6.5

$\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$ er en ortonormal basis.

Set $Q = [\vec{q}_1 \cdots \vec{q}_n]$ $n \times n$ matrix

Q siges at være ortogonal matrix.

Eks

$$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$ ortogonal matrix

Eks

$$Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} \quad \text{ortogonal matrix}$$

$$Q = \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix} \quad n \times n \text{ matrix}$$

Sæt $A = Q^T Q$

$$a_{ij} = \begin{cases} \vec{q}_i \cdot \vec{q}_j & i \neq j \\ \vec{q}_i \cdot \vec{q}_i = \|\vec{q}_i\|^2, & i = j \end{cases}$$

Hvis Q er ortogonal så er

$$A = Q^T Q = I_n$$

Omvendt hvis $Q^T Q = I_n$ så

er Q ortogonal.

Hvis Q er ortogonal så er

$$\det(Q^T Q) = \det I_n = 1$$

$$\det(Q^T Q) = \det Q^T \cdot \det Q =$$

$$\det Q \cdot \det Q = (\det Q)^2$$

Altså $(\det Q)^2 = 1$ og $\det Q = \pm 1$

$Q^T Q = I_n$ betyder at $Q^{-1} = Q^T$

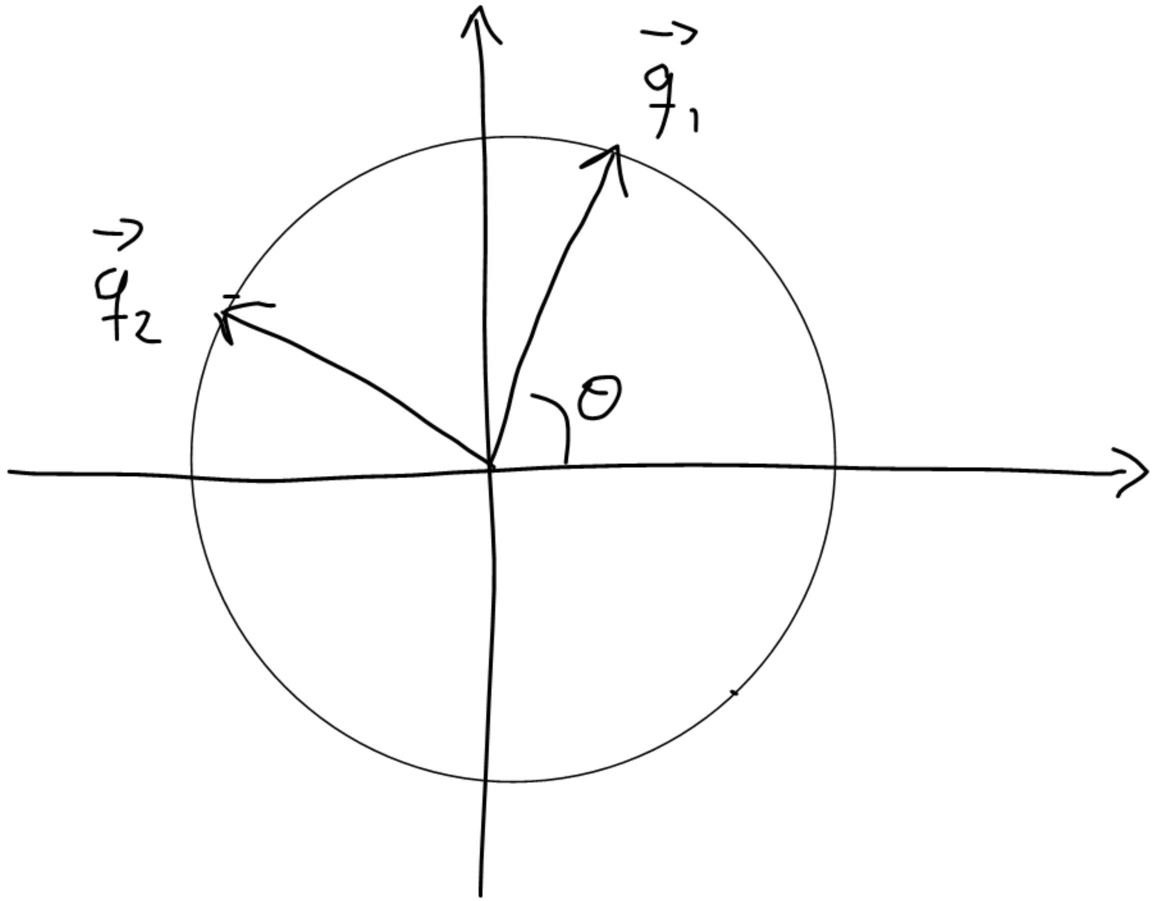
Desuden er

$$Q Q^{-1} = Q Q^T = I_n$$

Rækkerne i Q er en ortonormal mængde

Ortogonal 2×2 matrix

$$Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$$



$$\vec{q}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\vec{q}_2 = \begin{bmatrix} \cos(\theta + 90^\circ) \\ \sin(\theta + 90^\circ) \end{bmatrix}$$

