An Introduction to Pseudospectra and their Applications

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Definition of Pseudospectra

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Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. The ε-pseudospectrum of A is given by

$$\sigma_\varepsilon(A) = \sigma(A) \cup \{z \in \mathbf{C} \setminus \sigma(A) \mid \|(A - zI)^{-1}\| > \varepsilon^{-1}\}.$$

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Theorem

Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Then the following three statements are equivalent.

(i) $z \in \sigma_\varepsilon(A)$.

(ii) *There exists $B \in \mathcal{B}(\mathcal{H})$ with $\|B\| < \varepsilon$ such that $z \in \sigma(A + B)$.*

(iii) *$z \in \sigma(A)$ or there exists $v \in \mathcal{H}$ with $\|v\| = 1$ such that $\|(A - zI)v\| < \varepsilon$.*

Result on Pseudospectra, Finite Dimension

Let T be an $n \times n$ matrix. The square roots of the eigenvalues of T^*T are called the singular values of T. The smallest singular value is denoted $s_{\min}(T)$.

Theorem

Assume that $\mathcal{H}$ is finite dimensional and $T \in \mathcal{B}(\mathcal{H})$. Let $\varepsilon > 0$. Then $z \in \sigma_\varepsilon(T)$ if and only if $s_{\min}(T - zI) < \varepsilon$.

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Since the singular values of a matrix can be computed numerically, this result provides a method for plotting the pseudospectra of a given matrix. One chooses a finite grid of points in the complex plane, and evaluates $s_{\min}(T - zI)$ at each point. Plotting level curves for these points provides a picture of the pseudospectra of T.

Properties of $\sigma_\varepsilon(A)$

Define $D_\delta = \{z \in \mathbf{C} \mid |z| < \delta\}$.

Proposition

Let $A \in \mathcal{B}(\mathcal{H})$. Each $\sigma_\varepsilon(A)$ is a bounded open subset of $\mathbf{C}$. We have $\sigma_{\varepsilon_1}(A) \subset \sigma_{\varepsilon_2}(A)$ for $0 < \varepsilon_1 < \varepsilon_2$. Furthermore, $\cap_{\varepsilon>0}\sigma_\varepsilon(A) = \sigma(A)$. For $\delta > 0$ we have $D_\delta + \sigma_\varepsilon(A) \subseteq \sigma_{\varepsilon+\delta}(A)$. We have $\sigma_\varepsilon(A^*) = \overline{\sigma_\varepsilon(A)}$.

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Proposition

Let $A \in \mathcal{B}(\mathcal{H})$ and assume that $V \in \mathcal{B}(\mathcal{H})$ is invertible. Let $\kappa = \operatorname{cond}(V)(= \|V\| \cdot \|V^{-1}\|)$. Let $B = VAV^{-1}$. Then

$$\sigma(B) = \sigma(A),$$

and for $\varepsilon > 0$ we have

$$\sigma_{\varepsilon/\kappa}(A) \subseteq \sigma_\varepsilon(B) \subseteq \sigma_{\kappa\varepsilon}(A).$$

Properties of $\sigma_\varepsilon(A)$

Proposition

Let $A \in \mathcal{B}(\mathcal{H})$ *and* $\varepsilon > 0$. *Then*

$$\{z \mid \operatorname{dist}(z, \sigma(A)) < \varepsilon\} \subseteq \sigma_\varepsilon(A).$$

If A *is normal, then*

$$\sigma_\varepsilon(A) = \{z \mid \operatorname{dist}(z, \sigma(A)) < \varepsilon\}.$$

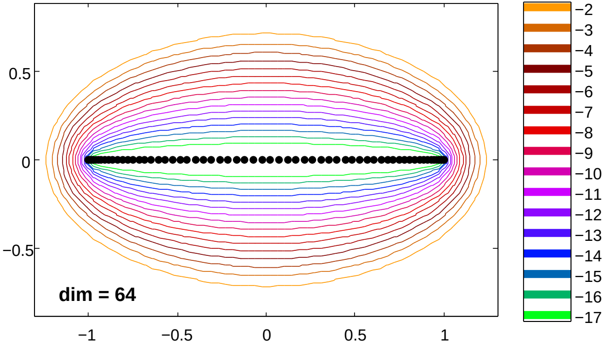
Example

A Toeplitz matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1/4 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1/4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1/4 & 0 \end{bmatrix}$$

We have $A = SDS^{-1}$ with D diagonal.

Example



Example

$A + E$, E random matrix with $\|E\| < 10^{-10}$. Plot of spectra: blue. Spectrum of A: red.