entydige $\vec{w} \in W$ og $\vec{z} \in W^\perp$

som opfylder $\vec{w} + \vec{z} = \vec{u}$

$\vec{w}$ kaldes ortogonal projektion af $\vec{u}$ på W

Hvis $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ er ortonormal basis for W

og $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ er ortonormal basis for $W^\perp$

så $\{\vec{v}_1 \dots \vec{v}_k \dots \vec{v}_n\}$ ortonormal basis for $\mathbb{R}^n$

og

$$\vec{u} = \underbrace{(\vec{u}\cdot\vec{v}_1)\vec{v}_1 + \dots + (\vec{u}\cdot\vec{v}_k)\vec{v}_k}_{\vec{w} \in W} + \underbrace{\dots + (\vec{u}\cdot\vec{v}_n)\vec{v}_n}_{\vec{z} \in W^\perp}$$

Eks

$$W = \mathrm{Span}\left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2\}$$

$\{\vec{v}_1, \vec{v}_2\}$ er ortonormal basis for W

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

ortogonal projektion af $\vec{u}$ på W

$$(\vec{u} \cdot \vec{v}_1)\vec{v}_1 + (\vec{u} \cdot \vec{v}_2) v_2 = \left(\frac{1}{3} + \frac{2}{3} + \frac{4}{3}\right)\vec{v}_1 + \left(\frac{2}{3} + \frac{1}{3} - \frac{4}{3}\right)\vec{v}_2$$

$$\frac{7}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2 = \begin{pmatrix} \frac{5}{9} \\ \frac{13}{9} \\ \frac{16}{9} \end{pmatrix}$$

Teori

Ortogonal projektion af $\vec{u}$ på W skrives $U_W(\vec{u})$

$U_W : \mathbb{R}^n \to \mathbb{R}^n$ er funktion

U_W er lineær (lineær operator)

Standardmatrix P_W

Hvis $\{\vec{b}_1, \ldots, \vec{b}_k\}$ er basis for W

og $C = [\vec{b}_1 \ \ldots \ \vec{b}_k]$ $\quad n \times k$ matrix

Så er $$P_W = \underbrace{C\,(\underbrace{C^T C}_{k \times k})^{-1}\,C^T}_{n \times n}$$

Eks

$$W = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$C^T C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(C^T C)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$P_W = C\left(C^TC\right)^{-1}C^T = C\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\vec{u} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

$$U_W(\vec{u}) = P_W\vec{u} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\ \frac{3}{2}\\ \frac{7}{2}\\ \frac{7}{2}\end{bmatrix}.$$

$U_W(\vec{u})$ er den vektor i W der er nærmest $\vec{u}$