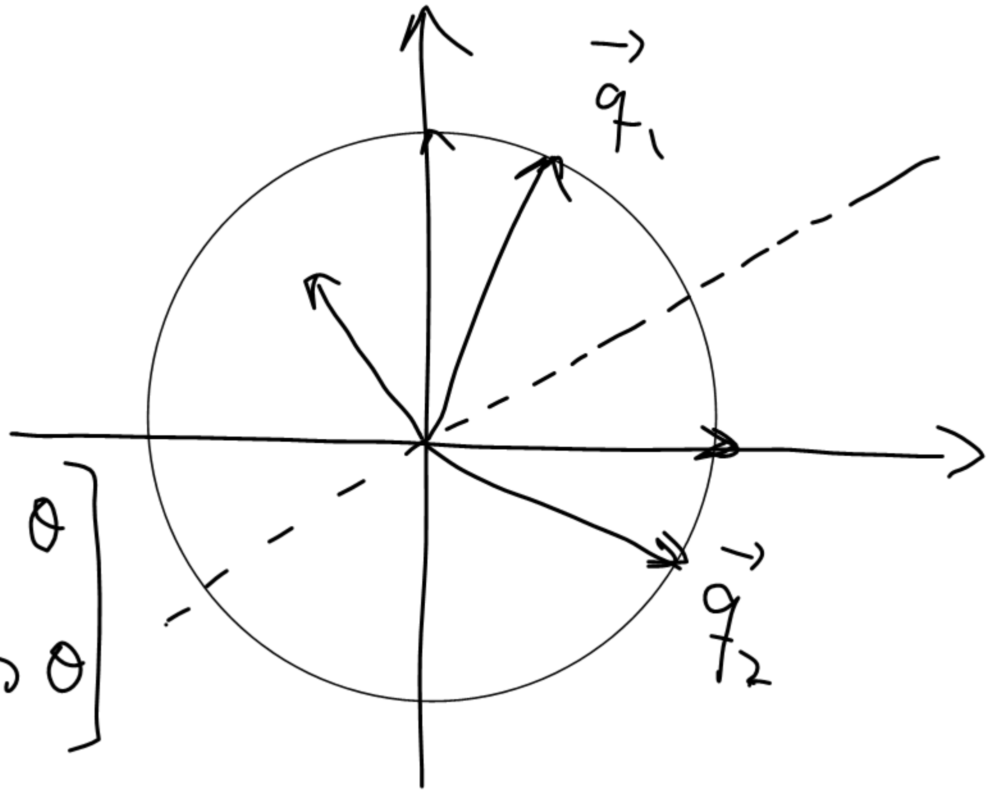
$$= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A_\theta \quad \text{rotation med vinkel } \theta$$

$$\det Q = (\cos\theta)^2 + (\sin\theta)^2 = 1$$

$$q_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$q_2 = \begin{bmatrix} \cos(\theta - 90^\circ) \\ \sin(\theta - 90^\circ) \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$



$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

$$\det Q = -(\cos\theta)^2 - (\sin\theta)^2 = -1$$

Q er standardmatrix for en spejling om en akse W (underrum)

W er egenrum hørende egenværdi 1

$W^{\perp}$ er egenrum hørende til egenværdi -1

Eks

$$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$\det Q = \frac{3}{5} \cdot \frac{3}{5} - \frac{4}{5}\left(-\frac{4}{5}\right) = 1$$

Q er rotation med vinkel θ

$$\cos\theta = \frac{3}{5}, \quad \sin\theta = \frac{4}{5}$$

$$\theta \approx 53^\circ$$

Eks

$$Q = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

$$\det Q = \frac{3}{5} \cdot \left(-\frac{3}{5}\right) - \frac{4}{5} \cdot \frac{4}{5} = -1$$

