Matematisk modellering og numeriske metoder

Hints to the exercises related to Lecture 2

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Section 6.1

Exercise 3

- Use the table on page 207.

Exercise 5

- Notice that this case is not covered by the table!
- Use Theorem 2 on page 208.
- Combine this with what the table tells you about $\sinh$.

Section 6.2

Exercise 7

- First use Theorem 1 to convert the problem to an algebraic one by applying the Laplace transform on both sides of the equality.
- Solve the resulting equation in Y by isolating Y.
- Write the expression for Y in a way that can be recognized as the the sum of Laplace transforms of some functions.

- If one does it right, one will probably at some point arrive at an expression equivalent to

$$Y(s) = \frac{3.5s^2 + 4s - 22.5}{(s^2 + 7s + 12)(s - 3)}.$$

This monster may seem hard to attack. The standard tool is *partial fraction expansions*, which is briefly discussed on pages 228–230. We'll here treat the concrete example. First, we factor the denominator, e.g. by finding the roots of the quadratic expression:

$$(s^2 + 7s + 12)(s - 3) = (s + 4)(s + 3)(s - 3). \tag{1}$$

We would then like to write $Y(s)$ as a sum of the form

$$\frac{A}{s+4} + \frac{B}{s+3} + \frac{C}{s-3}$$

for some some suitably chosen, simple A, B and C. Clearly, we need

$$\frac{3.5s^2 + 4s - 22.5}{(s^2 + 7s + 12)(s - 3)} = \frac{(s+3)(s-3)A + (s+4)(s-3)B + (s+4)(s+3)C}{(s+4)(s+3)(s-3)}$$

or, more compactly (using (1)),

$$\begin{aligned} 3.5s^2 + 4s - 22.5 &= (s+3)(s-3)A + (s+4)(s-3)B + (s+4)(s+3)C \\ &= (A + B + C)s^2 + (B + 7C)s + (-9A - 12B + 12C). \end{aligned}$$

One should now solve for A, B and C:

$$\begin{aligned} 4 &= B + 7C \\ 3.5 &= A + B + C \\ -22.5 &= -9A - 12C + 12C, \end{aligned}$$

which via linear algebra gives $A = 2.5$, $B = 0.5$ and $C = 0.5$. We conclude that

$$Y(s) = \frac{2.5}{s+4} + \frac{0.5}{s+3} + \frac{0.5}{s-3}.$$

- Find a function y so that $Y = \mathcal{L}(y)$. This is done using Theorem 2 on page 208, the table on page 207 and the linearity of (the inverse of) the Laplace transform, which means that we can pull constants and sums through:

$$\mathcal{L}^{-1}\Big(2.5\frac{1}{s+4} + 0.5\frac{1}{s+3} + 0.5\frac{1}{s-3}\Big) = 2.5\,\mathcal{L}^{-1}\Big(\frac{1}{s+4}\Big) + 0.5\,\mathcal{L}^{-1}\Big(\frac{1}{s+3}\Big) + 0.5\,\mathcal{L}^{-1}\Big(\frac{1}{s-3}\Big).$$

Exercise 15

- Follow Example 6 found on the same page.