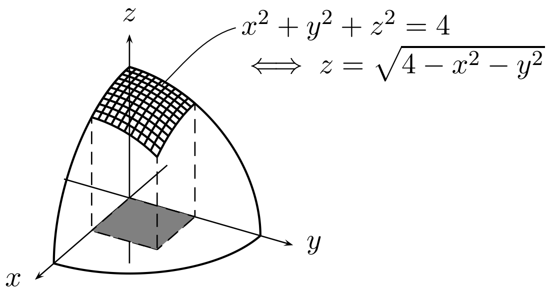
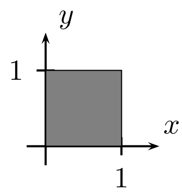
Opgave 13.3.52 i Edwards & Penney

I opgave 13.3.52 gennemskæres en kugle med centrum i $(0,0,0)$ og radius 2 symmetrisk af et kvadratisk prisme med sidelængde 2. Volumen af den del af kuglen, der fjernes, skal beregnes. Ved udnyttelse af symmetri om alle tre koordinatplaner kan vi nøjes med at betragte den del af voluminet, der ligger i første oktant.





$$
\begin{aligned}
V &= 8\int_0^1\int_0^1 \sqrt{4-x^2-y^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= 8\int_0^1 \left[\frac{y}{2}\sqrt{4-x^2-y^2}+\frac{4-x^2}{2}\arcsin\frac{y}{\sqrt{4-x^2}}\right]_0^1 \mathrm{d}x \\
&= 8\int_0^1 \left(\frac{1}{2}\sqrt{3-x^2}+\frac{4-x^2}{2}\arcsin\frac{1}{\sqrt{4-x^2}}\right)\mathrm{d}x \\
&= 4\int_0^1 \left(\sqrt{3-x^2}+(4-x^2)\arcsin\frac{1}{\sqrt{4-x^2}}\right)\mathrm{d}x \\
&= 4\left(\left[\frac{x}{2}\sqrt{3-x^2}+\frac{3}{2}\arcsin\frac{x}{\sqrt{3}}\right]_0^1+\int_0^1(4-x^2)\arctan\frac{1}{\sqrt{3-x^2}}\,\mathrm{d}x\right), \qquad \text{se (1)} \\
&= 4\Bigg(\left(\frac{\sqrt{2}}{2}+\frac{3}{2}\arcsin\frac{1}{\sqrt{3}}\right)+\left[\left(4x-\frac{x^3}{3}\right)\arctan\frac{1}{\sqrt{3-x^2}}\right]_0^1 \\
&\qquad -\int_0^1\left(\left(4x-\frac{x^3}{3}\right)\frac{1}{1+\frac{1}{3-x^2}}\left(-\frac{1}{2}\frac{1}{(3-x^2)\sqrt{3-x^2}}\right)(-2x)\,\mathrm{d}x\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4}{3}\biggl(\frac{3}{2}\sqrt{2} + \frac{9}{2}\arcsin\frac{1}{\sqrt{3}} + 11\arctan\frac{1}{\sqrt{2}} - \int_0^1 \frac{12x^2 - x^4}{(4-x^2)\sqrt{3-x^2}}\,\mathrm{d}x\biggr) \\
&= \frac{4}{3}\biggl(\frac{3}{2}\sqrt{2} + \frac{9}{2}\arctan\frac{1}{\sqrt{2}} + 11\arctan\frac{1}{\sqrt{2}} \\
&\qquad - \int_0^1 \biggl(\frac{x^2}{\sqrt{3-x^2}} - \frac{8}{\sqrt{3-x^2}} + \frac{32}{(4-x^2)\sqrt{3-x^2}}\biggr)\,\mathrm{d}x\biggr) \\
&= \frac{4}{3}\biggl(\frac{3}{2}\sqrt{2} + \frac{31}{2}\arctan\frac{1}{\sqrt{2}} - \biggl[-\frac{x}{2}\sqrt{3-x^2} + \frac{3}{2}\arcsin\frac{x}{\sqrt{3}} \\
&\qquad - 8\arcsin\frac{x}{\sqrt{3}} + 32\cdot\frac{1}{2}\arctan\frac{x}{2\sqrt{3-x^2}}\biggr]_0^1\biggr), \qquad \text{se (2)} \\
&= \frac{4}{3}\biggl(\frac{3}{2}\sqrt{2} + \frac{31}{2}\arctan\frac{1}{\sqrt{2}} + \frac{1}{2}\sqrt{2} + \frac{13}{2}\arcsin\frac{1}{\sqrt{3}} - 16\arctan\frac{1}{2\sqrt{2}}\biggr) \\
&= \frac{4}{3}\biggl(2\sqrt{2} + \frac{31}{2}\arctan\frac{1}{\sqrt{2}} + \frac{13}{2}\arctan\frac{1}{\sqrt{2}} - 16\Bigl(\frac{\pi}{2} - 2\arctan\frac{1}{\sqrt{2}}\Bigr)\biggr), \qquad \text{se (3)} \\
&= \frac{4}{3}\Bigl(2\sqrt{2} - 8\pi + 54\arctan\frac{1}{\sqrt{2}}\Bigr) \\
&= \frac{4}{3}\Bigl(2\sqrt{2} - 8\pi + 54\Bigl(\frac{\pi}{2} - \arctan\sqrt{2}\Bigr)\Bigr), \qquad \text{se (4)} \\
&= \frac{4}{3}\Bigl(2\sqrt{2} + 19\pi - 54\arctan\sqrt{2}\Bigr).
\end{aligned}
$$