Egenrum $\lambda = 1$

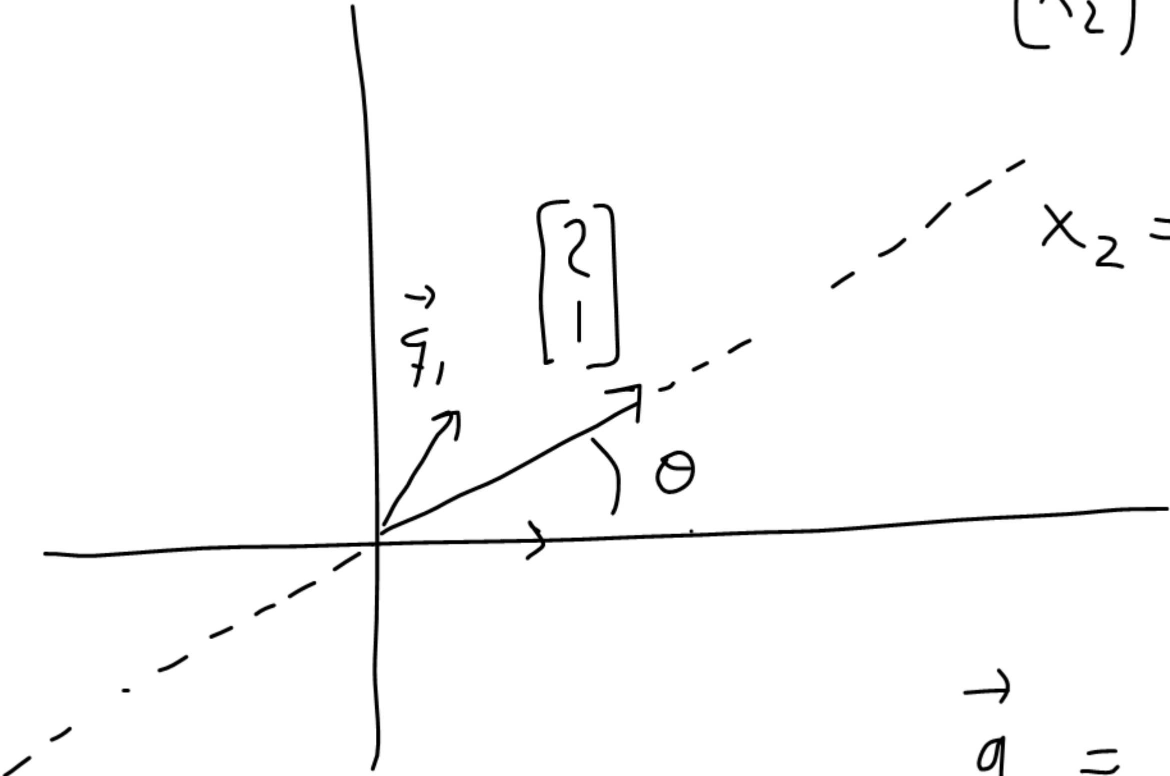
$$Q - I_2 = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} - \begin{pmatrix} \frac{5}{5} & 0 \\ 0 & \frac{5}{5} \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{8}{5} \end{pmatrix}$$

$$\text{rref} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$x_1 - 2x_2 = 0$

x_2 fri

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$x_2 = \frac{1}{2} x_1$

$$\vec{q}_1 = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$

Eks

$$Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} \qquad \text{ortogonal}$$

$R_2 - 2R_1 \to R_2$
$R_3 - 2R_1 \to R_3$

$$\det Q = \left(\frac{1}{3}\right)^3 \det \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} =$$

$$\frac{1}{27} \det \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 6 \\ 0 & -6 & 3 \end{bmatrix} = \frac{1}{27} \det \begin{bmatrix} -3 & 6 \\ -6 & 3 \end{bmatrix} =$$

$$\frac{1}{27}\left(-3 \cdot 3 - (-6) \cdot 6 \right) = 1$$

Da $\det Q = 1$ er Q standardmatrix for en rotation om akse W

W er egenrum hørende til egenværdi 1

Egenrum $\lambda = 1$

$$Q - I_3 = \frac{1}{3}\left(\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

$$\frac{1}{3}\begin{bmatrix} -2 & 2 & -2 \\ 2 & -2 & 2 \\ 2 & -2 & -4 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 = 0$, $x_3 = 0$, x_2 fri

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Hvis Q er en ortogonal $n \times n$ matrix og $\vec{u}, \vec{v} \in \mathbb{R}^n$

så er $Q\vec{u} \cdot Q\vec{v} = (Q\vec{u})^T Q\vec{v} =$

$\vec{u}^T Q^T Q \vec{v} = \vec{u}^T I_n \vec{v} = \vec{u}^T \vec{v} = \vec{u} \cdot \vec{v}$

$$\|Q\vec{u}\| = \sqrt{Q\vec{u}\cdot Q\vec{u}} = \sqrt{\vec{u}\cdot\vec{u}} = \|\vec{u}\|$$

Q bevarer længde, afstand (og vinkel).

