# Matematisk modellering og numeriske metoder

# Hints to the exercises related to Lecture 7

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## Section 9.8

## Exercise 5

- Just use the definition.

## Exercise 11

- That $v = y\mathbf{i}$ means that
$$v = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}.$$
- That the flow is incompressible means that $\operatorname{div} v = 0$.
- Let $\left(\begin{smallmatrix} x_0 \\ y_0 \\ z_0 \end{smallmatrix}\right)$ lie in the given cube (i.e. $0 \leq x_0 \leq 1$, $0 \leq y_0 \leq 1$, and $0 \leq z_0 \leq 1$). If this point is the coordinates of a particle at time $t = 0$, then where would this particle be at time $t = 1$? Well, the velocity vector at $\left(\begin{smallmatrix} x_0 \\ y_0 \\ z_0 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} y_0 \\ 0 \\ 0 \end{smallmatrix}\right)$, it is easy to see that this particle in fact has constant speed (why?), so to see where the particle is at $t = 1$, all we need to do is to add $1$ times the velocity vector to the position vector:
$$\begin{pmatrix} x_0 + y_0 \\ y_0 \\ z_0 \end{pmatrix}.$$
Alternatively, one can see this by writing the problem as a system of ODE's (this is how it is treated in Appendix 2 "Answers to Selected Problems").
- Try drawing a 2-dimensional section (including the $x$-axis) of what happens. See if you can get a feel of what happens to the particles.

- Find a mathematical expression of the region occupied at time $t = 1$ by the particles from the cube at $t = 0$. Hint: what happens to the corners?

## Exercise 13

- From Example 5 on page 372, we learn that $r$ is the position vector. This means that the velocity vector is of the form
$$v(r) = w \times r,$$
where $w$ is some constant vector. If we write $r = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} w_1 \\ w_2 \\ w_3 \end{smallmatrix}\right)$ , then this becomes
$$v(r) = \begin{pmatrix} w_2 z - w_3 y \\ w_3 x - w_1 z \\ w_1 y - w_2 x \end{pmatrix}.$$

- Find $\operatorname{div}(v)$ by direct computation.

- They ask if it seems plausible that nothing gets squeezed, disappears or appears if things are just rotating.

# Section 9.9

## Exercise 3

- In this case, the proof is just a direct computation.

- First, let $f$ be some twice continuously differentiable scalar field (a function taking real values but depending on the coordinates $x$, $y$, and $z$ and which is twice continuously differentiable).

- Write out what $\operatorname{grad}(f)$ is.

- Apply $\operatorname{curl}$ to this expression.

- Second, let $v$ be some twice continuously differentiable vector field (i.e. a function $v = \left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right)$, where $v_i$ takes real values and depends on $x$, $y$, and $z$ and is twice continuously differentiable, for $i = 1, 2, 3$).

- Write out what $\operatorname{curl} v$ is.

- Apply $\operatorname{div}$ to this expression.

## Exercise 5

- This means that
$$v(x, y, z) = \begin{pmatrix} x^3yz \\ xy^3z \\ xyz^3 \end{pmatrix}.$$

- Just use the definition.

## Exercise 7

- Again, just use the definition.

- Remember that $\sin(y)$ behaves like a constant when you differentiate with respect to $x$ and $z$, while $e^{-x}$ behaves like a constant when differentiating with respect to $y$ and $z$.

## Exercise 9

- The flow is incompressible if $\operatorname{div} v = 0$ and irrotational if $\operatorname{curl} v = 0$.

- The streamlines can be found by solving the system of ODE's $v = r'$, where $r$ is the position of a particle of the fluid.

## Exercise 11

- Exactly as above, only with different functions.

- See Appendix 2 for help on (setting up and) solving the system of ODE's.

## Exercise 14

(a) This follows by direct computation and basically just depends on the linearity of differentiation.

(b) This has already been proven in Exercise 3.

(c) Again, this is a matter of direct computation. Try calculating each term individually and see that the sum on the right equals the expression on the left.

(d) Again, this was done in Exercise 3.

(e) As with (a) and (c).

## Exercise 15

- There are two ways: either use (a) of Exercise 14 or:
- First calculate

$$u + v = v + u = \begin{pmatrix} z \\ x \\ y \end{pmatrix} + \begin{pmatrix} y + z \\ z + x \\ x + y \end{pmatrix}$$

## Exercise 19

- This can be done directly or by using (a) and (c) of Exercise 14.
- Obviously, the results are the same, as $v$ clearly is irrotational.