(1): Der gælder, at

$$
\arcsin\frac{1}{\sqrt{4-x^2}} = \arctan\frac{\frac{1}{\sqrt{4-x^2}}}{\sqrt{1-\frac{1}{4-x^2}}} = \arctan\frac{1}{\sqrt{4-x^2-1}} = \arctan\frac{1}{\sqrt{3-x^2}}.
$$

(2): Det ses, at

$$
\frac{1}{(4-x^2)\sqrt{3-x^2}} = \frac{3}{\bigl(4(3-x^2)+x^2\bigr)\sqrt{3-x^2}} = \frac{1}{2}\frac{1}{1+\bigl(\frac{x}{2\sqrt{3-x^2}}\bigr)^2}\frac{3}{2(3-x^2)\sqrt{3-x^2}}.
$$

Lad $u = \frac{x}{2\sqrt{3-x^2}}$. Da er

$$
\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}\frac{\sqrt{3-x^2}\cdot 1 - x\frac{1}{2\sqrt{3-x^2}}(-2x)}{3-x^2} = \frac{3-x^2+x^2}{2(3-x^2)\sqrt{3-x^2}} = \frac{3}{2(3-x^2)\sqrt{3-x^2}}.
$$

Dvs.

$$
\int \frac{1}{(4-x^2)\sqrt{3-x^2}}\,\mathrm{d}x = \frac{1}{2}\int\frac{1}{1+u^2}\,\mathrm{d}u = \frac{1}{2}\arctan u + c.
$$

(3): Der gælder, at

$$\arctan\frac{1}{2\sqrt{2}} = \arcsin\frac{\frac{1}{2\sqrt{2}}}{\sqrt{1+\left(\frac{1}{2\sqrt{2}}\right)^2}} = \arcsin\frac{1}{3} = \frac{\pi}{2} - \arccos\frac{1}{3}$$

$$= \frac{\pi}{2} - \arccos\frac{1-\left(\frac{1}{\sqrt{2}}\right)^2}{1+\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\pi}{2} - 2\arctan\frac{1}{\sqrt{2}}.$$

(4): For $0 < x < \infty$ har vi

$$\arctan\frac{1}{x} = y \iff \frac{1}{x} = \tan y \iff x = \cot y \iff y = \operatorname{arccot} x = \frac{\pi}{2} - \arctan x.$$

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