<u>Sætning 6.9</u> $\quad Q : n \times n$

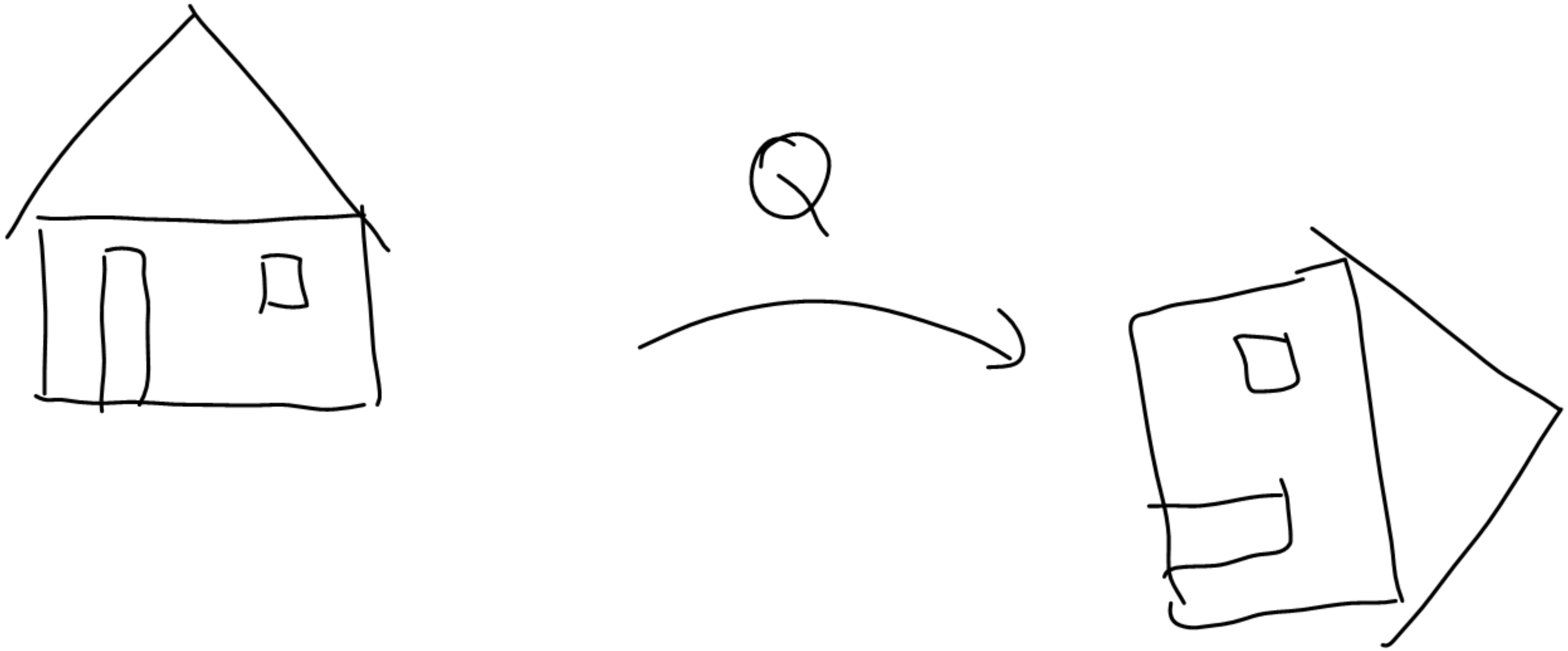
Q er ortogonal matrix

$$\Updownarrow$$

$Q\vec{u}\cdot Q\vec{v} = \vec{u}\cdot\vec{v}$ for alle $\vec{u},\vec{v}\in\mathbb{R}^n$

$$\Updownarrow$$

$\|Q\vec{u}\| = \|\vec{u}\|$ for alle $\vec{u}\in\mathbb{R}^n$



$$Q\vec{0} = \vec{0}$$